Rational integrals of geodesic flows

Boris Kruglikov
(UiT the Arctic University of Norway)

Seminar: Geometry of Differential Equations (Moscow)

Geodesic flows: Hamiltonian setup

Metric on a manifold M^n, $x=(x^i)_{i=1}^n$, $g=g_{ij}(x)dx^idx^j$.
Hamiltonian on T^*M:

$$H=\frac{1}{2}g^{ij}(x)p_ip_j.$$

Hamiltonian flow:

$$\dot{x}^i=\frac{\partial H}{\partial p_i}=g^{ij}p_j,\quad \dot{p}_i=-\frac{\partial H}{\partial x^i}=-\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i}p_jp_k.$$

Equivalently for $v^k=g^{ks}p_s$ (passage T^*M to TM):

$$\dot{x}^k=v^k,\quad \dot{v}^k=-\Gamma^k_{ij}v^iv^j.$$

Integral $I\in C^\infty(T^*M)$:

$$\dot{I}=\{H,I\}=\frac{\partial H}{\partial x^i}\frac{\partial I}{\partial p_i}-\frac{\partial I}{\partial x^i}\frac{\partial H}{\partial p_i}=0.$$

Liouville integrability: $H=I_1,I_2,\dots,I_n$, $\{I_k,I_l\}=0$, indep. a.e.

Global integrability

Global obstructions to integrability on closed manifolds M of $\dim M = n$ (Kozlov, Kolokoltsev, Paternain, Taimanov, Matveev).
Sample of results:

- If (M, g) is a surface of genus $g > 1$ then non-integrable.
- If $\pi_1(M)$ is not almost Abelian then (analytically) non-integrable.
- If $\dim_{\mathbb{Q}} H_1(M, \mathbb{Q}) > n$ then (analytically) non-integrable.
- If $h_{\text{top}}(M, g) > 0$ then (analytically/Bott) non-integrable.
- If $\dim_{\mathbb{Q}} \pi_*(M, \mathbb{Q}) > n$ then (Stäckel/Benenti) non-integrable.
- If $\dim_{\mathbb{Q}} H_*(M, \mathbb{Q}) > 2^n$ then (Stäckel/Benenti) non-integrable.
- If $\chi(M) < 0$ then (Stäckel/Benenti) non-integrable.

Also counter-examples (Bolsinov, Taimanov, Paternain, Butler).

Local integrability

Theorem (Folklore)

Locally every (pseudo-Riemannian) metric g is integrable.

The integrals are analytic on $T^*U \setminus 0_U$, $U \subset M$, but not on T^*U. What about "good integrals"?

Theorem (Another folklore .. Darboux, Wittaker)

If the geodesic flow of g allows integrals analytic near $0_M \subset T^*M$ then it allows polynomial integrals.

Polynomial integrals a.k.a. Killing tensors (KT) $a^{i_1\dots i_d}(x)$:

$$I = k^{i_1\dots i_d} p_{i_1} \cdots p_{i_d} : \{H, I\} = 0 \Leftrightarrow K = k^{i_1\dots i_d} \partial_{i_1} \cdots \partial_{i_d} : \nabla_{(i_0} k_{i_1\dots i_d)} = 0$$

Theorem (Kruglikov-Matveev)

Locally a C^2-generic metric g admits no non-trivial KTs.

Existence of local KTs

Theorem (Kozlov, Ten)

Locally on surfaces M^2 there exist metrics g with irreducible KTs of degree d.

In isotermal (conformal) coordinates $g = e^\lambda(dx^2 + dy^2)$ and

$$I = u_d p_x^d + u_{d-1} p_x^{d-1} p_y + \cdots + u_1 p_x p_y^{d-1} + u_0 p_y^d$$

then the condition $\{H, I\} = 0$ is a determined system on functions $\lambda, u_0, \dots, u_d$ and so has solutions by Cauchy-Kovalevskaya.

Important: it has the form of semi-Hamiltonian system in $1\frac{1}{2}$ dimensions - system in conservation laws (Bialy, Mironov, Pavlov, Tsarev, Agapov).

Remark

The above statement holds clearly true also for $\dim = n > 2$ but I have not seen a formal proof. Semi-Hamiltonianity fails.

Explicit examples (Fomenko, Kyohara, Bolsinov, Matveev, Topalov).

