# Quantum Information Processing and $E_8$ Lattice: From Contextuality to “Magic”

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# Brief outline

- Introduction.
- Dimension eight (quocts).
  - Korkin-Zolotarev-Gosset ($E_8$) lattice.
  - $E_8$ configuration with 120 states and 2025 bases.
- Dimension four (ququarts).
  - Witting configuration with 40 states and 40 bases.
    - Penrose dodecahedra and quantum cards
    - Note on quantum key distribution
  - Another configuration with 120 states and 210 bases.
  - Yet another configuration with 60 stabilizer states and 105 bases.
- Conclusion.

# Introduction

Regular structures and symmetries can be useful for analysis of different configurations of states relevant for quantum information processing. Here are considered few themes related with eight-dimensional diamond lattice $E_8$.

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This lattice is defined in real vector space, but it was already used in applications relevant for quantum information science both for quantum systems with eight-dimensional space of states (quocts) and with four-dimensional one (ququarts).

***

Possible novel application are also discussed in the presented talk.

# Korkin-Zolotarev-Gosset ($E_8$) lattice

**Definition**

$E_8$ lattice is a set of points in $\mathbb{R}^8$ with properties:

- *All eight coordinates* are either integers or half-integers

$$(x_1, \ldots, x_8)\colon \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + {}^1\!/\!_2$$

- The sum of the eight coordinates is an *even* integer

$$\sum x_i \equiv 0 \pmod 2$$

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*J. H. Conway and N. J. A. Sloane*, ***Sphere Packings, Lattices and Groups***,
Springer–Verlag, New York, 1999.

# Minimal vectors of $E_8$ lattice

## Minimal vectors

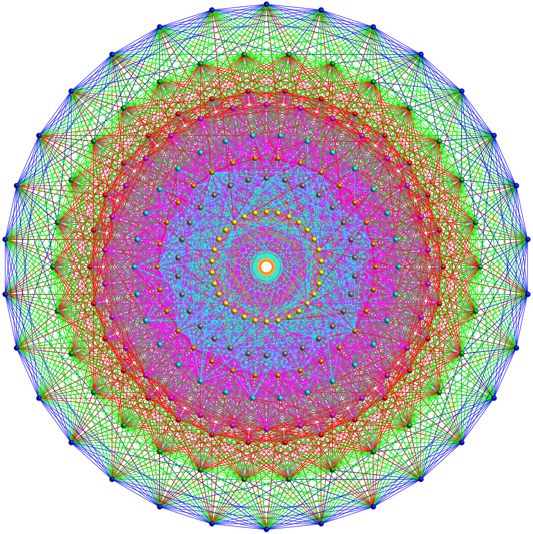
240 vectors (roots) with minimal norm = 2 include:

- 112 vectors $(\pm 1^{2\,\text{pos.}}, 0^{6\,\text{pos.}})$ with only two nonzero coefficients $\pm 1$ in some positions
- 128 vectors $(\pm {}^1\!/_2{}^{8\,\text{pos.}})$ with coefficients $\pm {}^1\!/_2$ in all positions and *even* number of minus signs

## Gosset polytope

$E_8$ root vectors correspond to semiregular Gosset polytope with 240 vertexes (connected by 6720 edges)

# Gosset ($E_8$) polytope



Picture source: Wikipedia

# Symmetries: automorphism group $W(E_8)$

Symmetries of $E_8$ lattice is Weyl group $W(E_8)$, generated by

- all possible permutations of eight coordinates
- all *even* sign changes
- matrix

$$\frac{1}{2}\begin{pmatrix} \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{matrix} \end{pmatrix},$$

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$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Number of symmetries: $|W(E_8)| = 696729600$.

# $W(E_8)$ and quantum circuits

Classical reversible circuits can be constructed using universal set with NOT ($X$), Controlled-NOT ($CX$) and Controlled-Controlled-NOT ($CCX$ or *Toffoli*) gates.

**3-qubit *Toffoli*–$H \otimes H$ circits**

Let us consider circuits composed of $X$, $CX$, $CCX$ gates (together with $SWAP$ gates between qubits) and **pairs** of Hadamard gates $H_j H_k$ acting on *two qubits* with different indexes $1 \leqslant j < k \leqslant 3$.

Transformations corresponding to even sign changes also can be composed of such a set of gates and such quantum circuits can implement any symmetry of $E_8$ lattice (yet, global $\pm 1$ multiplier formally is not physical).

This *nonuniversal* set of quantum gates may generate $|W(E_8)|/2 = 348364800$ different transformations.

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Some references: *M. Amy, N. J. Ross, and S. Wesley*, arXiv:2407.11152
*M. Planat*, arXiv:0904.3691

# $E_8$ roots as configuration of quantum states

Any pair of roots with opposite directions correspond to the same quantum state. Thus, configuration of 240 $E_8$ roots or vertexes of Gosset polytope corresponds to 120 different quantum states or rays in Hilbert space.

Sets with eight mutually orthogonal states (*bases*) have particular interest because laws of quantum mechanics allow such states to be measured simultaneously.

For the 120 rays produced from the 240 $E_8$ roots there are **2025** such bases.

An application to the analysis of quantum contextuality, was discussed in:

*M. Waegell and P.K. Aravind*, Parity proofs of the Kochen-Specker theorem based on the Lie algebra E8, `arXiv:1502.04350`, J. Phys. A: Math. Theor. **48**, 225301 (2015).

Kochen-Specker theorem places some constrains on hidden-variables theories and can have certain relations with foundations of quantum information science.

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**Quantum key distribution**

This configuration may also have applications in quantum communications. Choice of the same bases allows participants to exchange information or secret keys.

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However, the eight-dimensional configuration with 2025 bases is rather complicated and *four-dimensional complex* version of the eight-dimensional Gosset polytope is discussed next.

# Four-dimensional regular complex Witting polytope

## Witting polytope

Formal definition of Witting polytope uses four complex reflections, but let us first recollect the complex coordinates of 240 vertexes (norm $= 3$ is used):

$$(\pm\mathrm{i}\omega^\lambda\sqrt{3},0,0,0),\qquad (0,\pm\mathrm{i}\omega^\lambda\sqrt{3},0,0),\qquad (0,0,\pm\mathrm{i}\omega^\lambda\sqrt{3},0),\qquad (0,0,0,\pm\mathrm{i}\omega^\lambda\sqrt{3}),$$

$$(0,\pm\omega^\mu,\mp\omega^\nu,\pm\omega^\lambda),\ (\mp\omega^\mu,0,\pm\omega^\nu,\pm\omega^\lambda),\ (\pm\omega^\mu,\mp\omega^\nu,0,\pm\omega^\lambda),\ (\mp\omega^\mu,\mp\omega^\nu,\mp\omega^\lambda,0),$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ and $\lambda,\mu,\nu \in \{0,1,2\}$.

*H. S. M. Coxeter*, ***Regular Complex Polytopes***, CUP, 1991.

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$$(0,\pm\omega^\mu,\mp\omega^\nu,\pm\omega^\lambda),\ (\mp\omega^\mu,0,\pm\omega^\nu,\pm\omega^\lambda),\ (\pm\omega^\mu,\mp\omega^\nu,0,\pm\omega^\lambda),\ (\mp\omega^\mu,\mp\omega^\nu,\mp\omega^\lambda,0),$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$ and $\lambda,\mu,\nu \in \{0,1,2\}$.

These *complex* vertexes can be mapped into an eight-dimensional *real* vector space:
$(z_1,z_2,z_3,z_4) \equiv (x_1+iy_1,x_2+iy_2,x_3+iy_3,x_4+iy_4) \longrightarrow (x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)$.
After such *realification*, scaling by $\sqrt{2/3}$ and some 8-dimensional rotation, the configuration coincides with standard expressions for 240 vertexes of the Gosset polytope mentioned earlier.

