Cancellation Problems and Algebraic Groups

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Notation, conventions, and terminology

- k an algebraically closed field of characteristic zero.
- Variety means algebraic variety over k (so algebraic group means algebraic group over k)

- Below all algebraic groups are affine
- Action of an algebraic group means algebraic action
- A variety endowed with an action of algebraic group G is called a G-variety and this action is called a G-action

Notation, conventions, and terminology

- $X \cong Y$ denotes that varieties X and Y are isomorphic
- $X \overset{\text{bir}}{\approx} Y$ denotes that irreducible varieties X and Y are birationally isomorphic
- $\mathbb{A}^n$ n-dimensional affine space over k

Birational setting

Problem 1 (O. Zariski, 1949)

Let X be an irreducible n-dimensional variety.

$$\boxed{\textit{Does}\quad X \times \mathbb{A}^1 \overset{\text{bir}}{\approx} \mathbb{A}^{n+1} \ \textit{imply}\ \ X \overset{\text{bir}}{\approx} \mathbb{A}^n\ ?}$$

• J. Lüroth, 1876:
For $n = 1$, the answer to Problem 1 is **yes**.

• G. Castelnuovo, 1896:
For $n = 2$, the answer to Problem 1 is **yes**.

More generally, given a positive integer d, one asks:

Problem 2

Let X be an irreducible n-dimensional variety.

$$\boxed{\textit{Does}\quad X \times \mathbb{A}^d \overset{\text{bir}}{\approx} \mathbb{A}^{n+d} \ \textit{imply}\ \ X \overset{\text{bir}}{\approx} \mathbb{A}^n\ ?}$$

• A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer, 1985:

If $f(x_1, x_2) = x_1^3 + p(x_2)x_1 + q(x_2) \in k[x_1, x_2]$ is an irreducible polynomial, whose discriminant $\Delta(x_2) = 4p(x_2)^3 + 27q(x_2)^2$ has degree $\geqslant 5$, and V is the hypersurface in $\mathbb{A}^4$ defined by the equation

$$x_3^2 - \Delta(x_2)x_4^2 - f(x_1, x_2) = 0,$$

then $X = V$ yields the **negative answer to Problem 2** for $d = n = 3$, i.e.,

$$\boxed{V \times \mathbb{A}^3 \overset{\text{bir}}{\approx} \mathbb{A}^6, \quad \text{but } V \overset{\text{bir}}{\not\approx} \mathbb{A}^3.}$$

- N. Shephed-Barron, 2004:

The same $X = V$ yields the **negative answer to Problem 2** also for $d = 2$, $n = 3$, i.e.,

$$\boxed{V \times \mathbb{A}^2 \overset{\text{bir}}{\approx} \mathbb{A}^5, \quad \text{but } V \overset{\text{bir}}{\not\approx} \mathbb{A}^3.}$$

Corollary

At least for one of the values $n = 3$ or $n = 4$ the answer to Problem 1 is **negative**.

• It is **unknown** whether $X = V$ yields the negative answer to Problem 2 for $d = 1$, $n = 3$, i.e., whether $V \times \mathbb{A}^1 \overset{\text{bir}}{\approx} \mathbb{A}^4$.

Biregular setting

The following is usually called the Zariski cancellation problem:

Problem 3

Let X be an irreducible n-dimensional variety X.

$$\boxed{\textit{Does}\quad X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1} \textit{ imply}\quad X \cong \mathbb{A}^n?}$$

• For $n = 1$, the answer to Problem 3 is **yes** (follows from Lüroth's theorem).

• T. Fujita, M. Miyanishi, T. Sugie, 1979–80: For $n = 2$, the answer to Problem 3 is **yes**.

• For $n \geqslant 3$, the answer to Problem 3 is **unknown**.

The case of positive characteristic:

- N. Gupta, 2014:

Replace the condition $\operatorname{char} k = 0$ by $\operatorname{char} k = p > 0$. If

$$f(x_1, x_2) = x_1^{p^e} + x_2 + x_2^{sp} \in k[x_1, x_2], \quad \text{where } p^e \nmid sp,\ sp \nmid p^e,$$

and U is the hypersurface in $\mathbb{A}^{d+3}$ defined by the equation

$$f(x_1, x_2) - x_3^{r_3} \cdots x_{d+2}^{r_{d+2}} x_{d+3} = 0, \quad \text{where } r_i \geqslant 2 \text{ for every } i,$$

then $X = U$ yields the **negative answer to Problem 3** for $n = d + 2$, i.e.,

$$\boxed{U \times \mathbb{A}^1 \cong \mathbb{A}^{d+3}, \quad \text{but } U \ncong \mathbb{A}^{d+2}.}$$

