Constructing Regular Polygons via Minimal Angle Section (Third Talk in a Series)

Semjon Adlaj
https://SemjonAdlaj.com

A.A. Dorodnicyn Computing Centre of the Russian Academy of Sciences
Moscow, Russia

Contemporary Problems in Number Theory
Seminar of the Steklov Mathematical Institute of the Russian Academy of Sciences
Moscow, Russia

October 9, 2025

Юрий Иванович Мерзляков
(01.06.1940 -- 23.01.1995)

Профессор Ю.И.Мерзляков был незаурядным математиком, крупнейшим специалистом в области линейных групп, автором более 100 научных трудов. А главное, он был замечательным русским человеком, патриотом и гражданином.

В начале 1980-х годов он написал статью «Право на память», опубликованную в еженедельной газете Президиума СО АН СССР «Наука в Сибири» № 7 от 17 февраля 1983 г.

Статья чудом, по недосмотру надзирающих, попала в печать. Она имела подзаголовок «Размышления в связи с одной человеческой судьбой» и была посвящена французскому математику и патриоту Эваристу Галуа. Этот юноша, погибший в расцвете лет, был любимым героем и, в какой-то мере, предшественником Юрия Ивановича.

Однако, в статье речь шла не столько о математике, сколько о гражданственности и патриотизме. Соответственно, статье предшествовало предисловие редакции «Заветы Галуа - служение Родине, духовность, бескомпромиссность».

Возможно не все помнят, что Л.С.Понтрягину принадлежала решительная роль в борьбе против реформы курса школьной математики в сторону чрезмерной формализации, затеянной в 1967 году под водительством академика А.Н.Колмогорова.

Громить реформаторов Льву Семеновичу удавалось только в партийной прессе. Вот что он писал, к примеру, в журнале «Коммунист», 1980, № 14:

> ***«С большой досадой приходится констатировать, что вместо того, чтобы прививать учащимся практические умения и навыки в использовании обретаемых знаний, учителя подавляющую часть учебного времени тратят на разъяснение смысла вводимых отвлечённых понятий, трудных для восприятия в силу своей абстрактной постановки, никак не «стыкующихся» с собственным опытом детей и подростков, не способствующих развитию их математического мышления и, главное, ни для кого не нужных.»***

Доктор физико-математических наук, профессор Мерзляков Ю.И. скоропостижно скончался 23 января 1995 года в расцвете творческих сил. Институт математики им. С.Л.Соболева Сибирского отделения РАН, где он работал, числит его уволенным. Некролог отважился поместить лишь Пермский госуниверситет.

Отметим, что публикация такой статьи на пороге «разгула демократии» была мужественным поступком не только Ю.И.Мерзлякова, но и редактора газеты «Наука в Сибири» Ю.А.Ворончихина, а также многих незаурядных академгорожан, поплатившихся карьерой, а то и жизнью, за противодействие «пятой колонне». Помянем их добрым словом!

Русские люди, не теряйте память!

Applying Polynomial Computer Algebra to Geometrically Construct Regular Polygons

A highly critical review of the “classical” history of cyclotomy by Nikolai Vavilov

Math-Net.Ru

SEMINARS

RUS ENG

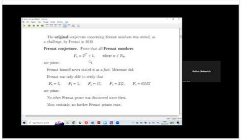
Full version

Seminar on the History of Mathematics

April 6, 2023 18:00, St. Peterburg, online

Fermat numbers and cyclotomy

N. A. Vavilov



Abstract: The talk is devoted to the history of Fermat numbers and their kin, as well as their role in cyclotomy. Fermat conjecture that Fermat numbers $F_n = 2^{2^n} + 1$ are prime, turned out to be both false and wrong (actually, Fermat never asserted this as a fact, Mersenne did!) Nevertheless, it is a GREAT conjecture, which played crucial role in the development of number theory, and algebra at large. Rebuttal of this conjecture, factorisation of F_6, was the first published paper of Leonhard Euler in number theory. Similarly, a claim that the regular 17-gon can be constructed by ruler and compass, also intimately related to Fermat numbers, was the first mathematical paper by Carl Friedrich Gauss, after which he decided to become a mathematician. After a brief introduction I outline the history of factorisation of Fermat numbers, including the recent progress due to the advent of distributed computations. The Gauss—Wanzel theorem asserts that a regular n-gon is constructible by ruler and compass, if and only if n is a product of 2^m and pair-wise distinct Fermat primes. It was an absolute shock for me to discover that the whole story of cyclotomy according to Klein, presented in all textbooks, is a COMPLETE FAKE, which ignores the contributions by French, Russian and even Prussian mathematicians. Thus, in "Disquisitiones" Gauss didn't prove necessity in the above theorem, this was only done by Pierre Wanzel some 40 years later. Gauss has not given an actual construction of a 17-gon, he computed $cos(2\pi/17)$. The first such geometric construction was published by Egor Andreevich von Pauker. Also, he computed $cos(2\pi/257)$ some 10 years before Friedrich Richelot and Fischer, Johann Hermes constructed the regular 65537-gon in Koenigsberg, and not in Linden or Goettingen, and so on. However, Gauss has established the sufficiency part of the Gauss—Pierpoint theorem, which is seldom mentioned either.

Computing radical expressions for roots of unity (conclusion)

In [6] a special algorithm to obtain radical expressions for roots of unity is given. This algorithm is also based on ideas of GAUSS, and is similar in some aspects to the one we have described in this paper. The main difference is that it does not use a chain of factors of $p-1$ but works in one single step on $p-1$, which allows certain simplifications. Its time complexity is $O(p^6)$ which makes is asymptotically worse for the class of primes p with the property that the prime factors of $p-1$ can be bound by a function that is sublinear. We have also implemented this algorithm in MAPLE. We found that for the examples we could compute it was much slower than the algorithm of this paper and gave much larger radical expressions for the roots of unity, even in the cases of prime numbers p with the property that $(p-1)/2$ is prime.

We know of an algorithm developed by B. TRAGER, which also computes radical expressions for a p-th root of unity [18]. This algorithm is entirely different from the one of GAUSS. The major computational task of this algorithm consists of inverting a matrix of size $O(p)$ over $\mathbb{Q}(\zeta_q)$, where q is a divisor of $p-1$. Thus if $p-1$ is smooth, the asymptotic complexity of the algorithm presented in this paper is much better. But in special cases such as the one that $(p-1)/2$ is prime the algorithm of TRAGER might be an interesting alternative.[9]

Several methods treat the more general case of giving radical expressions for general polynomials with a solvable GALOIS group—or a cyclic GALOIS group—and leave the case of roots of unity as a special one.

In [7] a method to solve cyclic equations which is based on the method of LAGRANGE is described. However, the described method uses an auxiliary expression which is obtained by applying the full group of permutations S_n to the set of roots $r_1, \ldots, r_n$. Thus this method would give an algorithm with exponential complexity.