Counting Killing vectors (KV: $d = 1$) and Killing tensors

Theorem (old one: Ricci, Fubini, Cartan, Egorov)

Maximal # of indep. KVs (isometry dimension) on M^n is $\binom{n+1}{2}$. The sub-maximal isometry dimension is $\binom{n}{2} + 1$ except for $n = 2$ when it is 1 and $n = 4$ when it is 8 (or 7 in Lorentzian signature).

Let $\mathcal{K}_{n,d}(g)$ be the space of polynomial integrals of $\deg K = d$.

Theorem (Delong, Thompson)

This space is finite dimensional with the sharp bound

$$\dim \mathcal{K}_{n,d}(g) \leq \Lambda_{n,d} := \frac{(n+d-1)!(n+d)!}{(n-1)!n!d!(d+1)!}$$

attained precisely on spaceforms (metrics of constant curvature; any metric signature and curvature sign).

Related results: max.dim. for conformal Killing tensors (Eastwood), Killing tensors on symmetric spaces (Eastwood, Matveev, Nikolaevsky)

Non-polynomial integrals

Consider the following (non-metric) example on $M = \mathbb{R}^n(x)$

$$H = f_0(p) + \sum_{i=1}^{n} x^i f_i(p), \quad I = \psi(p)$$

Then I is an integral $\{H, I\} = 0$ iff

$$\sum_{i=1}^{n} f_i(p) \frac{\partial \psi}{\partial p_i} = 0.$$

This first order PDE can be readily solved in many cases, for instance $n = 2$: $f_1 = f(p) \cdot p_1^a$, $f_2 = f(p) \cdot k\, p_2^b$. Then

$$\psi = \psi\Big(p_1^k p_2\Big) \text{ if } a = b = 1, \quad \psi = \psi\Big(k(1-b)\log p_1 + p_2^{1-b}\Big) \text{ if } a = 1, b, \neq 1;$$

$$\psi = \psi\Big(k(1-b)p_1^{1-a} + (1-a)p_2^{1-b}\Big) \text{ if } a, b, \neq 1.$$

Varying parameters a, b, k we get polynomial, rational, algebraic/analytic, non-meromorphic cases. Particular metric examples before (Maciejewski-Przybylska, Aoki-Houri-Tomoda).

Rational integrals

By a rational integral of bidegree (r, s) we will understand an expression of the form

$$F(x,p) = \frac{P(x,p)}{Q(x,p)} = \frac{P^{i_1 \dots i_r}(x) p_{i_1} \cdots p_{i_r}}{Q^{i_1 \dots i_s}(x) p_{i_1} \cdots p_{i_s}}, \qquad (1)$$

where P and Q are relatively prime homogeneous polynomials in p of degrees r and s, respectively. Denote the (nonlinear) space of such integrals by $\mathcal{R}_{n,r,s}(g)$.

Theorem (Kozlov)

Locally on surfaces M^2 there exist metrics g with irreducible rational integrals of bi-degree r, s.

We suggest to investigate the following.

Conjecture. *The space $\mathcal{R}_{n,r,s}(g)$ is finite-dimensional with the dimension bounded by that of $\mathcal{R}_{n,r,s}(g_0)$.*

Relative Killing tensors

Definition

A function $K = K(x, p)$ polynomial in momenta of $\deg K = d$ is a relative Killing tensor of degree d if

$$\{H, K\} = L \cdot K$$

for some function L (cofactor) linear in momenta: $\deg L = 1$.

This means that the Hamiltonian vector field X_H is tangent to the submanifold $\{K = 0\} \subset T^*M$.

Remark

The same concept appeared previously under the names generalized Killing tensor and Darboux polynomial. We prefer the above nomenclature, as it is analogous to relative invariants for group actions. Here the group $\mathbb{R}$ acts through the Hamiltonian flow.

Relative Killing tensors: cont'd

Note that if K is a relative Killing d-tensor, then so is $e^\varphi K$ for any $\varphi \in C^\infty(M)$ with L changed to $L + d\varphi$ (understood L as a one-form). In particular, K is (locally) conformal to a Killing d-tensor iff $dL = 0$ (this two-form dL is an analog of the Chern curvature in web theory). We will not distinguish between conformal solutions and so identify the pairs

$$(K, L) \sim (e^\varphi K, L + d\varphi).$$

Remark

It is instructive to compare the above definition with the definition of conformal Killing tensor K, which is $\{H, K\} = L \cdot H$. In this case scaling of H (or equivalently g) keeps this condition invariant (modulo modification of L, which now becomes a $(d-1)$-form).