More generally,

Problem 4

Let X and Y be the irreducible n-dimensional varieties.

$$\boxed{\textit{Does}\quad X\times\mathbb{A}^1\cong Y\times\mathbb{A}^1\quad\textit{imply}\quad X\cong Y\,?}$$

- W. Danielewski, 1989:

The following surfaces X and Y in $\mathbb{A}^3$

$$X : \ x_1x_3 - x_2^2 - x_2 = 0,$$
$$Y : \ x_1^2x_3 - x_2^2 - x_2 = 0$$

yield the **negative answer to Problem 4** for $n = 2$, i.e.,

$$\boxed{X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1, \text{ but } X \not\cong Y.}$$

Below is discussed a local version of the Zariski cancellation problem. Making precise its formulation leads to distinguishing the following class of varieties:

Definition (V.P., 2013)

An irreducible variety X is called

- **flattenable** *if X isomorphic to an open subset of some $\mathbb{A}^d$;*
- **locally flattenable** *if every point of X has a flattenable open neighbourhood.*

Locally flattenable varieties: other names

Under other names, locally flattenable varieties appeared in the literature long ago:

• C. Chevalley (1958) and Y. Manin (1974) call them **special varieties**

• S. Akbulut (1992) calls them **algebraic spaces**

• G. Bodnár, H. Hauser, J. Schicho, O. Villamayor (2008) call them **plain varieties**

• F. Bogomolov, C. Böhning (2014) call them **uniformly rational varieties.** The latter term is used in several recent papers by other authors.

The definition implies that

every locally flattenable variety is rational and smooth.

Whether the converse is true is **unknown**:

Problem 5

Let X be an irreducible smooth rational variety. Is X locally flattenable?

For projective X, Problem 5 was raised in 1989 by M. Gromov.

Examples of locally flattenable varieties

Below are some examples of locally flattenable varieties.

First example: curves and surfaces

Theorem (G. Bodnár, H. Hauser, J. Schicho, O. Villamayor, 2008)

For $\dim X \leqslant 2$*, the answer to Problem 5 is* **yes***, i.e,*
every irreducible smooth rational curve or surface is locally flattenable.

Examples of locally flattenable varieties: homogeneous spaces

Second example: homogeneous spaces

Theorem

Let X be an irreducible variety. If $\mathrm{Aut}(X)$ acts on X transitively, then the following are equivalent:

(a) *X is rational;*

(b) *X is locally flattenable.*

Corollary

Let G be a connected algebraic group and let H be a closed subgroup of G. Then the following are equivalent:

(a) *G/H is rational;*

(b) *G/H is locally flattenable.*

Examples of locally flattenable varieties: homogeneous spaces

In the notation of this Corollary, not every G/H is rational:

Theorem (V.P., 2013)

There are nonrational G/H with finite H.

Corollary

Not every G/H is locally flattenable.

Examples of locally flattenable varieties: homogeneous spaces

As a matter of fact, there are explicit examples of G and H such that G/H is nonrational (hence not locally flattenable).

Examples of locally flattenable varieties: homogeneous spaces

Example

Fix a prime integer $p \geqslant 5$ and let F be the finite group of order p^5 defined by following generators and relations:

$$G = \langle g_1, g_2, g_3, g_4, g_5 \rangle, \quad g_i^p = 1 \text{ for all } i,$$
$$\langle g_5 \rangle = \text{center of } G,$$
$$[g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = [g_3, g_2] = g_5,$$
$$[g_4, g_2] = [g_4, g_3] = 1$$

where $[a, b] = a^{-1}b^{-1}ab$. Let $\iota\colon F \to \mathrm{GL}(V)$ be any faithful linear representation where V is a finite-dimensional vector space over k. Put $G = \mathrm{GL}(V)$ and $H = \iota(F)$. Then G/H is nonrational.

Rationality of G/H for connected H is the **old open problem**:

Problem 6

Are there **nonrational** G/H *with* **connected** H*?*

Examples of locally flattenable varieties: homogeneous spaces

At the same time, there are many examples of rational G/H with connected H. The case of trivial H is among them, which gives

Examples of locally flattenable varieties: homogeneous spaces

Theorem

Every connected algebraic group is locally flattenable.