# Witting configuration with 40 quantum states

## 40 quantum states

The vertexes of the Witting polytope can be divided into 40 groups of six points, the coordinates of which differ only in the global phase factor $e^{2\pi i k/6}$, $k = 0, \ldots, 5$.
Each such group corresponds to the same quantum state.
After normalization, we have 40 states: four basic states

$$(1,0,0,0), \qquad (0,1,0,0), \qquad (0,0,1,0), \qquad (0,0,0,1),$$

and 36 states, which can be expressed as

$$\frac{1}{\sqrt{3}}(0,1,-\omega^\mu,\omega^\nu),\ \frac{1}{\sqrt{3}}(1,0,-\omega^\mu,-\omega^\nu),\ \frac{1}{\sqrt{3}}(1,-\omega^\mu,0,\omega^\nu),\ \frac{1}{\sqrt{3}}(1,\omega^\mu,\omega^\nu,0),$$

where $\omega = e^{2\pi i/3} = \dfrac{-1 + i\sqrt{3}}{2}$ and $\mu, \nu = 0, 1, 2$.

# Witting configuration and Penrose dodecahedra

Configuration with 40 states described above has direct correspondence with model of quantum contextuality described by Penrose using geometry of dodecahedron:

*M. Waegell and P. K. Aravind*, The Penrose dodecahedron and the Witting polytope are identical in $\mathrm{CP}^3$,* `arXiv:1701.06512 (2017)`, Phys. Lett. A **381**, 1853–1857 (2017).

* The appearance in the title of the projective space $\mathbb{C}P^3$, *i.e.*, the space of rays in the four-dimensional complex space, corresponds to the transition from 240 complex vectors to 40 quantum states discussed above.

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## Penrose dodecahedra

Penrose model uses two entangled spin-$3/2$ particles and configurations with 40 states described by geometry of dodecahedron. The states were defined using rather complicated Majorana representation of higher spin particle using sets of points on Bloch sphere corresponding to poles of polynomials. Witting configuration provides simpler expressions for description of the same set of states.

# Model with Penrose dodecahedron

*Rodger Penrose*,
On Bell non-locality without probabilities: some curious geometry,
Preprint, Oxford, 1992;
reprinted in: J. Ellis, D. Amati (eds.), *Quantum Reflections*, 1–27, CUP, 2000.

*J. Zimba and R. Penrose*,
On Bell non-locality without probabilities: more curious geometry,
Stud. Hist. Phil. Sci. **24**, 697–720 (1993).

*R. Penrose*, ***Shadows of the Mind***,
Oxford University Press, 1994.

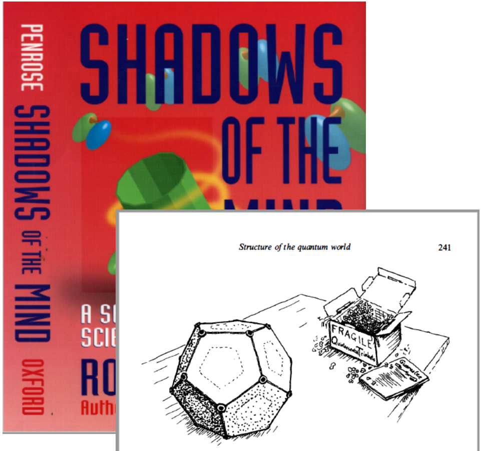


Fig. 5.2. The magic dodecahedron. My colleague has an identical copy on α-Centauri. On each vertex is a button, and pressing one *may* ring the bell and initiate the magnificent pyrotechnic display.

# Witting configuration: table with 40 states

## Notation for 40 states (without normalization)

| k | $\lvert\xi_0^k\rangle$ | $\lvert\xi_1^k\rangle$ | $\lvert\xi_2^k\rangle$ | $\lvert\xi_3^k\rangle$ |
|---|---|---|---|---|
| 1 | $(1, 0, 0, 0)$ | $(0, 1, 0, 0)$ | $(0, 0, 1, 0)$ | $(0, 0, 0, 1)$ |
| 2 | $(0, 1, -1, 1)$ | $(1, 0, -1, -1)$ | $(1, -1, 0, 1)$ | $(1, 1, 1, 0)$ |
| 3 | $(0, 1, -\omega, \bar\omega)$ | $(1, 0, -\bar\omega, -1)$ | $(1, -\omega, 0, 1)$ | $(1, \omega, \bar\omega, 0)$ |
| 4 | $(0, 1, -\bar\omega, \omega)$ | $(1, 0, -\omega, -1)$ | $(1, -\bar\omega, 0, 1)$ | $(1, \bar\omega, \omega, 0)$ |
| 5 | $(0, 1, -1, \omega)$ | $(1, 0, -\omega, -\bar\omega)$ | $(1, -\omega, 0, \bar\omega)$ | $(1, \omega, \omega, 0)$ |
| 6 | $(0, 1, -\omega, 1)$ | $(1, 0, -1, -\bar\omega)$ | $(1, -\bar\omega, 0, \bar\omega)$ | $(1, \bar\omega, 1, 0)$ |
| 7 | $(0, 1, -\bar\omega, \bar\omega)$ | $(1, 0, -\bar\omega, -\bar\omega)$ | $(1, -1, 0, \bar\omega)$ | $(1, 1, \bar\omega, 0)$ |
| 8 | $(0, 1, -1, \bar\omega)$ | $(1, 0, -\bar\omega, -\omega)$ | $(1, -\bar\omega, 0, \omega)$ | $(1, \bar\omega, \bar\omega, 0)$ |
| 9 | $(0, 1, -\omega, \omega)$ | $(1, 0, -\omega, -\omega)$ | $(1, -1, 0, \omega)$ | $(1, 1, \omega, 0)$ |
| 10 | $(0, 1, -\bar\omega, 1)$ | $(1, 0, -1, -\omega)$ | $(1, -\omega, 0, \omega)$ | $(1, \omega, 1, 0)$ |

# Coxeter definition of Witting polytope

**Complex reflection of order $k$**

$$R\langle {}^k\phi\rangle = \mathbf{1} + (\mathrm{e}^{2\pi\mathrm{i}/k} - 1)\frac{|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle}.$$

**Witting polytope, 4 'triflections'**

$$R_1 = R\langle {}^3\xi_0^1\rangle, \quad R_2 = R\langle {}^3\xi_3^2\rangle,$$
$$R_3 = R\langle {}^3\xi_2^1\rangle, \quad R_4 = R\langle {}^3\xi_0^2\rangle.$$

*H. S. M. Coxeter*, **Regular Complex Polytopes**, CUP, 1991.

Four 'triflections' $R_k$ generate group with 155520 symmetries of complex *Witting polytope* with 240 vertexes. For *Witting configuration* with 40 states it is enough to use four operators $r_k = \varpi R_k$, $k = 1, \ldots, 4$ with unit determinants.

Such operators produce smaller group $\mathcal{W}$ with 51840 elements corresponding to 25920 different quantum gates (up to $\pm\mathbf{1}$). Applications of different combinations of such gates to state $|0\rangle$ produce 40 states of Witting configuration.

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An entanglement between two particles is essential for initial Penrose model with *two dodecahedra*. Some notes can be found in *A. Y. Vlasov*, Penrose dodecahedron, Witting configuration and quantum entanglement, arXiv:2208.13644, Quanta **13**, 38–46 (2024).