Let us denote the space of L-relative Killing d-tensors of g by $\mathcal{K}^L_{n,d}(g)$. Note that this is a linear space due to linearity of the defining PDE system, and we have $\mathcal{K}^{L-d\varphi}_{n,d}(g) = e^\varphi \mathcal{K}^L_{n,d}(g)$.

Counting relative KTs

Theorem

For any L we have $\dim \mathcal{K}^L_{n,d}(g) \le \Lambda_{n,d}$.

Pf: The defining equation $\mathcal{E}_d \subset J^1(S^dTM)$ has the form

$$K_{(\sigma;i)} = L_{(i}K_{\sigma)},$$

where σ is a multiindex of length $|\sigma| = d$. We have: $\mathrm{Char}(\mathcal{E}_d) = \emptyset$, whence the system is of finite type. The prolongations $\mathcal{E}_d^{(k)}$ are:

$$K_{(\sigma;i)\varkappa} = \sum_{\nu\subset\varkappa} \binom{\varkappa}{\nu} L_{(i;\nu}K_{\sigma);\varkappa-\nu} \qquad (|\varkappa| = k),$$

where symmetrization of the rhs applies only to (i,σ). For $k = d$ this system is determined wrt $K_{\sigma;i\varkappa}$ (below $|\tau| = d+1$):

$$K_{\sigma;\tau} = F_{\sigma\tau}(\{L_{i;\nu}, K_{\mu;\nu} : 1 \le i \le n, |\mu| = d, |\nu| \le d\}).$$

This is a complete system, compatibile for $g = g_0$, in which case the dimension is that for KTs: $\dim \mathcal{K}^L_{n,d}(g_0) = \Lambda_{n,d}$.

Counting relative KTs: cont'd

The last part of the proof uses projective invariance of $\mathcal{E}_d$. In the case $L = 0$, this is known as the projective invariance of the Killing equation. The argument straightforwardly generalizes to our case.

Indeed, if $\hat{\nabla} = \nabla + \Upsilon \circ \mathrm{Id}$ is a projectively equivalent connection for some 1-form Υ, then we have

$$\hat{\nabla}_{(i}K_{\sigma)} - \hat{L}_{(i}K_{\sigma)} = \nabla_{(i}K_{\sigma)} - L_{(i}K_{\sigma)} \quad \text{for} \quad \hat{L} = L - \Upsilon.$$

The projective invariance implies that in the flat case ($g = g_0$, $L = 0$) the space of Cauchy data (equivalently: solution space $\mathcal{K}_{n,d}(g_0)$ is the representation $\Gamma(d \cdot \pi_2)$ of the group $SL(n+1)$, and exploiting the Weyl dimension formula we get

$$\dim \mathcal{K}_{n,d}(g_0) = \dim \Gamma(d \cdot \pi_2) = \Lambda_{n,d}.$$

Rational integrals vs Relative KTs (analytic case)

The ring of relative Killing tensors is doubly graded by d and L:

$$\mathcal{K}^{L_1}_{n,d_1}(g) \cdot \mathcal{K}^{L_2}_{n,d_2}(g) \subset \mathcal{K}^{L_1+L_2}_{n,d_1+d_2}(g).$$

Given $P \in \mathcal{K}^{L}_{n,r}(g)$ and $Q \in \mathcal{K}^{L}_{n,s}(g)$ we get $F = \frac{P}{Q} \in \mathcal{R}_{n,r,s}(g)$.

Lemma

Every factor of $K \in \mathcal{K}^{L}_{n,d}(g)$ belongs to some space $\mathcal{K}^{L'}_{n,d'}(g)$.
If $F = \frac{P}{Q} \in \mathcal{R}_{n,r,s}(g)$ then $P \in \mathcal{K}^{L}_{n,r}(g)$, $Q \in \mathcal{K}^{L}_{n,s}(g)$ for some L.