Examples of locally flattenable varieties: fiber bundles

Third example: fiber bundles

In the definition of fiber bundles, local triviality is understood in the étale topology. Namely, a morphism $\pi\colon X \to Z$ is called an algebraic fiber bundle over Z with fiber F if every point of Z has a neighbourhood U such that for some étale cover $\theta\colon \widetilde{U} \to U$ there is an isomorphism

$$\tau\colon \widetilde{U} \times_U \pi^{-1}(U) \to \widetilde{U} \times F$$

making the following diagram commutative

$$\begin{array}{ccc} \widetilde{U} \times_U \pi^{-1}(U) & \xrightarrow{\quad\tau\quad} & \widetilde{U} \times F \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \widetilde{U} & \end{array}$$

If for every U, this holds with $\widetilde{U} = U$, $\theta = \mathrm{id}$, then π is locally trivial in the Zariski topology.

Examples of locally flattenable varieties: fiber bundles

Theorem (J.-P. Serre, 1958)

Every algebraic vector bundle is locally trivial in the Zariski topology.

Examples of locally flattenable varieties: fiber bundles

The definition implies the following:

Theorem

Let

- X, Z, *and* F *be the irreducible varieties,*
- $X \to Z$ *be an algebraic fiber bundle over an* Z *with fiber* F *locally trivial in the Zariski topology.*

Then

$$\boxed{Z \text{ and } F \text{ are locally flattenable}} \Longrightarrow \boxed{X \text{ is locally flattenable}}.$$

Corollary

The total space of an algebraic vector bundle over a locally flattenable base is locally flattenable.

Combining this with some basic results of the algebraic transformation groups theory, one obtains the following:

Let G be a connected reductive algebraic group and let X be a smooth affine G-variety. Assume that

$$\text{every } G\text{-invariant regular function on } X \text{ is constant} \qquad (*)$$

(for instance, $(*)$ holds if X contains an open G-orbit). Some basic facts of algebraic transformation groups theory and $(*)$ imply that

$$X \text{ contains a unique closed } G\text{-orbit } \mathcal{O}.$$

Theorem

If the variety $\mathcal{O}$ is rational, then X is locally flattenable.

Fourth example: spherical varieties

Let G be a connected reductive algebraic group. Recall that a G-variety X is called spherical variety of G if X contains a a dense open orbit of a Borel subgroup of G.

Theorem (V.P., 2018)

Every smooth spherical variety is locally flattebable.

Examples of locally flattenable varieties: spherical varieties

Since every toric variety is spherical, this implies

Corollary (F. Bogomolov, C. Böhning, 2014)

Every smooth toric variety is locally flattenable.

Fifth example: blow-ups with nonsingular centers

Theorem (M. Gromov, 1989)

The blow-up of a locally flattenable variety along a smooth subvariety is locally flattenable.

Stable versions

The above concepts admit **stable versions**.

Namely, recall

Definition

A variety X is called **stably rational** *if there is a rational variety Y such that $X \times Y$ is rational.*

In a similar way one introduces

Definition

A variety X is called **stably flattenable** *(resp.* **stably locally flattenable***) if there is a flattenable (resp. locally flattenable) variety Y if $X \times Y$ is flattenable (resp. locally flattenable).*

Recenly **stable version of Problem 5** was answered in the **affirmative**:

Theorem (J. Banecki, 2025)

Let X be a smooth irreducible variety such that $X \times \mathbb{A}^1$ is rational. Then $X \times \mathbb{A}^2$ is locally flattenable.

Corollary (J. Banecki, 2025)

Every smooth irreducible stably rational variety is stably locally flattenable.

The local version of the Zariski cancellation problem

The affine spaces $\mathbb{A}^d$ involved in the formulation of the Zariski cancellation problem are flattenable varieties of a special kind. The local version of the Zariski cancellation problem is obtained by replacing the affine spaces with arbitrary flattenable varieties:

Problem 7

Let X and Y be the varieties such that

$$Y \text{ and } X \times Y \text{ are } \underline{\text{flattenable}}.$$

Does it follow from this that X is also <u>flattenable</u>?

This can be reformulated as follows:

Problem 7 (Reformulation)

Are there nonflattenable but stably flattenable varieties?

To answer this question, we explore flattenability of connected algebraic groups (recall that every such group is locally flattenable).

To do this, we first recall some basic definitions and properties of such groups.

About algebraic groups

Let G be a connected (affine) algebraic group.

