Main claim

We are presented with a certain (informal) proof.
It is claimed that this proof is a proof of consistency of PA "within PA" or "on the basis of PA axioms".

We need to answer two questions:
1) Have we really proved *consistency of PA*, or something else?
2) Is it really a proof *within PA*?

What we agree on

It seems that we agree on many things:

1) PA is the familiar formal system, formulated in the first order logic with equality.

2) The meaning of the term ‘consistency of PA’: There is no formal proof of $0 = 1$ from the axioms of PA by the rules of first order logic. A formal proof is a sequence of formulas satisfying certain standard conditions.

3) There is a *contentual version of* PA, denoted $\widehat{\mathrm{PA}}$ in the paper. It corresponds to the *informal axiomatic reasoning* on the basis of PA axioms, in the sense of axiomatic method of Hilbert. We seem to agree that any formal proof in PA can be read informally as a proof in $\widehat{\mathrm{PA}}$. Even more, any proof in $\widehat{\mathrm{PA}}$ can (after a suitable detailing) be formalized in PA.

This is the thesis that FO logic adequately formalizes informal axiomatic proofs. I stress that this is not a discussion of various less well-delineated concepts of proof such as finitism, etc. Whether any intuitively finitistic argument can be formalized in PA is a different claim that we are not discussing here.

4) We also seem to agree on the fact that the use of some fixed natural Gödel numbering of sequences, formulas, proofs etc. in PA and $\widehat{\mathsf{PA}}$ is harmless. We could also replace PA by some equivalent theory in which strings, formulas, proofs, etc. are first-class citizens.

Towards the point of disagreement

Formal axiomatic proofs in PA are, prima facie, uninterpreted. They are just sequences of symbols without any meaning attached. We know that we can interpret them, for example, in any first order model of PA, other kinds of models are also possible.

However, in the contentual $\widehat{\text{PA}}$ our (English) words refer to a specific *standard interpretation*: variables range over natural numbers, '$+$' means $+$, etc. One can say that $\widehat{\text{PA}}$-proofs are PA-proofs together with the contentual interpretation (in the standard model) of their statements. One can also say that $\widehat{\text{PA}}$-proofs are *regimented proofs*: contentual proofs simultaneously obeying the regimentation of the first order logical rules.

Thus, $\widehat{\text{PA}}$ statements have meaning, which is essentially their interpretation in the standard model.

Does G2 prohibit (some) proofs of consistency of PA?

I maintain that it does.

$\widehat{\text{PA}}$ presumes the standard interpretation. Under the standard interpretation Gödelian consistency statement $\neg\exists x \, \mathrm{proof}(x, \ulcorner 0 = 1 \urcorner)$ means precisely that there is no proof of the contradiction in PA.

Hence, G2 means that there cannot be a (regimented) $\widehat{\text{PA}}$ proof of consistency of PA. If there was such a proof, due to regimentation it would correspond to a formal proof of Con(PA) in PA.

A serial proof

We are presented a primitive recursive term $s(x)$ and a concrete PA-proof p of

$$\forall x\, \mathrm{proof}(s(x), \ulcorner \neg \mathrm{proof}(\dot{x}, \ulcorner 0 = 1 \urcorner) \urcorner).$$

We are also presented a $\widehat{\mathrm{PA}}$-proof of the (informal) claim that, for each n, $s(n)$ encodes a PA-proof of $\neg \mathrm{proof}(\bar{n}, \ulcorner 0 = 1 \urcorner)$. The latter is just a non-formalized version of the former.

The fact itself is undoubtedly true. Can we interpret this proof as a proof of consistency of PA in PA (or in $\widehat{\mathrm{PA}}$)?

There are two options:

1) The proof p is a proof in PA, but it proves a different claim, namely, that, for each n, the number $s(n)$ encodes a proof of $\neg\mathrm{proof}(\underline{n}, \ulcorner 0 = 1 \urcorner)$. To infer from this claim that n is not proof of $0 = 1$ involves reflection over PA-proofs. This is outside PA and $\widehat{\mathsf{PA}}$.

2) Artemov's position seems to be different and more refined. He maintains that consistency of PA is a 'serial property' and should not be expressed in the standard way as a Π_1-sentence in arithmetic. Instead, it is an infinite schema. This would extend the realm of PA-proofs to serial proofs of serial properties.

This is a possible stance. However, a few remarks are in order.

1. PA enhanced with serial properties is no longer the standard PA. It is PA enriched with additional expressive and proof power. It may be interesting to study the enhanced system, but identifying it with PA is misleading.
2. The point of formalization of proofs using first order logic was precisely to make provability an ordinary mathematical property, on a par with many others. There seems to be no reason to treat the consistency formula differently from any other Π_1-property of numbers.
3. The requirement that the proofs of serial properties must be computed by primitive recursive terms seems a bit arbitrary (why not take any other class of provably total computable functions)?

Alexander Pushkin, “Motion”

One bearded sage concluded: there’s no motion.
Without a word, another walked before him.
He couldn’t answer better; all adored him
And all agreed that he disproved that notion.

But one can see it all in a different light,
For me, another funny thought comes into play:
We watch the sun move all throughout the day
And yet the stubborn Galileo had it right.

1825