# Witting configuration: 40 bases

$1 \; \{\underline{|\xi_0^1\rangle, |\xi_1^1\rangle, |\xi_2^1\rangle, |\xi_3^1\rangle}\} \equiv \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\},$

| | | | | | |
|---|---|---|---|---|---|
| 2 | $\{\underline{\|\xi_0^2\rangle, \|\xi_1^2\rangle, \|\xi_2^2\rangle, \|\xi_3^2\rangle}\}$, | 3 | $\{\|\xi_0^2\rangle, \|\xi_1^7\rangle, \|\xi_2^6\rangle, \|\xi_3^8\rangle\}$, | 4 | $\{\|\xi_0^2\rangle, \|\xi_1^9\rangle, \|\xi_2^{10}\rangle, \|\xi_3^5\rangle\}$, |
| 5 | $\{\underline{\|\xi_0^3\rangle, \|\xi_1^3\rangle, \|\xi_2^3\rangle, \|\xi_3^3\rangle}\}$, | 6 | $\{\|\xi_0^3\rangle, \|\xi_1^5\rangle, \|\xi_2^7\rangle, \|\xi_3^9\rangle\}$, | 7 | $\{\|\xi_0^3\rangle, \|\xi_1^{10}\rangle, \|\xi_2^8\rangle, \|\xi_3^6\rangle\}$, |
| 8 | $\{\underline{\|\xi_0^4\rangle, \|\xi_1^4\rangle, \|\xi_2^4\rangle, \|\xi_3^4\rangle}\}$, | 9 | $\{\|\xi_0^4\rangle, \|\xi_1^6\rangle, \|\xi_2^5\rangle, \|\xi_3^{10}\rangle\}$, | 10 | $\{\|\xi_0^4\rangle, \|\xi_1^8\rangle, \|\xi_2^9\rangle, \|\xi_3^7\rangle\}$, |
| 11 | $\{\underline{\|\xi_0^5\rangle, \|\xi_1^5\rangle, \|\xi_2^5\rangle, \|\xi_3^5\rangle}\}$, | 12 | $\{\|\xi_0^5\rangle, \|\xi_1^{10}\rangle, \|\xi_2^9\rangle, \|\xi_3^2\rangle\}$, | 13 | $\{\|\xi_0^5\rangle, \|\xi_1^3\rangle, \|\xi_2^4\rangle, \|\xi_3^8\rangle\}$, |
| 14 | $\{\underline{\|\xi_0^6\rangle, \|\xi_1^6\rangle, \|\xi_2^6\rangle, \|\xi_3^6\rangle}\}$, | 15 | $\{\|\xi_0^6\rangle, \|\xi_1^8\rangle, \|\xi_2^{10}\rangle, \|\xi_3^3\rangle\}$, | 16 | $\{\|\xi_0^6\rangle, \|\xi_1^4\rangle, \|\xi_2^2\rangle, \|\xi_3^9\rangle\}$, |
| 17 | $\{\underline{\|\xi_0^7\rangle, \|\xi_1^7\rangle, \|\xi_2^7\rangle, \|\xi_3^7\rangle}\}$, | 18 | $\{\|\xi_0^7\rangle, \|\xi_1^9\rangle, \|\xi_2^8\rangle, \|\xi_3^4\rangle\}$, | 19 | $\{\|\xi_0^7\rangle, \|\xi_1^2\rangle, \|\xi_2^3\rangle, \|\xi_3^{10}\rangle\}$, |
| 20 | $\{\underline{\|\xi_0^8\rangle, \|\xi_1^8\rangle, \|\xi_2^8\rangle, \|\xi_3^8\rangle}\}$, | 21 | $\{\|\xi_0^8\rangle, \|\xi_1^4\rangle, \|\xi_2^3\rangle, \|\xi_3^5\rangle\}$, | 22 | $\{\|\xi_0^8\rangle, \|\xi_1^6\rangle, \|\xi_2^7\rangle, \|\xi_3^2\rangle\}$, |
| 23 | $\{\underline{\|\xi_0^9\rangle, \|\xi_1^9\rangle, \|\xi_2^9\rangle, \|\xi_3^9\rangle}\}$, | 24 | $\{\|\xi_0^9\rangle, \|\xi_1^2\rangle, \|\xi_2^4\rangle, \|\xi_3^6\rangle\}$, | 25 | $\{\|\xi_0^9\rangle, \|\xi_1^7\rangle, \|\xi_2^5\rangle, \|\xi_3^3\rangle\}$, |
| 26 | $\{\underline{\|\xi_0^{10}\rangle, \|\xi_1^{10}\rangle, \|\xi_2^{10}\rangle, \|\xi_3^{10}\rangle}\}$, | 27 | $\{\|\xi_0^{10}\rangle, \|\xi_1^3\rangle, \|\xi_2^2\rangle, \|\xi_3^7\rangle\}$, | 28 | $\{\|\xi_0^{10}\rangle, \|\xi_1^5\rangle, \|\xi_2^6\rangle, \|\xi_3^4\rangle\}$, |
| 29 | $\{\|\xi_0^1\rangle, \|\xi_0^2\rangle, \|\xi_0^3\rangle, \|\xi_0^4\rangle\}$, | 30 | $\{\|\xi_0^1\rangle, \|\xi_0^5\rangle, \|\xi_0^6\rangle, \|\xi_0^7\rangle\}$, | 31 | $\{\|\xi_0^1\rangle, \|\xi_0^8\rangle, \|\xi_0^9\rangle, \|\xi_0^{10}\rangle\}$, |
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| 35 | $\{\|\xi_2^1\rangle, \|\xi_2^2\rangle, \|\xi_2^3\rangle, \|\xi_2^8\rangle\}$, | 36 | $\{\|\xi_2^1\rangle, \|\xi_2^3\rangle, \|\xi_2^6\rangle, \|\xi_2^9\rangle\}$, | 37 | $\{\|\xi_2^1\rangle, \|\xi_2^4\rangle, \|\xi_2^7\rangle, \|\xi_2^{10}\rangle\}$, |
| 38 | $\{\|\xi_3^1\rangle, \|\xi_3^2\rangle, \|\xi_3^3\rangle, \|\xi_3^4\rangle\}$, | 39 | $\{\|\xi_3^1\rangle, \|\xi_3^5\rangle, \|\xi_3^6\rangle, \|\xi_3^7\rangle\}$, | 40 | $\{\|\xi_3^1\rangle, \|\xi_3^6\rangle, \|\xi_3^9\rangle, \|\xi_3^{10}\rangle\}$. |

All 40 bases can be produced by applications of symmetries from group $\mathcal{W}$ to 'computational basis' $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$.

Ten non-overlapping bases (underlined) $\{|\xi_0^k\rangle, |\xi_1^k\rangle, |\xi_2^k\rangle, |\xi_3^k\rangle\}$, $k = 1, \ldots, 10$ include all 40 states.

They can be produced from computational basis by some subgroup $\mathcal{W}_n \subset \mathcal{W}$.

# "Quantum cards"

More visual representation of quantum states and bases:

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\{\underline{\vert\xi_0^1\rangle, \vert\xi_1^1\rangle, \vert\xi_2^1\rangle, \vert\xi_3^1\rangle}\} \equiv \{\vert 0\rangle, \vert 1\rangle, \vert 2\rangle, \vert 3\rangle\}$, | | | | |
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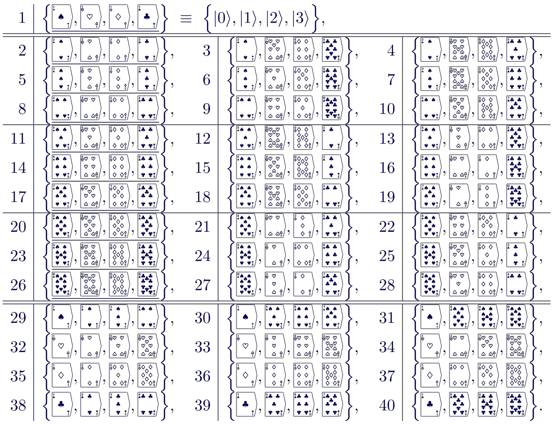
| 14 | $\{\underline{\vert\xi_0^6\rangle, \vert\xi_1^6\rangle, \vert\xi_2^6\rangle, \vert\xi_3^6\rangle}\}$, | 15 | $\{\vert\xi_0^6\rangle, \vert\xi_1^8\rangle, \vert\xi_2^{10}\rangle, \vert\xi_3^3\rangle\}$, | 16 | $\{\vert\xi_0^6\rangle, \vert\xi_1^4\rangle, \vert\xi_2^2\rangle, \vert\xi_3^9\rangle\}$, |
| 17 | $\{\underline{\vert\xi_0^7\rangle, \vert\xi_1^7\rangle, \vert\xi_2^7\rangle, \vert\xi_3^7\rangle}\}$, | 18 | $\{\vert\xi_0^7\rangle, \vert\xi_1^9\rangle, \vert\xi_2^8\rangle, \vert\xi_3^4\rangle\}$, | 19 | $\{\vert\xi_0^7\rangle, \vert\xi_1^2\rangle, \vert\xi_2^3\rangle, \vert\xi_3^{10}\rangle\}$, |
| 20 | $\{\underline{\vert\xi_0^8\rangle, \vert\xi_1^8\rangle, \vert\xi_2^8\rangle, \vert\xi_3^8\rangle}\}$, | 21 | $\{\vert\xi_0^8\rangle, \vert\xi_1^4\rangle, \vert\xi_2^3\rangle, \vert\xi_3^5\rangle\}$, | 22 | $\{\vert\xi_0^8\rangle, \vert\xi_1^6\rangle, \vert\xi_2^7\rangle, \vert\xi_3^2\rangle\}$, |
| 23 | $\{\underline{\vert\xi_0^9\rangle, \vert\xi_1^9\rangle, \vert\xi_2^9\rangle, \vert\xi_3^9\rangle}\}$, | 24 | $\{\vert\xi_0^9\rangle, \vert\xi_1^2\rangle, \vert\xi_2^4\rangle, \vert\xi_3^6\rangle\}$, | 25 | $\{\vert\xi_0^9\rangle, \vert\xi_1^7\rangle, \vert\xi_2^5\rangle, \vert\xi_3^3\rangle\}$, |
| 26 | $\{\underline{\vert\xi_0^{10}\rangle, \vert\xi_1^{10}\rangle, \vert\xi_2^{10}\rangle, \vert\xi_3^{10}\rangle}\}$, | 27 | $\{\vert\xi_0^{10}\rangle, \vert\xi_1^3\rangle, \vert\xi_2^2\rangle, \vert\xi_3^7\rangle\}$, | 28 | $\{\vert\xi_0^{10}\rangle, \vert\xi_1^5\rangle, \vert\xi_2^6\rangle, \vert\xi_3^4\rangle\}$, |
| 29 | $\{\vert\xi_0^1\rangle, \vert\xi_0^2\rangle, \vert\xi_0^3\rangle, \vert\xi_0^4\rangle\}$, | 30 | $\{\vert\xi_0^1\rangle, \vert\xi_0^5\rangle, \vert\xi_0^6\rangle, \vert\xi_0^7\rangle\}$, | 31 | $\{\vert\xi_0^1\rangle, \vert\xi_0^8\rangle, \vert\xi_0^9\rangle, \vert\xi_0^{10}\rangle\}$, |
| 32 | $\{\vert\xi_1^1\rangle, \vert\xi_1^2\rangle, \vert\xi_1^5\rangle, \vert\xi_1^8\rangle\}$, | 33 | $\{\vert\xi_1^1\rangle, \vert\xi_1^3\rangle, \vert\xi_1^6\rangle, \vert\xi_1^9\rangle\}$, | 34 | $\{\vert\xi_1^1\rangle, \vert\xi_1^4\rangle, \vert\xi_1^7\rangle, \vert\xi_1^{10}\rangle\}$, |
| 35 | $\{\vert\xi_2^1\rangle, \vert\xi_2^2\rangle, \vert\xi_2^3\rangle, \vert\xi_2^8\rangle\}$, | 36 | $\{\vert\xi_2^1\rangle, \vert\xi_2^3\rangle, \vert\xi_2^6\rangle, \vert\xi_2^9\rangle\}$, | 37 | $\{\vert\xi_2^1\rangle, \vert\xi_2^4\rangle, \vert\xi_2^7\rangle, \vert\xi_2^{10}\rangle\}$, |
| 38 | $\{\vert\xi_3^1\rangle, \vert\xi_3^2\rangle, \vert\xi_3^3\rangle, \vert\xi_3^4\rangle\}$, | 39 | $\{\vert\xi_3^1\rangle, \vert\xi_3^5\rangle, \vert\xi_3^6\rangle, \vert\xi_3^7\rangle\}$, | 40 | $\{\vert\xi_3^1\rangle, \vert\xi_3^6\rangle, \vert\xi_3^9\rangle, \vert\xi_3^{10}\rangle\}$. |

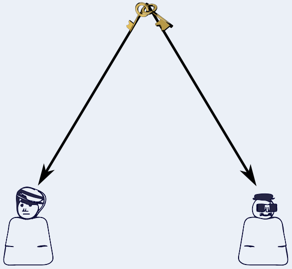


# Quantum key distribution

Witting configuration with 40 states and 40 bases can be used for quantum key distribution. Alice and Bob measure pairs of entangled ququarts to produce a pair of keys for private communications.

## Simplified scheme

$$|\Sigma\rangle = \frac{1}{2}\big(|0\rangle|0\rangle+|1\rangle|1\rangle+|2\rangle|2\rangle+|3\rangle|3\rangle\big)$$



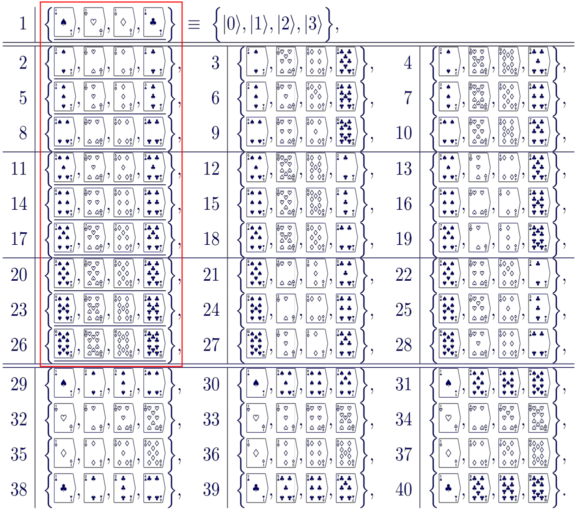

## More general case

$$|\Sigma^\ell\rangle = \frac{1}{2}\sum_{j=0,\ldots,3}|\phi^\ell_{j,A}\rangle|\varphi^\ell_{j,B}\rangle,$$

where $\ell$ denotes different pairs of bases and $|\Sigma^\ell\rangle = |\Sigma\rangle$, *if Bob uses complex-conjugated ('consistent') indexing* $|\varphi^\ell_{j,B}\rangle = \overline{|\phi^\ell_{j,A}\rangle}$.

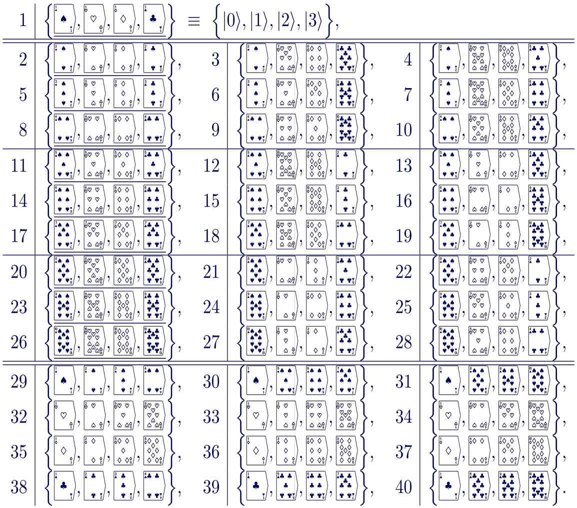
More details in *A. Y. Vlasov*, Scheme of quantum communications based on Witting polytope, `arXiv:2503.18431`.

# Contextuality




▷ If only ten non-overlapping bases are used each state appears only in one of them.

▷ For full set with 40 bases, any state may appear in four different bases, *i.e.*, in *different contexts*.




▷ If only ten non-overlapping bases are used each state appears only in one of them.

▷ For full set with 40 bases, any state may appear in four different bases, *i.e.*, in *different contexts*.

◇ It is impossible in advance to mark some cards in such a way, that *one and only one* such card would appear in any of 40 quadruples listed in the table.

◇ For set with only 10 underlined fours such choice is possible. For example, ten cards with *the same suit* can be marked.

◇ For full set such a method works only for first part of the table with 28 fours, but other 12 quadruples would have either four marked cards or none.