- The radical $\mathrm{Rad}(G)$ of G is its maximal connected solvable normal subgroup.
- $\mathrm{Rad}(G) = T \ltimes \mathrm{Rad}_u(G)$, where T is a torus and $\mathrm{Rad}_u(G)$ is a unipotent algebraic group (the unipotent radical of G).
- G is reductive means that $\mathrm{Rad}_u(G)$ is trivial, i.e., $\mathrm{Rad}(G)$ is a torus.
- If G is reductive, then the torus $\mathrm{Rad}(G)$ is the connected component of the center of G.
- G is semisimple means that $\mathrm{Rad}(G)$ is trivial.

Flattenable and nonflattenable algebraic groups

Theorem (V.P., 2018)

Let G be a connected algebraic group,

- *If G is solvable, then G is flattenable.*
- *If G is flattenable and nonsolvable, then*

$$\mathrm{Rad}_u(G) \neq \mathrm{Rad}(G).$$

Flattenable and nonflattenable algebraic groups

Corollary (V.P., 2018)

Let G be a nontrivial connected reductive algebraic group.

(i) *If G is flattenable, then the dimension of its center is positive.*

(ii) *In particular, every semisimple G is not flattenable.*

Since every connected algebraic group is locally flattenable, (ii) implies

Corollary

There are locally flattenable varieties that are not flattenable.

Answering the local version of the Zariski cancellation problem

Using this, one obtains the following answer to the local version of the Zariski cancellation problem:

Theorem (V.P., 2018)

There are affine varieties X and Y such that

- X *is not flattenable;*
- Y *and* $X \times Y$ *are flattenable.*

Reformulation:

There are nonflattenable, but stably flattenable varieties.

Namely, this is supported by the following concrete example:

Answering the local version of the Zariski cancellation problem: example

Example

Fix an integer $n > 1$ and take

$$X = \mathrm{SL}_n, \quad Y = \mathrm{GL}_1.$$

Both X and Y are irreducible.
Since SL_n is semisimple,

$$\underline{X \text{ is not flattenable}}.$$

Since GL_n is open in $\mathrm{Mat}_{n\times n} = \mathbb{A}^{n^2}$, it is flattenable.
In particular,

$$\underline{Y \text{ is flattenable}}.$$

Answering the local version of the Zariski cancellation problem: example

Claim:

$X \times Y$ and GL_n are isomorphic varieties, whence

$$X \times Y \text{ is flattenable.}$$

Warning:

The subtlety is that, as explained below,

$$X \times Y = \mathrm{SL}_n \times \mathrm{GL}_1 \quad \text{and} \quad \mathrm{GL}_n$$

are not isomorphic algebraic groups!

Answering the local version of the Zariski cancellation problem: example

Proof of Claim:

The morphism of varieties

$$\varphi\colon \mathrm{SL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n, \quad (s, a) \mapsto s\,\mathrm{diag}(a, 1, \ldots, 1),$$

is the isomorphism because the morphism

$$\mathrm{GL}_n \to \mathrm{SL}_n \times \mathrm{GL}_1, \quad g \mapsto \big(g\,\mathrm{diag}(1/\det(g), 1, \ldots, 1), \det(g)\big)$$

is its inverse.

Warning:

φ is not a group isomorphism!

Answering the local version of the Zariski cancellation problem: example

Moreover, the center of $\mathrm{SL}_n \times \mathrm{GL}_1$ is

$$\underbrace{\{\mathrm{diag}(\varepsilon, \ldots, \varepsilon) \mid \varepsilon \in k^\times, \varepsilon^n = 1\}}_{\text{cyclic group of order } n} \times \mathrm{GL}_1,$$

hence it is disconnected, and the center of GL_n is

$$\{\mathrm{diag}(\varepsilon, \ldots, \varepsilon) \mid \varepsilon \in k^\times\} \cong \mathrm{GL}_1,$$

hence it is connected. Therefore,

$\boxed{\mathrm{SL}_n \times \mathrm{GL}_1 \text{ and } \mathrm{GL}_n \text{ are not isomorphic algebraic groups}}$

Answering the local version of the Zariski cancellation problem: example

In fact, the essence of this example is a reflection of the following general phenomenon.

Answering the local version of the Zariski cancellation problem: example

We consider

- G a connected reductive algebraic group,
- D the derived group of G,
- Z be the identity component of the center of G.

In general, $D \times Z$ and G are **not** isomorphic algebraic groups. Namely, the following criterion holds:

The following properties are equivalent:

- *algebraic groups $D \times Z$ and G are isomorphic;*
- $D \cap Z = \{e\}$;
- *the isogeny $D \times Z \to G$, $(d, z) \mapsto dz$, is an isomorphism.*

However the following theorem shows that the **underlying varieties** of these groups are **always isomorphic**:

Theorem (V.P., 2022)

There is an embedding of algebraic groups $\iota\colon Z \hookrightarrow G$ *such that the map*

$$D \times Z \to G,\ (d, z) \mapsto d\cdot\iota(z),$$

is an isomorphism of algebraic varieties (but, in general, not a homomorphism of algebraic groups).