Denote by $\mathcal{S}^{L}_{r,s}$ the space of pairs

$$(P, Q) \in \mathcal{K}^{L}_{n,r}(g) \times \mathcal{K}^{L}_{n,s}(g)\colon\ P, Q \neq 0,\ \gcd(P, Q) = 1.$$

For $r = s$ this entails to $\dim \mathcal{K}^{L}_{n,r}(g) > 1$.

Let us call L admissible if $\mathcal{S}^{L}_{r,s}$ is nontrivial. The space of admissible cofactors L modulo the gauge equivalence is an algebraic variety.

Corollary

We have: $\mathcal{R}_{n,r,s}(g) = \left\{ \frac{P}{Q} : (P, Q) \in \mathcal{S}_{r,s} = \cup_L \mathcal{S}^{L}_{r,s} \right\}$.

Finite-dimensionality of the space of rational integrals

Theorem

For any metric g the space $\mathcal{R}_{n,r,s}(g)$ is finite-dimensional.

Pf: Any integral is determined by its values in $\pi^{-1}(U) \subset TM$ for a neighborhood $U \subset M$. A rational integral $F \in \mathcal{R}_{n,r,s}(g)$ on T_aM, is determined by a finite number of parameters, at most

$$N = \binom{r+n-1}{r} + \binom{s+n-1}{s} - 1.$$

Since F is homogeneous, it is determined by values on T^1M.

Choose N points $a_1, \dots, a_N \in U$ and a point $a \in M$. Connect a_i to a by geodesics γ_i. Since integrals are constant along geodesics, the values of F on γ_i at a_i give the values of F on γ_i at a. This determines the rational integral F on a dense set, and hence everywhere on TM. Moreover, we get $\dim \mathcal{R}_{n,r,s}(g) \leq N^2$.

Remark

The proof actually shows that $\mathcal{R}_{n,r,s}(g)$ is an algebraic variety.

Rational invariants for spaceforms

Consider now a metric g_0 of constant sectional curvature.

Theorem

$\mathcal{R}_{n,r,s}(g_0) = \{P/Q : P \in \mathcal{K}_{n,r}(g_0), Q \in \mathcal{K}_{n,s}(g_0)\}$.

Pf: Let $n = 2$, $H = \frac{1}{2}(p_1^2 + p_2^2)$. We have three KVs: $K_1 = p_1$, $K_2 = p_2$, $K_3 = x_1 p_2 - x_2 p_1$. These are independent in the complement to $2H = K_1^2 + K_2^2 = 0$. With $J = x_1 p_1 + x_2 p_2$ the transformation between coordinates (p_1, p_2, x_1, x_2) and (K_1, K_2, K_3, J) is an algebraic diffeomorphism outside $H = 0$, and

$$\xi = p_1 \partial_{p_1} + p_2 \partial_{p_2} = K_1 \partial_{K_1} + K_2 \partial_{K_2} + K_3 \partial_{K_3} + J \partial_J.$$

Since $\xi(F) = d \cdot F$ for KT $F = F(K_1, K_2, K_3)$, this implies that F is a homogeneous polynomial of degree d in K_1, K_2, K_3. Similar ideas work for rational integrals. The case $n > 2$ is more complicated as there are syzygies between KVs (superintegrability). We conclude:

$$\dim \mathcal{R}_{n,r,s}(g_0) = \Lambda_{n,r} + \Lambda_{n,s} - 1.$$

Particular case: Fractional-linear integrals on surfaces

It was proved by Agafonov-Alves (w/relation to webs) that $\dim \mathcal{R}_{2,1,1}(g)$ can be either 5 or 3. Their proof is for Lorentzian signature, we re-prove this in Riemannian case differently. Even though the Riemannian and Lorentzian cases are related by a Wick rotaiton, the problems of rational integrals (complexification of a nonlinear functions) are not readily equivalent.

Theorem

If non-empty, the space $\mathcal{R}_{2,1,1}(g)/PGL(2,\mathbb{R})$ is either $\mathbb{R}P^2$ or a finite number of points (no more than 6).

We note that $PGL(2)$ acts on the space of integrals and the quotient is a rational algebraic variety.

Theorem

A surface of revolution (M,g) with non-constant curvature does not possess a (nontrivial) fractional-linear integral.