# Contextuality and groups

- ▷ The full set with 40 bases used for contextual measurements can be obtained from computational basis by symmetry group $\mathcal{W}$ of Witting configuration generated by four operators (gates) $r_k$ defined earlier.
- ▷ The subgroup $\mathcal{W}_{\rm n}$, generated by two elements $h_1 = r_1 r_3$ and $h_2 = r_1/r_4$ with 720 symmetries can transform computational bases only into one of ten non-overlapping quadruples discussed above.
- ▷ Thus, $\mathcal{W}_{\rm n}$ could be treated as 'non-contextual subgroup' of $\mathcal{W}$ and, in turn, $\mathcal{W}$ is some 'contextual closure' of $\mathcal{W}_{\rm n}$.

# Contextuality and groups

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▷ Thus, $\mathcal{W}_\mathrm{n}$ could be treated as 'non-contextual subgroup' of $\mathcal{W}$ and, in turn, $\mathcal{W}$ is some 'contextual closure' of $\mathcal{W}_\mathrm{n}$.

▷ The subgroup $\mathcal{W}_\mathrm{n}$ is also of particular interest in connection with the original Penrose model, which used 'asymmetric' entangled state

$$|\Omega_a\rangle = \frac{1}{2}\bigl(|0\rangle|3\rangle - |1\rangle|2\rangle + |2\rangle|1\rangle - |3\rangle|0\rangle\bigr),$$

because such a state is invariant with respect to $\mathcal{W}_\mathrm{n}$.

## $U_4(2) \cong S_4(3)$

Unitary group $U_4(2) \cong {}^2A_3(2) \cong O_6^-(2)$; Symplectic group $S_4(3) \cong C_2(3) \cong O_5(3)$

Order = 25,920 = $2^6.3^4.5$ Mult = 2 Out = 2

**Constructions**

Unitary: $GU_4(2) \cong 3 \times G$ : all 4 x 4 matrices over $\mathbb{F}_4$ preserving a non-singular Hermitian form;
$PGU_4(2) \cong SU_4(2) \cong PSU_4(2) \cong G$

Orthogonal (2): $GO_6^-(2) \cong PGO_6^-(2) \cong SO_6^-(2) \cong PSO_6^-(2) \cong G.2$ : all 6 x 6 matrices over $\mathbb{F}_2$ preserving a non-singular quadratic form of Witt defect 1, for example $x_1^2+x_1x_2+x_2^2+x_3x_4+x_5x_6$; $O_6^-(2) \cong G$

Orthogonal (3): $GO_5(3) \cong 2 \times G.2$ : all 5 x 5 matrices over $\mathbb{F}_3$ preserving a non-singular quadratic form;
$PGO_5(3) \cong SO_5(3) \cong PSO_5(3) \cong G.2$; $O_5(3) \cong G$
G : the automorphism group of a generalized 4-gon of order (3,3) consisting of 40 vertices and 40 edges, each object being incident with 4 of the other type

Symplectic: $Sp_4(3) \cong 2.G$ : all 4 x 4 matrices over $\mathbb{F}_3$ preserving a non-singular symplectic form;
$PSp_4(3) \cong S_4(3) \cong G$;

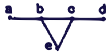
Weyl: G.2 : the Weyl group of $E_6$, generated by the 36 reflections in the minimal (root) vectors of the $E_6$ lattice; these may be taken as
$(0,0,0,0,0,0;1,-1)$ (1), $(1,-1,0,0,0,0;0,0)$ (15), $(\frac12,\frac12,\frac12,-\frac12,-\frac12,-\frac12;\frac12,-\frac12)$ (20) (first base)
or $(\pm1,\pm1,0,0,0,0;0,0,0)$ (20), $\pm(\pm\frac12,\pm\frac12,\pm\frac12,\pm\frac12,\pm\frac12;\frac12,\frac12,\frac12)$ (even sum) (16) (second base)
where the coordinates before the semicolon may be freely permuted.
The coordinates are obtained from the $E_8$ root lattice (see $O_8^+(2)$) by imposing the conditions
$x_1+x_2+x_3+x_4+x_5+x_6=0=x_7+x_8$ (first base) or $x_6=x_7=x_8$ (second base).
The Weyl group permutes the 27 vectors (which belong to one coset of the lattice in its dual $E_6^*$)
$a_i = (\frac56,-\frac16,-\frac16,-\frac16,-\frac16,-\frac16;\frac12,-\frac12)$ (6), $b_i = (\frac56,-\frac16,-\frac16,-\frac16,-\frac16,-\frac16;-\frac12,\frac12)$ (6), $c_{ij} = (-\frac23,-\frac23,\frac13,\frac13,\frac13,\frac13;0,0)$ (15);
the full automorphism group of the lattice is 2 x G.2 and is obtained by adjoining negation.
Reducing modulo 2 shows the isomorphism with $O_6^-(2)$.

Biflection: 2 x G : the group generated by the 45 complex reflections of order 2 in the vectors
$(\pm2,0,0,0,0)^C$ (5), $(0,\pm w,\pm\bar{w},\pm\bar{w},\pm w)^C$ (40), where $w = \sqrt[3]{}$3.
Reducing mod $\theta$ = $\sqrt{-3}$ shows the isomorphism with $O_5(3)$.
6 x G : the full automorphism group of the lattice these generate.

Triflection: 3 x 2.G : the group generated by the 40 complex reflections of order 3 (triflections) in the ($\pm w$)vectors $(\theta;0,0,0)$ $(0;\theta,0,0)$ $(0;w^a,w^b,w^c)$ $(w^a;0,w^b,-w^c)$, where the last 3 coordinates may be cyclically permuted; the triflection in r takes v to $v - (1-w)(v.r)r/(r.r)$, where $x.y = \Sigma x_i\bar{y}_i$. Reducing mod 2 shows the isomorphism with $U_4(2)$; reducing mod $\theta$ shows the isomorphism with $S_4(3)$.

Schläfli: G.2 : the group of the incidence-preserving permutations of the 27 lines on the general cubic surface in projective 3-space : $a_i$ (6), $b_i$ (6), $c_{ij}$ (15) ($i,j = 1, \ldots 6, i\neq j$). These lie in threes in 45 tritangent planes $(a_i,b_j,c_{ij})$ (30), $(c_{ij},c_{kl},c_{mn})$ (15) (i,j,k,l,m,n, distinct), and two lines intersect if and only if they lie in some tritangent plane. There are 36 "double sixes" such as $\{a_i|b_j\}$, in which each line of either "six" meets all but one line of the other "six".

Hesse: G.2 : the stabilizer of a bitangent in Hesse's group (see $S_6(2)$) of symmetries of the 28 bitangents to the general quartic curve in the complex projective plane; the bitangents may be labelled with unordered pairs from {1,2,3,4,5,6,a,b} and the 27 bitangents other than ab then correspond naturally with Schläfli's 27 lines: ai $\leftrightarrow a_i$, bj $\leftrightarrow b_j$, ij $\leftrightarrow c_{ij}$. Generating permutations may easily be obtained from Hesse's bifid maps (see $S_6(2)$) e.g. (a123|b456) and (ab12|3456)(a 1)(b 2). The 45 quartets {ab,bj,ji,ia} (30) and {ab,ij,kl,mn} (15) containing ab yield the 45 tritangent planes.

Presentations: G.2 $\cong$  ; G x 2 $\cong$ < 

| $(bcbe)^3=1$ >

**Maximal subgroups** / **Specifications**

| Order | Index | Structure | G.2 | Character | Abstract | Unitary | Orthogonal (2) |
|---|---|---|---|---|---|---|---|
| 960 | 27 | $2^4{:}A_5$ | : $2^4{:}S_5$ | 1a+6a+20a | $N(2^4) = N(2A_5B_{10})$ | isotropic line | isotropic point |
| 720 | 36 | $S_6$ | : $S_6 \times 2$ | 1a+15b+20a | N(2B,3C,3D,4B,5A), C(2C) | $S_4(2)$ | non-isotropic point |
| 648 | 40 | $3_+^{1+2}{:}2A_4$ | : $3_+^{1+2}{:}2S_4$ | 1a+15a+24a | N(3AB) | non-isotropic point | $U_3(2)$ |
| 648 | 40 | $3^3{:}S_4$ | : $3^3{:}(S_4 \times 2)$ | 1a+15b+24a | $N(3^3) = N(3AB_4C_3D_6)$ | base | $O_2^-(2) \wr S_3$ |
| 576 | 45 | $2^{\cdot}(A_4 \times A_4).2$ | : H.2 | 1a+20a+24a | N(2A) | isotropic point | isotropic line |

*J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson,* ***ATLAS of Finite Groups***, p. 26, OUP, 1985.