LANDAU and MILLER [14] have shown that the general problem to solve solvable polynomial equations by radicals can be solved in polynomial time. The general algorithm described in [14] can be applied to this more specific problem. However, this algorithm has a time complexity of more than $O(p^{12})$.

An interesting algorithm—which is based on invariant theory—to obtain radical expressions for the roots of a solvable polynomial is sketched in [16, Section 2.7]. However, we do not know of an implementation of this algorithm nor of a discussion of its computational complexity. For the general

[9] We do not know of an actual implementation of this algorithm and so we could not compare its actual behavior on small instances of the "hard cases" such as $p = 11, 23, 47, 59, 83, \ldots$

case the algorithm requires the computation of a GRÖBNER basis for a relative orbit variety which suggests that its computational complexity is quite large.

References

[1] ABARTH, O., AND SCHAEFER, M. J. Precise computation using range arithmetic, via C++. *ACM Trans. Math. Software 18* (1992), 481–491.

[2] CHAR, B. W., GEDDES, K. O., GONNET, G. H., BENTON, L. L., MONAGAN, M. B., AND WATT, S. M. *Maple V Language Reference Manual.* Springer-Verlag, New York, 1991.

[3] CHAR, B. W., GEDDES, K. O., GONNET, G. H., BENTON, L. L., MONAGAN, M. B., AND WATT, S. M. *Maple V Library Reference Manual.* Springer-Verlag, New York, 1991.

[4] COHEN, H. *A Course in Computational Algebraic Number Theory*, vol. 138 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, 1993.

[5] DAVENPORT, H. *Multiplicative Number Theory*, second ed., vol. 74 of *Graduate Texts in Mathematics.* Springer-Verlag, 1980.

[6] EDWARDS, H. M. *Galois Theory*, vol. 101 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1984.

[7] GAAL, L. *Classical Galois Theory*, fourth ed. Chelsea Publishing Company, New York, 1988.

[8] GAUSS, C. F. *Disquisitiones Arithmeticae.* G. Fleischer Jun., Göttingen, 1801. In Latin. Reprinted in [11]. German translation: [10]; english translation: [12].

[9] GAUSS, C. F. *Mathematisches Tagebuch 1796–1814*, vol. 256 of *Oswalds Klassiker der exakten Wissenschaften.* Akademische Verlagsgesellschaft, Leipzig, 1976.

[10] GAUSS, C. F. *Untersuchungen über höhere Arithmetik*, second ed. Chelsea Publishing Company, 1981.

[11] GAUSS, C. F. *Werke*, vol. I. Georg Olms Verlag, Hildesheim, New York, 1981.

[12] GAUSS, C. F. *Disquisitiones Arithmeticae — English Edition.* Springer-Verlag, 1986.

[13] IRELAND, K., AND ROSEN, M. *A Classical Introduction to Modern Number Theory*, vol. 84 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1982.

[14] LANDAU, S., AND MILLER, G. L. Solvability by radicals is in polynomial time. *Journal of Computer and System Sciences 30*, 2 (Apr. 1985), 179–208.

[15] LENSTRA, A. K., AND LENSTRA, JR, H. W. Algorithms in number theory. In *Algorithms and Complexity*, J. van Leeuwen, Ed., vol. A of *Handbook of Theoretical Computer Science.* Elsevier, Amsterdam, 1990, chapter 12, pp. 673–715.

[16] STURMFELS, B. *Algorithms in Invariant Theory.* Texts and Monographs in Symbolic Computation. Springer-Verlag, Wien, 1993.

[17] VAN DER WAERDEN, B. L. *Modern Algebra*, vol. 1. Frederick Ungar Publishing Co., 1949.

[18] ZIPPEL, R. Computer algebra. Unpublished Lecture Notes, 1994.

Table 1: Summary of Computations

The following computations times refer to our MAPLE implementation of the algorithm on a SUN SPARC 10 workstation.

p	$p-1$	comp. time (in sec.)	size of term (tree rep.)		size of term (dag rep.)	
			rational operations	radical operations	rational operations	radical operations
3	2	<1	3	2	3	2
5	2^2	1	9	6	8	4
7	$2\cdot 3$	2	86	38	30	6
11	$2\cdot 5$	10	566	189	94	8
13	$2^2\cdot 3$	5	196	59	47	5
17	2^4	5	36	19	28	6
19	$2\cdot 3^2$	13	767	221	90	5
23	$2\cdot 11$	758	22357	73731	450	6
29	$2^2\cdot 7$	352	27225	8133	147	7
31	$2\cdot 3\cdot 5$	206	8634	2753	216	7
37	$2^2\cdot 3^2$	267	2429	691	158	8
41	$2^3\cdot 5$	418	7316	2349	259	11
43	$2\cdot 3\cdot 7$	1379	144977	43207	410	7
53	$2^2\cdot 13$	9190	352608	101319	679	9
61	$2^2\cdot 3\cdot 5$	639	28194	8959	318	12
67	$2\cdot 3\cdot 11$	27435	3655262	1204289	872	9
71	$2\cdot 5\cdot 7$	2058	441479	131457	684	9
73	$2^3\cdot 3^2$	506	11996	3401	293	13
79	$2\cdot 3\cdot 13$	27693	1883327	542751	1049	9
89	$2\cdot 3\cdot 11$	38739	347236	1132419	1055	13
97	$2^5\cdot 3$	2573	6702	2053	237	19
101	$2^2\cdot 5^2$	3700	72796	23271	708	11
109	$2^2\cdot 3^3$	4372	63949	17943	405	11
113	$2^4\cdot 7$	16419	656641	195381	904	19
127	$2\cdot 3^2\cdot 7$	38256	3070482	913347	868	14
151	$2\cdot 3\cdot 5^2$	24544	400742	127933	920	11
163	$2\cdot 3^4$	33739	357895	100287	805	16
181	$2^2\cdot 3^2\cdot 5$	74043	1109706	352045	1063	17

Computing radical expressions for roots of unity (29)

We will use assignments to intermediate expressions to print the dag representation of the radical expression for the 29-th root of unity. The root of the dag will be the last auxiliary variable that is printed, i. e. `t82`; this format corresponds to the output of the `optimize` command of MAPLE [3].