Equivariant versions of flattenability: definitions

We now consider the **equivariant** versions of flattenability.

Let G be an algebraic group.

Definition

A G-variety X is called

equivariantly (resp., linearly equivariantly) flattenable

if there are

- *a G-action (resp., a linear G-action) on some* $\mathbb{A}^n$;
- *a G-equivariant open embedding of X in this G-variety* $\mathbb{A}^n$.

Such X is necessarily irreducible.

For $X = G$, this leads to the following

Definition

An algebraic group G is called

equivariantly (resp., linearly equivariantly) flattenable

if the variety G endowed with the G-action by left multiplication shares this property.

Equivariant versions of flattenability: definitions

The assumption $\mathrm{char}\, k = 0$ implies the following reformulation of the latter definition:

The following properties of an algebraic group G are equivalent:

- *G is equivariantly (resp., linearly equivariantly) flattenable;*
- *there is a G-action (resp., a linear G-action) on $\mathbb{A}^{\dim(G)}$ such that the G-stabilizer of some point of $\mathbb{A}^{\dim(G)}$ is trivial.*

Our further discussion centers around the following **open** problem:

Problem 8

Obtain a group-theoretic classification of equivariantly (resp. linearly equivariantly) flattenable algebraic groups.

Example

Every GL_n is linearly equivariantly flattenable because

$$\dim(\mathrm{GL}_n) = n^2$$

and for the linear action of GL_n on $\mathbb{A}^{n^2} = \mathrm{Mat}_{n\times n}$ by left multiplication, the GL_n-stabilizer of the identity matrix from $\mathrm{Mat}_{n\times n}$ is trivial.

This example admits the following generalization.

Example

Let A be a finite-dimensional associative k-algebra with identity. The group A^* of its invertible elements is a connected algebraic group. It is open in A, therefore, $\dim(A^*) = \dim(A)$. For the linear action of A^* on A by left multiplication, the A^*-stabilizer of the identity is trivial. Hence

$$\underline{A^* \text{ is a linearly equivariantly flattenable group}}.$$

For $A = \mathrm{Mat}_{n\times n}$, we obtain $A^* = \mathrm{GL}_n$. More generally, if A is semisimple, then A^* is a reductive group of type

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s},$$

and all groups of this type are obtained in this way.

Equivariantly flattenable groups: examples

Example

If $G_1, \dots, G_m$ are equivariantly (resp., linearly equivariantly) flattenable groups, then, clearly, $G_1 \times \cdots \times G_m$ is equivariantly (resp., linearly equivariantly) flattenable as well. In particular, this and the previous example imply that

the group $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s}$ is linearly equivariantly flattenable

for any $n_1, \dots, n_s$. Taking $n_1 = \dots = n_s = 1$ implies that

every torus is linearly equivariantly flattenable.

Equivariantly flattenable groups: examples

The next example shows that

there are equivariantly flattenable groups
that are not linearly equivariantly flattenable

Example

Every unipotent algebraic group G is, as a variety, isomorphic to $\mathbb{A}^{\dim(G)}$. Therefore,

G is equivariantly flattenable.

However, one proves that

G is linearly equivariantly flattenable only if it is trivial.

Equivariantly flattenable groups: examples

The next example shows the existence of

- linearly equivariantly flattenable **reductive** groups different from $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s}$;
- a nonflattenable group G_1 and
  a linearly equivariantly flattenable group G_2
  such that

$$G_1 \times G_2 \text{ is linearly equivariantly flattenable.}$$

Example

Take $G_1 = \mathrm{SL}_n$, $G_2 = \mathrm{GL}_1$, and $G = G_1 \times G_2$. Then
G_1 in nonflattenable and G_2 is linearly equivariantly flattenable.

We have $\dim(G) = n^2$. Consider the linear G-action on
$\mathbb{A}^{n^2} = \mathrm{Mat}_{n\times n}$ defined by the condition that

$$g = (s, \varepsilon) \in G, \text{ where } s \in G_1,\ \varepsilon \in G_2,$$

$$\text{transfoms } a \in \mathrm{Mat}_{n\times n} \text{ into}$$

$$\boxed{g(a) := s \cdot a \cdot \mathrm{diag}(\varepsilon, 1, \ldots, 1).}$$

Then the
G-stabilizer of the identity matrix from $\mathrm{Mat}_{n\times n}$ is trivial.
Hence the group G is linearly equivariantly flattenable.