# Yet another complexification of $E_8$ roots

The Witting configuration is not only possible configuration related with complexification of $E_8$ root vectors. Let's consider the group generated by matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\iota & \iota' & \iota \\ 0 & \iota' & \iota & -\iota'' \\ 0 & \iota & -\iota'' & -\bar{\iota}' \end{pmatrix},$$

where $\iota = \dfrac{\mathrm{i}}{\sqrt{7}}, \quad \iota' = \dfrac{1+\iota}{2}, \quad \iota'' = \dfrac{1+3\iota}{2}.$

Application of elements of such group to the vector $(1,0,0,0)$ generates configuration with 240 four-dimensional complex vectors and after *realification* and some eight-dimensional rotation, the configuration coincides with expressions for 240 vertexes of the Gosset polytope used earlier.

# Configuration with 120 states and 210 bases

- ▷ Configuration with 240 complex vectors described above corresponds to 120 quantum states or 120 complex rays, because global phase can be only $\pm 1$.
- ▷ The group with 5040 elements generated by matrices $P_1$, $P_2$ and $S$ is double cover of the *alternating group* $A_7$ of even permutations of seven elements ($\pm 1$ are mapped into trivial permutation) and denoted here $\tilde{\mathcal{A}}_7$.
- ▷ All states of configuration can be produced by transformations from group $\tilde{\mathcal{A}}_7$ from state $|0\rangle$. Application of $\tilde{\mathcal{A}}_7$ to $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ can produce 210 different bases. Each state of the configuration belongs to seven different bases.
- ▷ 'Non-contextual subgroup' with 120 elements $\tilde{\mathcal{A}}_5 \subset \tilde{\mathcal{A}}_7$ can be generated by matrices $J = P_1^{-1} P_2 P_1$ and $H = S P_2 P_1 S^{-1} P_1$. Application of subgroup $\tilde{\mathcal{A}}_5$ to $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ produces 30 non-overlapping (non-contextual) bases.

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More details in *A. Y. Vlasov*, Quantum entanglement and contextuality with complexifications of $E_8$ root system, `arXiv:2210.15338`.
Also: ***ATLAS***, p. 10.

# Relation between complex and real representation of $E_8$ roots

For two examples discussed above, transition between 4D complex and 8D real configurations can be described by a diagram that includes some ambiguity

$$|\xi\rangle \in \mathbb{C}P^3 \xrightarrow{?\,\mathrm{e}^{\mathrm{i}\varphi}\,(\text{phases ambiguity})} \boldsymbol{\xi} \in \mathbb{C}^4 \xrightarrow{\text{realification}} \mathbf{x} \in \mathbb{R}^8 \xrightarrow{\text{rotation } \mathcal{R} \in SO(8)} \mathbf{r} \in E_8.$$

Such ambiguity was avoided due to knowledge of 'correct' phases and vectors. After that, operation $\mathcal{R}$ may look like some formality with orientation.

An opposite transition can be described as

$$\mathbf{r} \in E_8 \xrightarrow{\text{rotation } \mathcal{R}^{-1} \in SO(8)} \mathbf{v} \in \mathbb{R}^8 \xrightarrow{\text{complexification}} \boldsymbol{\xi} \in \mathbb{C}^4 \xrightarrow{\text{normalization}} |\xi\rangle \in \mathbb{C}P^3.$$

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There is no phase ambiguity here, but knowledge of $\mathcal{R}$ is necessary to produce proper complex configuration from $E_8$ roots.

Let us consider complexification of $E_8$ roots *without any rotation* ($\mathcal{R} \equiv \mathbf{1}$).

# Complexification of $E_8$ roots

## Examples

$$
\begin{aligned}
(\pm1,\pm1,0,0,0,0,0,0) &\xrightarrow{\text{complexif.}} (\pm1\pm \mathrm{i},0,0,0) \xrightarrow{\text{normalization}} (1,0,0,0)\\
(\pm1,0,\pm1,0,0,0,0,0) &\xrightarrow{\text{complexif.}} (\pm1,\pm1,0,0) \xrightarrow{\text{normalization}} \tfrac{1}{\sqrt{2}}(1,\pm1,0,0)\\
(\pm1,0,0,\pm1,0,0,0,0) &\xrightarrow{\text{complexif.}} (\pm1,\pm\mathrm{i},0,0) \xrightarrow{\text{normalization}} \tfrac{1}{\sqrt{2}}(1,\pm\mathrm{i},0,0)\\
\tfrac12(1,1,1,1,1,1,1,1) &\xrightarrow{\text{complexif.}} \tfrac12(1+\mathrm{i},1+\mathrm{i},1+\mathrm{i},1+\mathrm{i}) \xrightarrow{\text{normalization}} \tfrac12(1,1,1,1)\\
\tfrac12(1,-1,1,-1,1,1,1,1) &\xrightarrow{\text{complexif.}} \tfrac12(1-\mathrm{i},1-\mathrm{i},1+\mathrm{i},1+\mathrm{i}) \xrightarrow{\text{normalization}} \tfrac12(1,1,\mathrm{i},\mathrm{i})\\
\tfrac12(1,-1,-1,1,1,1,1,1) &\xrightarrow{\text{complexif.}} \tfrac12(1-\mathrm{i},-1+\mathrm{i},1+\mathrm{i},1+\mathrm{i}) \xrightarrow{\text{normalization}} \tfrac12(1,-1,\mathrm{i},\mathrm{i})
\end{aligned}
$$

The complex configuration with 240 vectors corresponds to 60 different states.

# Complexification of $E_8$ roots

## Examples

$$(\pm1,\pm1,0,0,0,0,0,0) \xrightarrow{\text{complexif.}} (\pm1\pm\mathrm{i},0,0,0) \xrightarrow{\text{normalization}} (1,0,0,0)$$
$$(\pm1,0,\pm1,0,0,0,0,0) \xrightarrow{\text{complexif.}} (\pm1,\pm1,0,0) \xrightarrow{\text{normalization}} \frac{1}{\sqrt{2}}(1,\pm1,0,0)$$
$$(\pm1,0,0,\pm1,0,0,0,0) \xrightarrow{\text{complexif.}} (\pm1,\pm\mathrm{i},0,0) \xrightarrow{\text{normalization}} \frac{1}{\sqrt{2}}(1,\pm\mathrm{i},0,0)$$
$$\tfrac{1}{2}(1,1,1,1,1,1,1,1) \xrightarrow{\text{complexif.}} \tfrac{1}{2}(1{+}\mathrm{i},1{+}\mathrm{i},1{+}\mathrm{i},1{+}\mathrm{i}) \xrightarrow{\text{normalization}} \tfrac{1}{2}(1,1,1,1)$$
$$\tfrac{1}{2}(1,-1,1,-1,1,1,1,1) \xrightarrow{\text{complexif.}} \tfrac{1}{2}(1{-}\mathrm{i},1{-}\mathrm{i},1{+}\mathrm{i},1{+}\mathrm{i}) \xrightarrow{\text{normalization}} \tfrac{1}{2}(1,1,\mathrm{i},\mathrm{i})$$
$$\tfrac{1}{2}(1,-1,-1,1,1,1,1,1) \xrightarrow{\text{complexif.}} \tfrac{1}{2}(1{-}\mathrm{i},-1{+}\mathrm{i},1{+}\mathrm{i},1{+}\mathrm{i}) \xrightarrow{\text{normalization}} \tfrac{1}{2}(1,-1,\mathrm{i},\mathrm{i})$$

The complex configuration with 240 vectors corresponds to 60 different states. They coincide with 60 stabilizer states for two qubits.