$$t1 = \sqrt{-3},\ t2 = \frac{t1}{2} - 1/2,\ t3 = t2^2,\ t5 = \sqrt[3]{9\,t3 + 8 - 6\,t1},\ t8 = \frac{6 - 2\,t3 - t1}{t5},\ t10 = \sqrt{-7/3 + \frac{t5}{3} + \frac{t8}{3}},$$

$$t11 = t5\,t3,\ t12 = t8\,t2,\ t13 = \frac{t10}{2} + \frac{t11}{6} + \frac{t12}{6} - 1/6,$$

$$t14 = t13^2,\ t15 = t14^2,\ t16 = t14\,t13,\ t17 = t15\,t14,\ t18 = t15\,t13,$$

$$t20 = \sqrt[7]{-18529\,t10 - \frac{18529\,t11}{3} - \frac{18529\,t12}{3} - \frac{117701}{3} + 33383\,t15 + 63224\,t16 + 58352\,t17 - 26096\,t18 - 46396\,t14},$$

$$t21 = t15^2,\ t22 = t21\,t15,\ t25 = t20^2,\ t26 = t25^2,$$

$$t29 = \left(\frac{48710}{3} - 1570\,t14 + 3186\,t18 + 6521\,t17 - 1454\,t16 - 7832\,t10 - \frac{7832\,t11}{3} - \frac{7832\,t12}{3} - 4644\,t15\right) t26^{-1} t20^{-1},$$

$$t30 = t21\,t14,$$

$$t34 = \left(-259\,t10 - \frac{259\,t11}{3} - \frac{259\,t12}{3} - \frac{7385}{3} - 982\,t15 + 1628\,t16 - 257\,t17 + 2672\,t18 + 4\,t14\right) t26^{-1},$$

$$t39 = \left(\frac{251}{3} - 178\,t14 + 257\,t18 - 236\,t17 - 178\,t16 + 259\,t10 + \frac{259\,t11}{3} + \frac{259\,t12}{3} - 352\,t15\right) t25^{-1} t20^{-1},$$

$$t43 = \frac{2\,t10 + \frac{2\,t11}{3} + \frac{2\,t12}{3} + 10/3 + 62\,t15 + 4\,t16 + 62\,t17 - 25\,t18 - 112\,t14}{t25},$$

$$t47 = \left(\frac{77}{3} - 2\,t10 - \frac{2\,t11}{3} - \frac{2\,t12}{3} - 4\,t15 - 4\,t18 - 4\,t17 - 4\,t14 - 4\,t16\right) t20^{-1},\ t50 = t21^2,\ t56 = t50\,t15,$$

$$t63 = \frac{29}{7} + \frac{t20\,t22}{7} + \frac{t29\,t30}{7} + \frac{t34\,t21}{7} + \frac{t39\,t17}{7} + \frac{t43\,t15}{7} + \frac{t47\,t14}{7} - \frac{2\,t20\,t50\,t21\,t17}{7} - \frac{2\,t29\,t50\,t21\,t13}{7} - \frac{2\,t34\,t56}{7} - \frac{2\,t39\,t21\,t15\,t16}{7} - \frac{2\,t43\,t30}{7} - \frac{2\,t47\,t18}{7},$$

$$t65 = t20\,t17,\ t66 = t29\,t18,\ t67 = t34\,t15,\ t68 = t39\,t16,\ t69 = t43\,t14,\ t70 = t47\,t13,$$

$$t80 = \frac{29}{7} + \frac{t65}{7} + \frac{t66}{7} + \frac{t67}{7} + \frac{t68}{7} + \frac{t69}{7} + \frac{t70}{7} - \frac{2\,t20\,t50\,t21}{7} - \frac{2\,t29\,t56}{7} - \frac{2\,t34\,t50}{7} - \frac{2\,t39\,t22}{7} - \frac{2\,t43\,t21}{7} - \frac{2\,t47\,t15}{7},$$

$$t82 = \sqrt{-\frac{29}{14} + \frac{\sqrt{t63}}{2} + \frac{t65}{14} + \frac{t66}{14} + \frac{t67}{14} + \frac{t68}{14} + \frac{t69}{14} + \frac{t70}{14}}\,1/2 + \frac{\sqrt{t80}}{4} + \frac{t20}{28} + \frac{t29}{28} + \frac{t34}{28} + \frac{t39}{28} + \frac{t43}{28} + \frac{t47}{28} - 1/28$$

Figure 2: Radical expression for ζ_{29}

Computing radical expressions for roots of unity (23)

mathworld.wolfram.com Trigonometry...

TOPICS Wolfram MathWorld FROM THE MAKERS OF MATHEMATICA AND WOLFRAM|ALPHA

As already noted, a special type of expansion in terms of radicals with *real* arguments can be obtained if n is a power of two times a product of distinct Fermat primes. For other values of n, the situation becomes more complicated. It is now no longer possible to express trigonometric functions in a form that they are expressed as real radicals, but a certain minimal representation still exists. The simplest nontrivial example is for $n = 7$. The exact meaning of "minimal" is rather technical and is related to the Galois subgroups of certain cyclotomic polynomials (Weber 1996). As it turns out, for n prime, the expansions are especially interesting and difficult, and higher order Galois group calculations are both difficult and time-consuming. For example, $n = 23$ is a very difficult case and takes a long time to calculate. Some larger primes are easier again but the complexity grows with the size of the prime on average.

Computing radical expressions for roots of unity (19)

math.stackexchange.com abstract alge...

Search on Mathematics... Log in Sign up

Am I missing something that could help me to solve this second trisection? Thanks.

EDIT:

After some computations, I found this result (checked by WA, though):

$$2\cos\left(\frac{2\pi}{19}\right) = (2,1) = \frac{2}{3}\cdot\sqrt{2k+7}\cdot\cos\left(\frac{1}{3}\arccos\frac{(3k^2+17k+18)}{2\sqrt{19}\sqrt{4k^2+18k+21}}\right) + \frac{(-5-k+k^2)}{3}$$

Where $k = (6,2) = \frac{2\sqrt{19}}{3}\cos\left(\frac{1}{3}\arccos\frac{7}{2\sqrt{19}}\right) - \frac{1}{3}$

But, man, now I have no clue how to construct this.

abstract-algebra trigonometry complex-numbers geometric-construction

Share Cite Follow

edited May 6, 2020 at 14:15

asked May 5, 2020 at 20:20

Eduardo Muller

11 ▲2

UNIFIED CONSTRUCTIONS OF HEPTAGON AND TRISKAIDECAGON

HELMUT RUHLAND

1. Introduction

In [2] A. Gleason gave constructions of a heptagon and triskaidecagon using only square roots and the trisection of angles. Later S. Adlaj [1] presented a very elegant construction of the heptagon, that shows the action of a cyclic subgroup of order 3 in the related Galois group on the 3 constructed vertices.

In this article I present 3 new unified constructions of the 2 regular polygons, unified means here *all* based on S. Adlaj's geometric construction.

In section 2 I repeat the construction in [1] and show one more for the heptagon. In section 3 2 constructions for the trikaidecagon are shown.