Levy subgroups

We shall now discuss a relationship between equivariant flattenability of arbitrary algebraic groups and that of reductive ones.

For this purpose, we recall that maximal connected reductive subgroups L of a connected algebraic group G (called the Levi subgroups) are characterized (in view of $\mathrm{char}(k) = 0$) by the property

$$G = L \ltimes \mathrm{Rad}_u(G)$$

and are all conjugate in G.

Besides,

$$\boxed{\begin{array}{c}\text{all maximal connected}\\ \text{semisimple subgroups of } G\end{array}} = \boxed{\begin{array}{c}\text{derived subgroups of}\\ \text{all Levi subgroups of } G\end{array}}$$

These subgroups, too, are all conjugate in G.

Relation of equivariant flattenability of a group to that of its Levy subgroup

The following theorem reveals that equivariant flattenability of G is related to that of L.

Theorem (V.P., 2018)

Let G be a connected algebraic group and let L be a Levi subgroup of G. If L is equivariantly flattenable, then so is G.

Equivalent flattenability of solvable groups

We shall now discuss equivariant flattenability of connected **solvable** algebraic groups.

Equivariant flattenability of solvable groups

If G is a connected solvable algebraic group, then L is a torus. Since tori are equivariantly flattenable, the above theorem yields

Corollary (V.P., 2018)

Every connected solvable algebraic group is equivariantly flattenable.

Equivariant flattenability of solvable groups

So in the class of connected algebraic groups we have

solvable groups	$=$	linearly equivariantly flattenable solvable groups	$\bigsqcup$	equivariantly flattenable but not linearly equivariantly flattenable solvable groups

In the right-hand side of this equality both boxes are nonempty as the following examples show:

- all **tori** lie in the **first** box,
- all nontrivial **unipotent** groups lie in the **second** box.

One can show that both boxes contain groups that are neither tori nor unipotent.

The following problem is **open**:

Problem 9

Find a group-theoretic criterion for a connected solvable algebraic group to be linearly equivariantly flattenable.

We shall now discuss equivariant flattenability of connected **reductive** algebraic groups.

While for connected **solvable** groups,

the classes of **equivariantly flattenable** groups and
linearly equivariantly flattenable groups
do not coincide,

the following theorem shows that for connected **reductive** groups, the situation is **opposite**:

Equivariant flattenability of reductive groups

Theorem (V.P., 2018)

The following properties of a connected reductive algebraic group G are equivalent:

- *G is equivariantly flattenable;*
- *G is linearly equivariantly flattenable.*

Equivariant flattenability of reductive groups

Another difference is that

while every connected solvable group is equivariantly flattenable,

there are connected reductive groups that are not flattenable

(and a fortiori not equivariantly flattenable) — for example, these are nontrivial connected semisimple groups.

Equivariant flattenability of reductive groups

Moreover,

while for connected solvable groups

there is no difference between flattenability and equivariant flattenability,

it is unknown whether the latter property holds for connected reductive groups:

Problem 10

Are there connected reductive groups that are

flattenable but not equivariantly flattenable?

Equivariant flattenability of reductive groups

More can be said regarding the following

Problem 11

Obtain a group-theoretic classification of equivariantly flattenable connected reductive algebraic groups.

Namely, recall that an algebraic group G (not necessarily connected and affine) is called special in the sense of J.-P. Serre if every principal locally trivial in étale topology G-bundle is locally trivial in the Zariski topology.

Special groups are classified:

Theorem (J.-P. Serre and A. Grothendieck, 1958)

Every special group is automatically connected and affine. The following properties of a connected affine algebraic group G are equivalent:

(a) *G is special;*

(b) *every maximal connected semisimple subgroup of G is isomorphic to a group of type*

$$\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r} \times \mathrm{Sp}_{2m_1} \times \cdots \times \mathrm{Sp}_{2m_s}$$

The notion of special group was introduced by J.-P. Serre who also proved (b)⇒(a). A. Grothendieck proved (a)⇒(b).

The following theorem reveals the role of special groups in describing equivariantly flattened algebraic groups:

Theorem (V.P., 2018)

Every linearly equivariantly flattenable algebraic group is special.