*M. Ohta and K. Sakurai*, Extremal magic states from symmetric lattices, `arXiv:2506.11725` (2025).

# Table with 60 states

## 60 states (up to normalization)

| k | $\lvert\varsigma_0^k\rangle$ | $\lvert\varsigma_1^k\rangle$ | $\lvert\varsigma_2^k\rangle$ | $\lvert\varsigma_3^k\rangle$ |
|---|---|---|---|---|
| 1 | $(1, 0, 0, 0)$ | $(0, 1, 0, 0)$ | $(0, 0, 1, 0)$ | $(0, 0, 0, 1)$ |
| 2 | $(1, 1, 0, 0)$ | $(1, -1, 0, 0)$ | $(0, 0, 1, 1)$ | $(0, 0, 1, -1)$ |
| 3 | $(1, -i, 0, 0)$ | $(1, i, 0, 0)$ | $(0, 0, 1, -i)$ | $(0, 0, 1, i)$ |
| 4 | $(1, 0, 1, 0)$ | $(1, 0, -1, 0)$ | $(0, 1, 0, 1)$ | $(0, 1, 0, -1)$ |
| 5 | $(1, 0, -i, 0)$ | $(1, 0, i, 0)$ | $(0, 1, 0, -i)$ | $(0, 1, 0, i)$ |
| 6 | $(0, 1, 1, 0)$ | $(0, 1, -1, 0)$ | $(1, 0, 0, 1)$ | $(1, 0, 0, -1)$ |
| 7 | $(0, 1, -i, 0)$ | $(0, 1, i, 0)$ | $(1, 0, 0, -i)$ | $(1, 0, 0, i)$ |
| 8 | $(1, 1, 1, 1)$ | $(1, 1, -1, -1)$ | $(1, -1, 1, -1)$ | $(1, -1, -1, 1)$ |
| 9 | $(1, 1, 1, -1)$ | $(1, 1, -1, 1)$ | $(1, -1, 1, 1)$ | $(1, -1, -1, -1)$ |
| 10 | $(1, 1, -i, -i)$ | $(1, 1, i, i)$ | $(1, -1, -i, i)$ | $(1, -1, i, -i)$ |
| 11 | $(1, 1, -i, i)$ | $(1, 1, i, -i)$ | $(1, -1, -i, -i)$ | $(1, -1, i, i)$ |
| 12 | $(1, -i, 1, -i)$ | $(1, -i, -1, i)$ | $(1, i, 1, i)$ | $(1, i, -1, -i)$ |
| 13 | $(1, -i, 1, i)$ | $(1, -i, -1, -i)$ | $(1, i, 1, -i)$ | $(1, i, -1, i)$ |
| 14 | $(1, -i, -i, 1)$ | $(1, -i, i, -1)$ | $(1, i, -i, -1)$ | $(1, i, i, 1)$ |
| 15 | $(1, -i, -i, -1)$ | $(1, -i, i, 1)$ | $(1, i, -i, 1)$ | $(1, i, i, -1)$ |

# Scheme with stabilizers

## Stabilizers for 60 states (naturally arranged into 15 bases)

| k | $\lvert\varsigma_0^{\mathsf{k}}\rangle$ | $\lvert\varsigma_1^{\mathsf{k}}\rangle$ | $\lvert\varsigma_2^{\mathsf{k}}\rangle$ | $\lvert\varsigma_3^{\mathsf{k}}\rangle$ |
|---|---|---|---|---|
| 1 | $I{\otimes}Z,\ \ Z{\otimes}I$ | $-I{\otimes}Z,\ \ Z{\otimes}I$ | $I{\otimes}Z, -Z{\otimes}I$ | $-I{\otimes}Z, -Z{\otimes}I$ |
| 2 | $I{\otimes}X,\ \ Z{\otimes}I$ | $-I{\otimes}X,\ \ Z{\otimes}I$ | $I{\otimes}X, -Z{\otimes}I$ | $-I{\otimes}X, -Z{\otimes}I$ |
| 3 | $-I{\otimes}Y,\ \ Z{\otimes}I$ | $I{\otimes}Y,\ \ Z{\otimes}I$ | $-I{\otimes}Y, -Z{\otimes}I$ | $I{\otimes}Y, -Z{\otimes}I$ |
| 4 | $I{\otimes}Z,\ \ X{\otimes}I$ | $-I{\otimes}Z,\ \ X{\otimes}I$ | $I{\otimes}Z, -X{\otimes}I$ | $-I{\otimes}Z, -X{\otimes}I$ |
| 5 | $I{\otimes}Z, -Y{\otimes}I$ | $I{\otimes}Z,\ \ Y{\otimes}I$ | $-I{\otimes}Z, -Y{\otimes}I$ | $-I{\otimes}Z,\ \ Y{\otimes}I$ |
| 6 | $X{\otimes}X,\ \ Y{\otimes}Y$ | $-X{\otimes}X, -Y{\otimes}Y$ | $X{\otimes}X, -Y{\otimes}Y$ | $-X{\otimes}X,\ \ Y{\otimes}Y$ |
| 7 | $X{\otimes}Y, -Y{\otimes}X$ | $-X{\otimes}Y,\ \ Y{\otimes}X$ | $-X{\otimes}Y, -Y{\otimes}X$ | $X{\otimes}Y,\ \ Y{\otimes}X$ |
| 8 | $I{\otimes}X,\ \ X{\otimes}I$ | $I{\otimes}X, -X{\otimes}I$ | $-I{\otimes}X,\ \ X{\otimes}I$ | $-I{\otimes}X, -X{\otimes}I$ |
| 9 | $X{\otimes}Z,\ \ Z{\otimes}X$ | $-X{\otimes}Z,\ \ Z{\otimes}X$ | $X{\otimes}Z, -Z{\otimes}X$ | $-X{\otimes}Z, -Z{\otimes}X$ |
| 10 | $I{\otimes}X, -Y{\otimes}I$ | $I{\otimes}X,\ \ Y{\otimes}I$ | $-I{\otimes}X, -Y{\otimes}I$ | $-I{\otimes}X,\ \ Y{\otimes}I$ |
| 11 | $X{\otimes}Y, -Y{\otimes}Z$ | $-X{\otimes}Y,\ \ Y{\otimes}Z$ | $-X{\otimes}Y, -Y{\otimes}Z$ | $X{\otimes}Y,\ \ Y{\otimes}Z$ |
| 12 | $-I{\otimes}Y,\ \ X{\otimes}I$ | $-I{\otimes}Y, -X{\otimes}I$ | $I{\otimes}Y,\ \ X{\otimes}I$ | $I{\otimes}Y, -X{\otimes}I$ |
| 13 | $X{\otimes}Z,\ \ Y{\otimes}X$ | $-X{\otimes}Z, -Y{\otimes}X$ | $X{\otimes}Z, -Y{\otimes}X$ | $-X{\otimes}Z,\ \ Y{\otimes}X$ |
| 14 | $-Y{\otimes}Z, -Z{\otimes}Y$ | $Y{\otimes}Z, -Z{\otimes}Y$ | $-Y{\otimes}Z,\ \ Z{\otimes}Y$ | $Y{\otimes}Z,\ \ Z{\otimes}Y$ |
| 15 | $-I{\otimes}Y, -Y{\otimes}I$ | $-I{\otimes}Y,\ \ Y{\otimes}I$ | $I{\otimes}Y, -Y{\otimes}I$ | $I{\otimes}Y,\ \ Y{\otimes}I$ |

For each state $\lvert\varsigma_j^{\mathsf{k}}\rangle$ the table represents two commuting operators (stabilizers) with property

$$S\lvert\varsigma_j^{\mathsf{k}}\rangle = \lvert\varsigma_j^{\mathsf{k}}\rangle$$

# Bases for 60 states

There are 105 bases for 60 states, and they can be separated into two sets. In the first set, there are 15 bases $\left\{|\varsigma_0^k\rangle, |\varsigma_1^k\rangle, |\varsigma_2^k\rangle, |\varsigma_3^k\rangle\right\}$ $k = 1, \dots, 15$.