2. Constructions of the heptagon

2.1. The first construction, type I: S. Adlaj's from [1]. Let $\epsilon_k = e^{2\pi i/3\,k} = ((-1+\sqrt{3}\,i)/2)^k,\ \ k=0,1,2$ a third root of unity. Define the 3 third roots $\zeta_k,\ \ k = 0,1,2$:

$$\zeta_k = \epsilon_k \sqrt[3]{\zeta} \qquad \zeta = \frac{1-3\sqrt{3}\,i}{2\sqrt{7}} \tag{2.1}$$

Determining the third root of ζ is a equivalent to a trisection because $|\zeta| = 1$. The angle to trisect is $\theta = -\arctan(3\sqrt{3}) = -\arccos(1/(2\sqrt{7})) \approx -79.1066°$

Take as radii of the 2 grey, concentric circles in figure 1:

$$R_1, R_2 = \sqrt{(7 \pm \sqrt{21})/18} \tag{2.2}$$

Form three vertices of the heptagon, the three red, green, blue parallelograms in figure 1 realize the following complex additions:

$$V_0 = R_1\epsilon_0 + R_2\zeta_1 \qquad V_1 = R_1\epsilon_1 + R_2\zeta_0 \qquad V_2 = R_1\epsilon_2 + R_2\zeta_2{}^1 \tag{2.3}$$

The 3 constructed vertices V_0, V_1, V_2 represent the quadratic residues $PR_2 = \{1,2,4\}$ modulo 7. The triangle built by these 3 vertices has 3 different side lengths on so no symmetry.
The cyclic permutation $C_3 : \zeta_k \to \zeta_{k+1}$ of order 3, subscripts modulo 3, acts on these 3 vertices.

[1]There is no typing error here, in V_0, V_1 the ϵ and ζ subscripts are different

2.2. The second construction, Type II . Now the angle to trisect is $\theta = 0 = 0°$. Only a square root is necesary to get $\theta/3 = 0°, 120°, 240°$.

Set $R_1 = 1$ and R_2 equals one of the six real roots $r_k, k = 0,\dots,5$ of the palindromic, sextic polynomial $P = r^6 + 6r^5 - 6r^4 - 29r^3 - 6r^2 - 6r + 1$. Because P is palindromic, with every root r, the inverse is a root too. The 6 roots can be constructed by square roots and solving a cubic, because with $Q = (s^3+6s^2-9s-41)$ and $s = r + 1/r$ $P = Q(s)\,r^3$

$$r_k, r_{k+3} = \frac{s_k \pm \sqrt{s_k^2-4}}{2} \qquad s_k \text{ a root of } Q \qquad k = 0,1,2 \tag{2.4}$$

These 6 roots can also be expressed by the ζ_k gotten by the trisection 2.1:

$$r_k, r_{k+3} = \frac{s_k \pm \sqrt{s_k^2-4}}{2} \qquad s_k = -2 + \sqrt{7}\,(\zeta_k + \bar{\zeta_k}) \qquad k = 0,1,2 \tag{2.5}$$

The vertices of the heptagon are given by 2.3. Hint: a negative R_2 in this geometric construction means: a vector in the parallelogram has to be taken negative.

In contrary to the previous construction, the triangle built by the 3 constructed vertices V_0, V_1, V_2 is now s an *isosceles* triangle with symmetry axis a horizontal line through 0.
$C_3 : \zeta_k \to \zeta_{k+1}$ acts now on the 6 radii r_k and so on the 6 isosceles triangles. Up to scaling and rotation are only 3 different triangles.

2.3. The third construction, but nothing new, is of type II too. Now the angle to trisect is $\theta = \pi = 180°$. Only a square root is necesary to get $\theta/3 = 60°, 180°, 300°$.
Set $R_1 = 1$ and R_2 equals one of the *negative* six real roots r_k of the second construction in 2.2.

Because a negative radius R_2 is equivalent to a positive radius and a rotation by π of the triangle inscribed in this circle: this construction is equivalent to the previous subsection 2.2 .

3. Constructions of the triskaidecagon

3.1. The first construction, type I. Let $\epsilon_k = e^{2\pi i/3\,k} = ((-1+\sqrt{3}\,i)/2)^k,\ \ k = 0,1,2$ a third root of unity. Define the 3 third roots $\zeta_k,\ \ k = 0,1,2$:

$$\zeta_k = \epsilon_k \sqrt[3]{\zeta} \qquad \zeta = \frac{\sqrt{26+5\sqrt{13}} - \sqrt{26-5\sqrt{13}}\,i}{2\sqrt{13}} \tag{3.1}$$

Determining the third root of ζ is a equivalent to a trisection because $|\zeta| = 1$. The angle to trisect is $\theta = -\arccos\left(\frac{\sqrt{26+5\sqrt{13}}}{2\sqrt{13}}\right) \approx -23.0510°$

Take as radii of the 2 grey, concentric circles in figure 1:

$$R_1, R_2 = \sqrt{\frac{\sqrt{13+\sqrt{13}} \pm \sqrt{5+\sqrt{13}}}{2\sqrt{2}}} \qquad R_1 R_2 = 1 \tag{3.2}$$

The 3 constructed vertices V_0, V_1, V_2, see 2.3, represent the now quartic residues $PR_4 = \{1,3,9\}$ modulo 13. Using $-\theta/3$ the vertices represent $4\,PR_4 = \{4,10,12\}$. The triangle built by these 3 vertices has 3 different side lengths.
The cyclic permutation $C_3 : \zeta_k \to \zeta_{k+1}$ of order 3, subscripts modulo 3, acts on

these 3 vertices.

To get the vertices belonging to the 2 remaing multiplicative cosets of quartic residues $2\,PR_4 = \{2,5,6\}$ and $8\,PR_4 = \{7,8,11\}$, the following 2 radii and angle to trisect have to be used:

$$R_1, R_2 = \sqrt{\frac{\sqrt{13-\sqrt{13}} \pm \sqrt{5-\sqrt{13}}}{2\sqrt{2}}} \quad R_1\,R_2 = 1 \quad \theta = -\arccos\left(\frac{\sqrt{26-5\sqrt{13}}}{2\sqrt{13}}\right) \approx -66.9489^\circ \tag{3.3}$$

3.2. **The second construction, Type II .** Now the angle to trisect is $\theta = 0 = 0^\circ$. Only a square root is necesary to get $\theta/3 = 0^\circ, 120^\circ, 240^\circ$.

Set $R_1 = 1$ and R_2 equals one of the 12 real roots $r_k, k = 0,1,\ldots,11$ of the palindromic, degree 12 polynomial $P = r^{12}+12r^{11}-12r^{10}-274r^9-441r^8+441r^7+1275r^6+441r^5-441r^4-274r^3-12r^2+12r+1$. Because P is palindromic, with every root r, the inverse is a root too. The 12 roots can be constructed by trisections ? and square roots, because with $Q = (s^6+12s^5-18s^4-334s^3-384s^2+1323s+2131)$ and $s = r + 1/r$ $P = Q(s)\,r^6$. The sectic Q factorizes in $\mathbb{Q}(\sqrt{13})$ as product of $R = s^3 + 3\,(2+\sqrt{13})\,s^2 + 21\,(3+1\sqrt{13})/2\,s + (15+107\sqrt{13})/2)$ and its $\mathbb{Q}(\sqrt{13})$-conjugate $\bar{R}$.

$$\begin{aligned} r_k, r_{k+3} &= \frac{s_k \pm \sqrt{s_k^2-4}}{2} \qquad s_k \text{ a root of } R \qquad k = 0,1,2 \\ r_{k+6}, r_{k+9} &= \frac{s_k \pm \sqrt{s_k^2-4}}{2} \qquad s_k \text{ a root of } \bar{R} \qquad k = 0,1,2 \end{aligned} \tag{3.4}$$

The vertices of the trikaidecagon are also given by 2.3. Hint: a negative R_2 in this geometric construction means: a vector in the parallelogram has to be taken negative.