For Problem 11, this theorem yields the following:

Equivariant flattenability of reductive groups

Theorem (V.P., 2018)

For every nontrivial equivariantly flattenable reductive group G, the following properties hold:

- *the derived group L of G is a semisimple group of type*

$$\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r} \times \mathrm{Sp}_{2m_1} \times \cdots \times \mathrm{Sp}_{2m_s};$$

- $\mathrm{Rad}(G)$ *is a central torus C of positive dimension;*
- $G = L \cdot C$ *and $C \cap L$ is finite.*

Comment on Problem 11

Initially, the author of this talk was overoptimistic and put forward in Obewolfach the conjecture that equivariantly flattenable connected reductive groups are precisely all groups of the type

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}. \qquad (*)$$

This overoptimism was shared by some of the participants (C. Procesi, N. Ressayre) who even sketched a plan of possible proof.

Comment on Problem 11

However, the found later above example of $\mathrm{SL}_n \times \mathrm{GL}_1$ shows the existence of equivariantly flattenable reductive groups which are not of the type $(*)$).

Given this, for some time there was the belief in the weakened conjecture that the maximal connected semisimple subgroup of any equivariantly flattenable connected reductive group should be the type

$$\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}.$$

Comment on Problem 11

So it came as a surprise, when the examples of equivariantly flattenable reductive groups whose derived group contains factors of the type Sp have been revealed, making Problem 11 even more intriguing.

Here are these examples.

Theorem (D. Burde, W. Globke, A. Minchenko, 2017)

For every positive integer n, the group

$$G := \mathrm{Sp}_{2n} \times \mathrm{GL}_{2n-1} \times \mathrm{GL}_{2n-2} \times \cdots \times \mathrm{GL}_1$$

is equivariantly flattenable.

Equivariant flattenability of reductive groups

Remark

Since

Sp_{2n} is not flattenable, but

$\mathrm{GL}_{2n-1} \times \mathrm{GL}_{2n-2} \times \cdots \times \mathrm{GL}_1$ is flattenable,

this theorem provides new examples that solve in the negative the local version of the Zariski cancellation problem (Problem 7).

The key element of the proof of the last theorem is the explicit construction of

- a linear space V over k,
- a linear action of G on V,

such that

- $\dim V = \dim G$,
- the G-stabilizers of points in general position in V are trivial.

Here are these V and action.

- The vector space V is defined by the formula:

$$\boxed{V := \mathrm{Mat}_{2n\times 1} \oplus \mathrm{Mat}_{2n\times(2n-1)} \oplus \mathrm{Mat}_{(2n-1)\times(2n-2)} \oplus \cdots \oplus \mathrm{Mat}_{2\times 1}.}$$

- The linear action $G \times V \to V,\ \ (g, v) \mapsto g \cdot v$, of G on V, is defined by the formulas:

if

$$g = (A, B_{2n-1}, B_{2n-2}, \ldots, B_2, B_1) \in G,\ A \in \mathrm{Sp}_n,\ B_d \in \mathrm{GL}_d,$$
$$v = (X, Y_{2n-1}, Y_{2n-2}, \ldots, Y_1) \in V,\ X \in \mathrm{Mat}_{2n\times 1},\ Y_d \in \mathrm{Mat}_{(d+1)\times d},$$

then

$$\boxed{g \cdot v = (AX, AY_{2n-1}B_{2n-1}^{\top}, B_{2n-1}Y_{2n-2}B_{2n-2}^{\top}, \ldots, B_2 Y_1 B_1^{\top}).}$$

We shall now briefly dwell on flattenability of **arbitrary** G-varieties X for connected **reductive** group G.

For homogeneous X, this topic has received much attention in the literature since eighties of the last century in the form of classifying so-called prehomogeneous vector spaces of connected reductive groups G, i.e., linear G-actions on $\mathbb{A}^n$ such that $\mathbb{A}^n$ contains an open G-orbit $\mathcal{O}$. This $\mathcal{O}$ is then a homogeneous linearly flattenable G-variety.

Equivariant flattenability of arbitrary G-varieties

Such classifications have been obtained (A. Ehlashvili, 1972) in the following two cases:

(i) G is simple,

(ii) G is semisimple and G-action of $\mathbb{A}^n$ is irreducible.

Example

If both (i) and (ii) hold, then, up to isomorphism, there are only the following four prehomogeneous vector spaces:

- SL_n naturally acting on $\mathbb{A}^n$,
- SL_{2n+1} naturally acting on the space $\mathbb{A}^{n(n-1)/2}$ of skew-symmetric bilinear forms in n variables,
- Sp_{2n} naturally acting on $\mathbb{A}^{2n}$,
- Spin_{10} acting on $\mathbb{A}^{16}$ by means of a half-spinor representation.