Example 1

Consider the metric $g = (x^2 + 4y^2)(dx^2 + dy^2)$ on $\mathbb{R}^2(x, y)$:

$$H = \frac{p^2 + q^2}{2(x^2 + 4y^2)}.$$

No KVs, but 3 quadratic KTs:

$$F_1 = \frac{(2yp - xq)(2yp + xq)}{x^2 + 4y^2}, \quad F_2 = \frac{(2yp - xq)(x^2 p + 2y^2 p - xyq)}{x^2 + 4y^2}$$

and also one cubic integral, which is automatically irreducible:

$$F_3 = \frac{(2yp - xq)(2xp^2 - 2ypq + xq^2)}{x^2 + 4y^2}.$$

All quartic integrals are reducible, and apparently the same concerns the fifth degree integrals. Here are some rational integrals:

$$G_1 = \frac{x^2 p + 2y^2 p - xyq}{2yp + xq}, \quad G_2 = \frac{2xp^2 - 2ypq + xq^2}{x^2 p + 2y^2 p - xyq}.$$

The above H is equivalent to an example of Agapov-Shubin. Their integral corresponds to our G_1.

Example 1 cont'd

Let us observe a relation to relative Killing tensors

$$R_0 = x^2p+2y^2p-xyq,\ R_1 = 2yp-xq,\ R_2 = 2yp+xq,\ R_3 = 2xp^2-2ypq+xq^2$$

and a direct verification shows

$$\{H,R_0\} = L_-R_0,\ \{H,R_1\} = L_+R_1,\ \{H,R_2\} = L_-R_2,\ \{H,R_3\} = L_-R_3,$$

$$L_- = -\frac{6(xp+2yq)}{(x^2+4y^2)^2},\ L_+ = \frac{2(xp-2yq)}{(x^2+4y^2)^2}.$$

However since $\{H, -\frac{1}{4}\ln(x^2+4y^2)\} = \frac{xp+4yq}{(x^2+4y^2)^2}$ we get

$$L_- \sim -L,\ L_+ \sim +L \ \text{ for } \ L = \frac{3xp}{(x^2+4y^2)^2}.$$

Now we can see that R_0R_1, R_1R_2, R_1R_3, with proper factors $f(x,y)$, give polynomial integrals, while R_0/R_2 and R_3/R_2 give rise to rational integrals.

In particular, we notice that $G_1 = F_2/F_1$, so the rational integral of Agapov and Shubin is a ratio of Killing tensors.

Example 2

The metric $g = (x^4 + 4y^4)(dx^2 + dy^2)$ possesses no KVs and no cubic integrals, but two quadratic integrals:

$$2H = \frac{p^2 + q^2}{x^4 + 4y^4}, \quad F_2 = \frac{(2y^2p - x^2q)(2y^2p + x^2q)}{x^4 + 4y^4}.$$

Relative KVs of type $P(x,y)p + Q(x,y)q$, $\deg P, \deg Q \leq 3$:

$$R_1 = 2y^2p - x^2q,\ R_2 = 2y^2p + x^2q,$$
$$R_3 = (x^2 + 2y^2)yp + x^3q,\ R_4 = (x^2 - 2y^2)p - 2xyq;$$

The corresponding cofactors $L_i = \{H, \log R_i\}$ satisfy: $-L_1 \sim L_2 \not\sim L_3 = L_4$. We conclude:

$$F_2 \sim R_1R_2 \ \text{ and } \ F_3 = \frac{R_3}{R_4} = \frac{(x^2 + 2y^2)yp + x^3q}{(x^2 - 2y^2)p - 2xyq}$$

The fractional-linear integral F_3 is algebraically independent of the energy and F_2, hence H is super-integrable.

Many more explicit examples in recent works by Agapov-Shubin.

References

- Boris Kruglikov, *On fractional-linear integrals of geodesics on surfaces*, Russian Journal of Mathematical Physics (2025)
- Boris Kruglikov, *Rational first integrals and relative Killing tensors*, arXiv (2024)
- Boris Kruglikov, Vladimir Matveev, *The geodesic flow of a generic metric does not admit nontrivial integrals polynomial in momenta*, Nonlinearity (2016).
- Boris Kruglikov, Vladimir Matveev, *Non-degenerate geodesic equivalence implies* $h_{top}(g) = 0$, Ergodic Theory and Dynamical Systems (2006).

See also citations therein and many more...

Thanks for your attention!