In contrary to the previous construction, the triangle built by the 3 constructed vertices V_0, V_1, V_2 is now s an *isosceles* triangle with symmetry axis a horizontal line through 0.
$C_3 : \zeta_k \to \zeta_{k+1}$ acts now on the 12 radii r_k and so on the 12 isosceles triangles. Up to scaling and rotation are only 6 different triangles.

3.3. **The third construction, but nothing new, is of type II too.** Now the angle to trisect is $\theta = \pi = 180^\circ$. Only a square root is necesary to get $\theta/3 = 60^\circ, 180^\circ, 300^\circ$.
Set $R_1 = 1$ and R_2 equals one of the *negative* 12 real roots r_k of the second construction in 3.2.

Because a negative radius R_2 is equivalent to a positive radius and a rotation of the triangle inscribed in this circle: this construction is equivalent to the previous subsection 3.2 .

4. Open questions

Are the constructions of type I, type II or both also possible for other p-gons with p a prime of the form $6n+1$? The next primes p to investigate would be $p = 19, 31, \ldots$.

Appendices

Appendix A. S. Adlaj's geometric construction

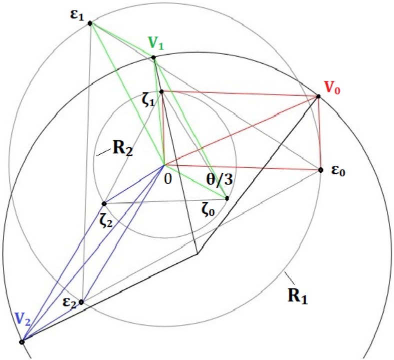


Figure 1. A scan of the heptagon construction in [1] with the denomination of vertices added by the author of this article. The vertices ϵ_k, ζ_k do not have unit distance to 0, instead they should have been denominated $\epsilon_k R_1, \zeta_k R_2$

References

1. S. Adlaj *The Heptagon constructed!*. Avialable_at_the_author's_website
2. A. Gleason *Angle Trisection, the Heptagona, and the Triskaidecagon*. The American Mathematical Monthly, Vol. 95, No. 3 (March, 1988): 185-194. Available_at_JSTORE

Santa Fé, La Habana, Cuba
Email address: helmut.ruhland50@web.de

A few curiosities concerning the prime 11

- **Évariste Galois, 1832:** The modular equation of level 11 (among all modular equations of prime level) possesses the highest depressable (from 12 to 11) degree.
- **Srinivasa Ramanujan, 1916:** The Dirichlet series $L(s) = L(s, \tau) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$, where $\tau(n)$ is the coefficient of q^n in $\Delta(q) := q \prod_{n=1}^{\infty}(1-q^n)^{24}$, possesses the Euler product $L(s) = \prod_{p \text{ is prime}}(1 - \tau(p)\, p^{-s} + p^{11-2s})^{-1}$.
- **Barry Mazur, 1977:** 11 groups $\mathbb{Z}/n\mathbb{Z}$, for n ranging from 1 to 12 but excluding 11, are the possible cyclic torsion subgroups of a rational elliptic curve.
- **Lisa Piccirillo, 2018:** The minimal crossing number for a non-slice yet trivial Alexander polynomial knot is 11. That knot is the Conway "mutant" of the Kinoshita–Terasaka knot (which is a slice knot). It is the only such knot among all knots with no more that 12 crossings.
- The 11-gon is the "smallest" polygon for which no neusis nor (single-fold) origami construction has ever been demonstrated, although the existence (without its explicit demonstration) has been alleged by "Wikipedia" since October 3, 2016.

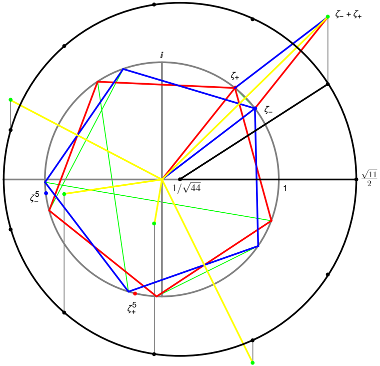
A hendecagon construction via quintisection

Semjon Adlaj

The construction, as offered, requires angle quintisections. So, assume that we are given the ten vertices of two regular pentagons. The vertices are the fifth roots of the two (complex) numbers ζ_+^5 and ζ_-^5, where

$$\zeta_\pm^5 := \frac{\pm 25\sqrt{5} - 89 - 5\,i\sqrt{410 \pm 178\sqrt{5}}}{44\sqrt{11}},\ i := \sqrt{-1}.$$

Note that both numbers ζ_+^5 and ζ_-^5 lie in the third quadrant, and let ζ_+ and ζ_- denote their corresponding fifth roots in the first quadrant. Construct a circle centered at $1/\sqrt{44}$ with radius $5/\sqrt{11}$, and place a vertex on it at $\sqrt{11}/2$. Then the real part of the sum $\zeta_- + \zeta_+$ turns out to match the real part of a "next" vertex of a hendecagon thereby inscribed in the just-constructed circle.

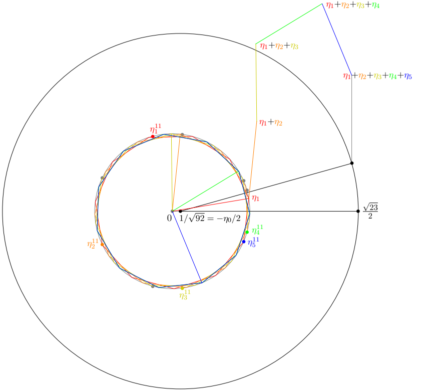


A 23-gon construction via 5 and 11 angle section

Semjon Adlaj

Let ξ denote a primitive 11^{th} root of unity. Its construction via (no more than) fifth root extraction was explicitly shown here. Introduce the (complex) numbers $\eta_n^{11} := p(\xi^n)$, where $p(x) := \frac{4810728 - 18426793\,x + 12996313\,x^2 + 15151367\,x^3 + 4283752\,x^4 - 4039948\,x^5 + 290906\,x^6 - 5853705\,x^7 - 2099438\,x^8 - 1781164\,x^9 - 5332019\,x^{10}}{23^{11/2}}$.