It appears that there is a relation of the topic of flattenability of arbitrary G-varieties to the following old

Problem 12 (Linearization problem)

Is any algebraic action of a (not necessarily connected) reductive algebraic group R on an affine space $\mathbb{A}^n$ linearizable, i.e., linear with respect to an appropriate polynomial coordinate system on $\mathbb{A}^n$?

Replacing R with its quotient by the kernel of action, one may assume that this action is faithful, i.e., R is a subgroup of $\mathrm{Aut}(\mathbb{A}^n)$. Then the Linearization problem admits the following equivalent reformulation:

Problem 13 (Reformulation)

Given a reductive algebraic subgroup R of the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of the algebraic variety $\mathbb{A}^n$, is there an element $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ such that

$$\varphi R \varphi^{-1} \subseteq \mathrm{GL}_n?$$

Equivariant flattenability of arbitrary G-varieties

At the moment, the following is known about this problem:

(a) For every connected noncommutative reductive R, there is a faithful nonlinearizable R-action on some affine space $\mathbb{A}^n$. [F. Knop, 1991].

(b) There are finite groups R admitting nonlinearizable actions on some affine spaces [M. Masuda, L. Moser-Jauslin, T. Petrie, 1991–1995].

(c) It is unknown whether there are nonlinearizable actions of commutative reductive groups on affine spaces.

From this, one deduces:

If G is a connected noncommutative reductive group, then there is a G-variety X that is

equivariantly flattenable, but not linearly equivariantly flattenable.

Example

Let X be an affine space endowed with a nonlinearizable G-action (by (a), it exists). One shows that X has the specified properties.

Equivariant flattenability of arbitrary G-varieties

We conclude with an intriguing construction (P., 2018) of some flattenable F-varieties X for some finite groups F, whose consideration leads to the following remarkable alternative:

Using such X, one obtains

(i) either an example of

> a flattenable, but not equivariantly flattenable F-variety,

(ii) or an example of

> a nonlinearizable F-action on an affine space.

Some of these F are commutative, so if (ii) holds, it yields the affirmative answer to the open part of the Linearization problem about the existence of nonlinearizable actions of finite commutative groups.

Construction (V.P., 2020)

Take a pair S, F such that

(c_1) S is a connected semisimple algebraic group;

(c_2) F is a finite subgroup of S;

(c_3) $\mathcal{Z}_S(F)$ (the centralizer of F in S) is finite and nontrivial.

There are many examples of such pairs S, F, including the ones with commutative F.

(c_2) implies the existence of a Borel subgroup B of S containing two different elements of $\mathcal{Z}_S(G)$.

Let B^- be the Borel subgroup of S opposite to B. Then the "big cell"

$$\Theta := B^- B$$

is open in S and isomorphic to the complement of a union of several coordinate hyperplanes in $\mathbb{A}^{\dim(S)}$; in particular, Θ is an affine flattenable variety. We put

$$\boxed{X := \bigcap_{g \in F} g\Theta g^{-1}.}$$

Theorem (V.P.,2018)

- *X is a flattenable affine open subset of S.*
- *X is F-stable with respect to the conjugating F-action on S.*

The F-variety X shares the properties:

(i) *X is not linearly equivariantly flattenable.*

(ii) *The alternative holds:*

— *either X is not equivariantly flattenable,*

— *or X is equivariantly flattenable, but the action of G on an affine space, extending that on X, is nonlinearizable.*

Open question

Is each of these two a priori possibilities realizable for an appropriate pair S, F?

Open question

Take $S = \mathrm{PGL}_2$. Its subgroup F whose nonidentity elements are

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \text{ where } i = \sqrt{-1},$$

is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and satisfies property (c_3). The variety X is isomorphic to an affine open subset of $\mathbb{A}^3$. Which of these two a priori possibilities holds in this case?

Similarly to flattenability, equivariant flattenability admits an evident local version:

Definition (C. Petitjean, 2017 (up to change of terminology))

A variety X endowed with an action of an algebraic group G is called

locally equivariantly flattenable (resp., locally linearly equivariantly flattenable)

if for every point $x \in X$ there is an equivariantly flattenable (resp., linearly equivariantly flattenable) G-stable open subset of X containing x.

Locally equivariantly flattenable varieties

This definition leads to the following equivariant version of Problem 5 (Gromov's question):

> *Is every irreducible smooth rational G-variety equivariantly locally flattenable?*

In 2017, C. Petitjean, found the examples showing that the answer to this question is **negative**.