The construction of the icositrigon which we propose assumes that we have extracted the five 11^{th} roots η_1, η_2, η_3, η_4, η_5 which correspond to particular vertices of five (colored in red, orange, yellow, green and blue) hendecagons, inscribed in a unit circle.



Construct a circle centered at $1/\sqrt{92}$ (which would coincide with $-\eta_0/2$) with radius $11/\sqrt{23}$, and place a vertex on it at $\sqrt{23}/2$. Then the real part of the sum $\eta_1+\eta_2+\eta_3+\eta_4+\eta_5$ turns out to match the real part of a "next" vertex of a 23-gon thereby inscribed in the just-constructed circle.

A 29-gon construction via 3 and 7 angle section

Semjon Adlaj

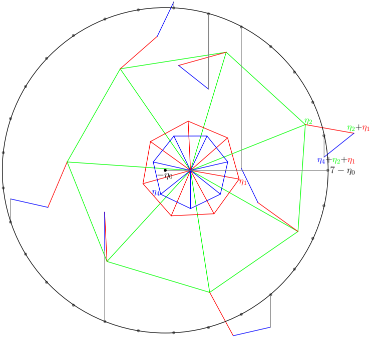
Let $\xi := e^{2\pi\sqrt{-1}/7}$. The construction of a primitive 7^{th} root of unity via (no more than) cube root extraction was explicitly shown here. Introduce the four (complex) numbers

$$\eta_0 := \frac{\sqrt{29}-1}{4},\ \eta_1^7 := p(\xi),\ \eta_2^7 := p(\xi^2),\ \eta_4^7 := p(\xi^4),$$

where

$$p(x) := \frac{7(18879 - 1709\sqrt{29} - 6641\sqrt{-7} - 2237\sqrt{-203})}{4}\,x^2 + \\ + \frac{7(19981 - 6411\sqrt{29} - 8845\sqrt{-7} + 439\sqrt{-203})}{4}\,x - \frac{84071}{2} - 8557\sqrt{29} - 10962\sqrt{-7} + \frac{4445\sqrt{-203}}{2}.$$

The construction of the icosienneagon which we propose assumes that we have extracted the 7^{th} roots η_1, η_2 and η_4 which correspond to particular vertices of three (colored in red, green and blue) heptagons.



Construct a circle, centered at $-\eta_0$, with radius 7, and place a vertex on it at $(29-\sqrt{29})/4$. Then the real part of the sum $\eta_1+\eta_2+\eta_4$ turns out to match the real part of a "next" vertex of a 29-gon thereby inscribed in the just-constructed circle.

A heiskaitriacontagon construction via trisection and quintisection

Semjon Adlaj

Introduce the (three) complex numbers

$$\eta_0 := \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\right)} + \sqrt[3]{31\left(2-3\sqrt{-3}\right)} - 1}{6},\ \eta_1^5 := p(\xi),\ \eta_2^5 := p(\xi^2),$$

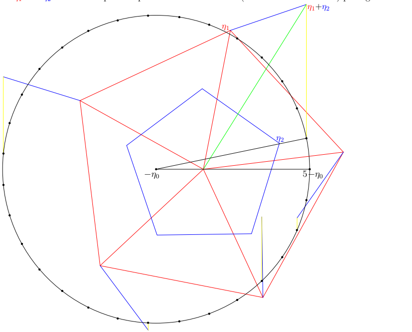
where

$$\xi := e^{\frac{2\pi\sqrt{-1}}{5}} = \frac{\sqrt{-2(5+\sqrt{5})}+\sqrt{5}-1}{4},\ p(x) := \frac{\sqrt[3]{31\left(2+3\sqrt{-3}\right)}\,z_+(x) + \sqrt[3]{31\left(2-3\sqrt{-3}\right)}\,z_-(x) - z_0(x)}{6},$$

$$z_0(x) := 280\,x^4 + 2140\,x^3 - 30\,x^2 - 960\,x - 1208,$$

$$z_\pm(x) := -50\,x^4 - 215\,x^3 - 60\,x^2 + 255\,x + 274 \pm \sqrt{-3}\left(60\,x^4 - 5\,x^3 + 130\,x^2 - 115\,x - 44\right).$$

The construction of the heiskaitriacontagon which we propose assumes that we have extracted the 5^{th} roots η_1 and η_2 which correspond to particular vertices of two (colored in red and blue) pentagons.



Construct a circle, centered at $-\eta_0$, with radius 5, and place a vertex on it at

$$5-\eta_0 = \frac{31 - \sqrt[3]{31\left(2+3\sqrt{-3}\right)} - \sqrt[3]{31\left(2-3\sqrt{-3}\right)}}{6}.$$

Then the real part of the sum $\eta_1+\eta_2$ turns out to match the real part of a "next" vertex of a 31-gon thereby inscribed in the just-constructed circle.

$$\rho_0 = 2\left(\cos\left(\frac{8\pi}{41}\right) + \cos\left(\frac{32\pi}{41}\right) + \cos\left(\frac{36\pi}{41}\right) + \cos\left(\frac{20\pi}{41}\right) + \cos\left(\frac{40\pi}{41}\right)\right) = \frac{-1+\sqrt{41}-\sqrt{82-10\sqrt{41}}}{4},$$

$$\rho_1^5 = \frac{-2000\,\xi_5^4 - 2000\,\xi_5^3 + 460\,\xi_5^2 + 1895\,\xi_5 - 1016 + \sqrt{41}\left(-120\,\xi_5^4 - 140\,\xi_5^3 + 230\,\xi_5^2 + 485\,\xi_5 - 14\right)}{4} +$$

$$+\frac{\sqrt{82-10\sqrt{41}}\left(-240\,\xi_5^4 - 130\,\xi_5^3 + 5\,\xi_5^2 - 45\,\xi_5 + 1\right) + 5\sqrt{82+10\sqrt{41}}\left(-64\,\xi_5^4 - 26\,\xi_5^3 + 43\,\xi_5^2 + 50\,\xi_5 + 23\right)}{4},$$

$$\xi_5 := e^{2\pi\sqrt{-1}/5},$$

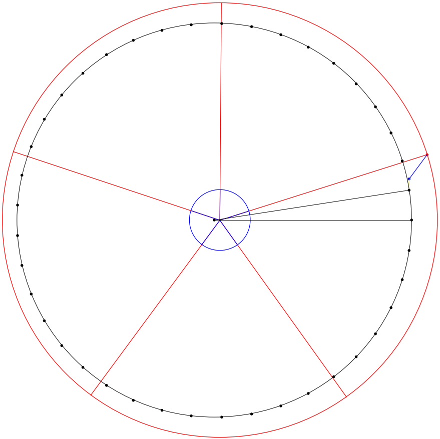
$$\rho_1 = \rho(\xi_5),$$

$$\rho_2 = \rho(\xi_5^2),$$

$$\rho_3 = \rho(\xi_5^3),$$

$$\rho_4 = \rho(\xi_5^4),$$

$$\boxed{\cos\left(\frac{2\pi}{41}\right) = \frac{\rho_0+\rho_1+\rho_2+\rho_3+\rho_4}{10}}$$



$$\rho_0 = \frac{\sqrt[3]{43\,(-4+3\sqrt{-3}\,)} + \sqrt[3]{43\,(-4-3\sqrt{-3}\,)} - 1}{3},$$

$$\rho_1^7 = \frac{-116963\,\xi_7^6 + 39557\,\xi_7^5 + 7952\,\xi_7^4 - 44422\,\xi_7^3 + 147014\,\xi_7^2 + 90727\,\xi_7 - 159748}{3} +$$

$$+\frac{\sqrt[3]{43\,(-4+3\sqrt{-3}\,)}\,z_+(\xi_7) + \sqrt[3]{43\,(-4-3\sqrt{-3}\,)}\,z_-(\xi_7)}{6},$$

$$z_\pm(x) := -21329\,x^6 - 2149\,x^5 + 2891\,x^4 + 13328\,x^3 + 23114\,x^2 + 11641\,x - 16192 +$$

$$\pm\sqrt{-3}\left(4431\,x^6 + 27307\,x^5 - 5845\,x^4 - 26138\,x^3 + 1092\,x^2 + 11193\,x - 10254\right).$$

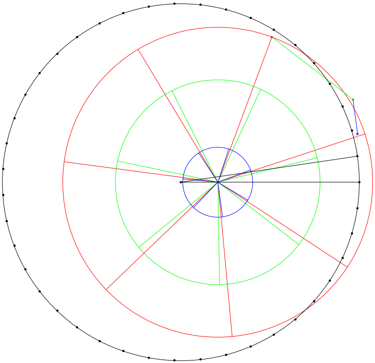
$\xi_7 := e^{2\pi\sqrt{-1}/7}$,

$\rho_1 = \rho(\xi_7)$,

$\rho_2 = \rho(\xi_7^2)$, $\rho_3 = \rho(\xi_7^3)$,

$\rho_4 = \rho(\xi_7^4)$, $\rho_5 = \rho(\xi_7^5)$, $\rho_6 = \rho(\xi_7^6)$,

$$\cos\left(\frac{2\pi}{43}\right) = \frac{\rho_0+\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6}{14}$$



Chronologically ordered dates corresponding to prime numbers of sides of explicitly constructed regular polygons

- Pythagoras of Samos (570–495 BC) 5
- Archimedes of Syracuse (287–212 BC) 7
- 1796 Carl Friedrich Gauss 17
- 1822 Magnus Georg Paucker 257
- 1894 Johann Gustav Hermes 65537
- 1988 Andrew Mattei Gleason 13
- 2024 11
- 2025 19, 23, 29, 31, 37, 41, 43

The colors signify the maximum number of equal parts into which an angle must be divided for a corresponding construction: {2,3,5,7,11}.

$$J\left(\frac{5i}{4}\right)=\left(1+\frac{9(1+\sqrt{5})^{38}}{2^{41}\sqrt{2}}\left(7485+762\sqrt{2}+1479\sqrt{5}+3072\sqrt{10}-\sqrt[4]{5}\left(178+2221\sqrt{2}+3148\sqrt{5}+1289\sqrt{10}\right)\right)^2\right)^3$$

$$J(20i)=\left(1+\frac{9(1+\sqrt{5})^{38}}{2^{41}\sqrt{2}}\left(7485+762\sqrt{2}+1479\sqrt{5}+3072\sqrt{10}+\sqrt[4]{5}\left(178+2221\sqrt{2}+3148\sqrt{5}+1289\sqrt{10}\right)\right)^2\right)^3$$

これ以前に示したすべての値は実数である。複素共役のペアは、$J(10i)$ と $J(5i/2)$ に対し、参考文献のように値に沿って、上記のように対称的になっていると推察される。

$$J\left(\frac{5i\pm1}{4}\right)=\left(1-\frac{9}{8}\left((2402-1074\sqrt{5})i\pm(1607-719\sqrt{5})\sqrt[4]{5}\right)^2\right)^3$$

4つの特殊値は、2つの複素共役のペアにより与えられる[9]。

$$J\left(\frac{4(5i\pm1)}{13}\right)=\left(1-\frac{9(1-\sqrt{5})^{38}}{2^{41}\sqrt{2}}\left(7485-762\sqrt{2}-1479\sqrt{5}+3072\sqrt{10}\pm i\sqrt[4]{5}\left(178-2221\sqrt{2}-3148\sqrt{5}+1289\sqrt{10}\right)\right)^2\right)^3$$

$$J\left(\frac{5(4i\pm1)}{17}\right)=\left(1+\frac{9(1-\sqrt{5})^{38}}{2^{41}\sqrt{2}}\left(7485+762\sqrt{2}-1479\sqrt{5}-3072\sqrt{10}\pm i\sqrt[4]{5}\left(178+2221\sqrt{2}-3148\sqrt{5}-1289\sqrt{10}\right)\right)^2\right)^3$$

参考文献 [編集]

1. ^ Silverman, Joseph H. (1986). *The Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics. **106**. Springer-Verlag. p. 339. ISBN 0-387-96203-4. Zbl 0585.14026
2. ^ Petersson, Hans (1932). *Über die Entwicklungskoeffizienten der automorphen Formen*. **58**. 169–215. doi:10.1007/BF02547776. MR1555346
3. ^ Rademacher, Hans (1938). *The Fourier coefficients of the modular invariant j(τ)*. **60**. The Johns Hopkins University Press. 501–512. doi:10.2307/2371313. JSTOR 2371313. MR1507331
4. ^ Chandrasekharan (1985) p.108
5. ^ Chandrasekharan, K. (1985), *Elliptic Functions*, Grundlehren der mathematischen Wissenschaften, **281**, Springer-Verlag, p. 110, ISBN 3-540-15295-4, Zbl 0575.33001
6. ^ Girondo, Ernesto; González-Diez, Gabino (2012), *Introduction to compact Riemann surfaces and dessins d'enfants*, London Mathematical Society Student Texts, **79**, Cambridge: Cambridge University Press, p. 267, ISBN 978-0-521-74022-7, Zbl 1253.30001
7. ^ Borcherds, R.E. (1992). *Monstrous moonshine and monstrous Lie superalgebras*. **60**. 405– 444.
8. ^ Adlaj, Semjon. “Multiplication and division on elliptic curves, torsion points and roots of modular equations”. 2014年10月17日閲覧。
9. ^ Adlaj, Semjon (2014年). “Torsion points on elliptic curves and modular polynomial symmetries”. 2014年10月15日閲覧。

From a letter by a well-informed friend

Hi! Semjon,

Only today did I have time to read your "DellaDumbAugh.pdf" article. Apologies.

I think I understand why you spell "dumbaugh" with capital D and capital A. Indeed, she is dumb not to purchase your article when you are clearly ahead in this field...

There are lots of things I don’t understand about the problem, let alone the solution that you give and which contrasts with misguided statements by her and some of her colleagues...

However, let me give this element of thought:

Creating a solution is more useful than stating that a solution exists.
In my current field of Engineering, this is recognized by the Patents system, where discovering is rewarded by a Patent, so that others cannot "claim" your invention.
I don’t know if this is practical, but I would recommend that you "file" a patent application about your method, with just the purpose of establishing your priority on the discovery.

Now, Science is not Engineering. The ideal case in Science is that you should be paid by academic institutions to make discoveries, that are then shared free-of-charge.

However, this model of "purely humanistic science" is quickly falling apart. I think a case can and should be made to DumbAugh that your contribution to a successful publication (which do generate revenue, it seems) should be rewarding to you.

Let me emphasize that I am really impressed with your article, and the construction of even a 31-gon – but that does not mean that I understand it. To me, it is mysterious that you are able to reduce a N-gon’s coordinates calculation into the calculation of smaller roots, for (it would seem) any N value...

Amities,

Some References and Internet Sources I

[1] Semjon Adlaj. Back to solving the quintic, depression and Galois primes. Polynomial Computer Algebra International Conference, St. Petersburg, Russia, 2018, April 16-21, Ed. by N.N. Vassiliev, VVM Pubishing: 12-16. Available at https://pca-pdmi.ru/2018/files/13/PCA2018SA.pdf

[2] Semjon Adlaj. Modular equations and Galois elliptic function. Seminar on the History of Mathematics, St. Petersburg, 2023, March 2, online. Available at https://www.mathnet.ru/php/seminars.phtml?presentid=37946&option_lang=eng

[3] Semjon Adlaj. A hendecagon construction via quintisection. Available at https://SemjonAdlaj.com/Excerpts/Hendecagon.pdf

[4] Semjon Adlaj. A 23-gon construction via 5 and 11 angle section. Available at https://SemjonAdlaj.com/Excerpts/Icositrigon.pdf

[5] Semjon Adlaj. A 29-gon construction via 3 and 7 angle section. Available at https://SemjonAdlaj.com/Excerpts/Icosienneagon.pdf

[6] Semjon Adlaj. A heiskaitriacontagon construction via trisection and quintisection. Available at https://SemjonAdlaj.com/Excerpts/Heiskaitriakontagon.pdf

[7] Semjon Adlaj. Some regular polygons and Della Dumbaugh. Available at https://SemjonAdlaj.com/Excerpts/DellaDumbAugh.pdf

[8] Semjon Adlaj. Constructing regular polygons via minimal angle section. First talk in a series, given on May 5, 2025 at the 939th session of the "Évariste Galois" seminar (founded by Yuri Ivanovich Merzlyakov in April 1987). Available at https://SemjonAdlaj.com/Talks/2025-05-05_15-13-21.mp4

Some References and Internet Sources II

[9] Semjon Adlaj. Applying Polynomial Computer Algebra to Geometrically Construct Regular Polygons. Second talk in a series. Polynomial Computer Algebra International Conference, St. Petersburg, Russia, 2025, September 29 – October 04, Ed. by N.N. Vassiliev, VVM Pubishing: 12-16. Available at https://pca.conf-pdmi.ru/2025/files/24/PCA2025SA.pd

[10] Johann Carl Friedrich Gauß. Disquisitiones Arithmeticae. Yale University Press edition of 1966, New Haven and London. Translation from Latin language edition of 1870: Disquisitiones Arithmeticae edited by E. C. J. Schering. Translated by Arthur A. Clarke. Revised by William C. Waterhouse with the help of Cornelius Greither and A. W. Grootendorst. Available at https://doi.org/10.1007/978-1-4939-7560-0

[11] Andrew Mattei Gleason. Angle Trisection, the Heptagon, and the Triskaidecagon. The American Mathematical Monthly, Vol. 95, No. 3 (March, 1988): 185-194. Available at https://www.jstor.org/stable/2323624

[12] Johann Gustav Hermes. Über die Teilung des Kreises in 65537 gleiche Teile. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse. Göttingen, 1894: 170–186.

[13] Magnus Georg Paucker. Geometrische Verzeichnung des regelmäßigen Siebzehn-Ecks und Zweyhundersiebenundfunfzig-Ecks in den Kreis. Jahresverhandlungen der Kurländischen Gesellschaft für Literatur und Kunst. Zweyter Band. Mitau, 1822: 160–219.

[14] James Pierpont. On an undemonstrated theorem of the Disquisitiones Arithmeticæ. Bull. Amer. Math. Soc. 2 (3), December 1895: 77–83.

[15] Helmut Ruhland. Unified Constructions of the Regular Heptagon and Triskaidecagon. Computer Tools in Education, 2025 (1): 27-32. Available at http://cte.eltech.ru/ojs/index.php/kio/article/view/1889. An updated (and further expanded) version of this article is made available at https://SemjonAdlaj.com/Excerpts/Heptagon,Triskaidecagon.pdf

Some References and Internet Sources III

[16] Игорь Соколов. От Ньютона к искусственному интеллекту. Выступление в апреле 2020 года в частной англо-американской школе “Марина”, “образовательная” деятельность которой была прекращена: https://www.youtube.com/watch?v=SYMDbYEZQng&list=PLpJDufKCOYVfOS4ORaazqLhTImOkp_weL&index=3&pp=iAQB. Утверждение сего “академика РАН” о том, что “сто пятьдесят лет назад был доказан в окончательной формулировке факт” – “любое уравнение степени выше четвёртой не имеет алгортма решения” доступно на канале “О науке и имитации учёности” по ссылке https://rutube.ru/video/efc75e5f695051b52e16738cb4c119e0/?playlist=474773

[17] Bernd Sturmfels. Algorithms in Invariant Theory. Springer-Verlag/Wien. ISSN 0943-853X, ISBN 3-211-82445-6 (first edn. 1993), ISBN 978-3-211-77416-8 (second edn. 2008).

[18] Nikolai Vavilov. Fermat numbers and cyclotomy. Seminar on the History of Mathematics, St. Petersburg, 2023, April 6, online. Available at https://www.mathnet.ru/php/seminars.phtml?presentid=38204&option_lang=eng

[19] Pierre Laurent Wantzel. Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas. Journal de Mathématiques Pures et Appliquées (1837), 2: 366–372.

[20] Andreas Günter Weber. Computing radical expressions for roots of unity. ACM Communications in Computer Algebra, Volume 30, Issue 3, Sept. 1996: 11-20.

[21] Eric Weisstein at Wolfram Research. Trigonometry Angles at Wolfram MathWorld. Last Updated: September 29, 2025. Available at https://mathworld.wolfram.com/TrigonometryAngles.html