## First set with 15 bases for 60 states

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\left\{|\varsigma_0^1\rangle, |\varsigma_1^1\rangle, |\varsigma_2^1\rangle, |\varsigma_3^1\rangle\right\}$ | 2 | $\left\{|\varsigma_0^2\rangle, |\varsigma_1^2\rangle, |\varsigma_2^2\rangle, |\varsigma_3^2\rangle\right\}$ | 3 | $\left\{|\varsigma_0^3\rangle, |\varsigma_1^3\rangle, |\varsigma_2^3\rangle, |\varsigma_3^3\rangle\right\}$ |
| 4 | $\left\{|\varsigma_0^4\rangle, |\varsigma_1^4\rangle, |\varsigma_2^4\rangle, |\varsigma_3^4\rangle\right\}$ | 5 | $\left\{|\varsigma_0^5\rangle, |\varsigma_1^5\rangle, |\varsigma_2^5\rangle, |\varsigma_3^5\rangle\right\}$ | 6 | $\left\{|\varsigma_0^6\rangle, |\varsigma_1^6\rangle, |\varsigma_2^6\rangle, |\varsigma_3^6\rangle\right\}$ |
| 7 | $\left\{|\varsigma_0^7\rangle, |\varsigma_1^7\rangle, |\varsigma_2^7\rangle, |\varsigma_3^7\rangle\right\}$ | 8 | $\left\{|\varsigma_0^8\rangle, |\varsigma_1^8\rangle, |\varsigma_2^8\rangle, |\varsigma_3^8\rangle\right\}$ | 9 | $\left\{|\varsigma_0^9\rangle, |\varsigma_1^9\rangle, |\varsigma_2^9\rangle, |\varsigma_3^9\rangle\right\}$ |
| 10 | $\left\{|\varsigma_0^{10}\rangle, |\varsigma_1^{10}\rangle, |\varsigma_2^{10}\rangle, |\varsigma_3^{10}\rangle\right\}$ | 11 | $\left\{|\varsigma_0^{11}\rangle, |\varsigma_1^{11}\rangle, |\varsigma_2^{11}\rangle, |\varsigma_3^{11}\rangle\right\}$ | 12 | $\left\{|\varsigma_0^{12}\rangle, |\varsigma_1^{12}\rangle, |\varsigma_2^{12}\rangle, |\varsigma_3^{12}\rangle\right\}$ |
| 13 | $\left\{|\varsigma_0^{13}\rangle, |\varsigma_1^{13}\rangle, |\varsigma_2^{13}\rangle, |\varsigma_3^{13}\rangle\right\}$ | 14 | $\left\{|\varsigma_0^{14}\rangle, |\varsigma_1^{14}\rangle, |\varsigma_2^{14}\rangle, |\varsigma_3^{14}\rangle\right\}$ | 15 | $\left\{|\varsigma_0^{15}\rangle, |\varsigma_1^{15}\rangle, |\varsigma_2^{15}\rangle, |\varsigma_3^{15}\rangle\right\}$ |

# Bases for 60 states

There are 105 bases for 60 states, and they can be separated into two sets. In the first set, there are 15 bases $\left\{|\varsigma_0^k\rangle, |\varsigma_1^k\rangle, |\varsigma_2^k\rangle, |\varsigma_3^k\rangle\right\}$ $k = 1, \ldots, 15$.

**First set with 15 bases for 60 states**

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\left\{|\varsigma_0^1\rangle, |\varsigma_1^1\rangle, |\varsigma_2^1\rangle, |\varsigma_3^1\rangle\right\}$ | 2 | $\left\{|\varsigma_0^2\rangle, |\varsigma_1^2\rangle, |\varsigma_2^2\rangle, |\varsigma_3^2\rangle\right\}$ | 3 | $\left\{|\varsigma_0^3\rangle, |\varsigma_1^3\rangle, |\varsigma_2^3\rangle, |\varsigma_3^3\rangle\right\}$ |
| 4 | $\left\{|\varsigma_0^4\rangle, |\varsigma_1^4\rangle, |\varsigma_2^4\rangle, |\varsigma_3^4\rangle\right\}$ | 5 | $\left\{|\varsigma_0^5\rangle, |\varsigma_1^5\rangle, |\varsigma_2^5\rangle, |\varsigma_3^5\rangle\right\}$ | 6 | $\left\{|\varsigma_0^6\rangle, |\varsigma_1^6\rangle, |\varsigma_2^6\rangle, |\varsigma_3^6\rangle\right\}$ |
| 7 | $\left\{|\varsigma_0^7\rangle, |\varsigma_1^7\rangle, |\varsigma_2^7\rangle, |\varsigma_3^7\rangle\right\}$ | 8 | $\left\{|\varsigma_0^8\rangle, |\varsigma_1^8\rangle, |\varsigma_2^8\rangle, |\varsigma_3^8\rangle\right\}$ | 9 | $\left\{|\varsigma_0^9\rangle, |\varsigma_1^9\rangle, |\varsigma_2^9\rangle, |\varsigma_3^9\rangle\right\}$ |
| 10 | $\left\{|\varsigma_0^{10}\rangle, |\varsigma_1^{10}\rangle, |\varsigma_2^{10}\rangle, |\varsigma_3^{10}\rangle\right\}$ | 11 | $\left\{|\varsigma_0^{11}\rangle, |\varsigma_1^{11}\rangle, |\varsigma_2^{11}\rangle, |\varsigma_3^{11}\rangle\right\}$ | 12 | $\left\{|\varsigma_0^{12}\rangle, |\varsigma_1^{12}\rangle, |\varsigma_2^{12}\rangle, |\varsigma_3^{12}\rangle\right\}$ |
| 13 | $\left\{|\varsigma_0^{13}\rangle, |\varsigma_1^{13}\rangle, |\varsigma_2^{13}\rangle, |\varsigma_3^{13}\rangle\right\}$ | 14 | $\left\{|\varsigma_0^{14}\rangle, |\varsigma_1^{14}\rangle, |\varsigma_2^{14}\rangle, |\varsigma_3^{14}\rangle\right\}$ | 15 | $\left\{|\varsigma_0^{15}\rangle, |\varsigma_1^{15}\rangle, |\varsigma_2^{15}\rangle, |\varsigma_3^{15}\rangle\right\}$ |

The second set includes 90 bases.

# Yet another 90 bases for 60 states

## Second set with 90 bases for 60 states

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\{\lvert\varsigma_{0}^{1}\rangle,\lvert\varsigma_{1}^{1}\rangle,\lvert\varsigma_{2}^{2}\rangle,\lvert\varsigma_{3}^{2}\rangle\}$ | 2 | $\{\lvert\varsigma_{0}^{1}\rangle,\lvert\varsigma_{1}^{1}\rangle,\lvert\varsigma_{2}^{3}\rangle,\lvert\varsigma_{3}^{3}\rangle\}$ | 3 | $\{\lvert\varsigma_{0}^{1}\rangle,\lvert\varsigma_{2}^{1}\rangle,\lvert\varsigma_{2}^{4}\rangle,\lvert\varsigma_{3}^{4}\rangle\}$ |
| 4 | $\{\lvert\varsigma_{0}^{1}\rangle,\lvert\varsigma_{2}^{1}\rangle,\lvert\varsigma_{2}^{5}\rangle,\lvert\varsigma_{3}^{5}\rangle\}$ | 5 | $\{\lvert\varsigma_{0}^{1}\rangle,\lvert\varsigma_{3}^{1}\rangle,\lvert\varsigma_{0}^{6}\rangle,\lvert\varsigma_{1}^{6}\rangle\}$ | 6 | $\{\lvert\varsigma_{0}^{1}\rangle,\lvert\varsigma_{3}^{1}\rangle,\lvert\varsigma_{0}^{7}\rangle,\lvert\varsigma_{1}^{7}\rangle\}$ |
| 7 | $\{\lvert\varsigma_{1}^{1}\rangle,\lvert\varsigma_{2}^{1}\rangle,\lvert\varsigma_{2}^{6}\rangle,\lvert\varsigma_{3}^{6}\rangle\}$ | 8 | $\{\lvert\varsigma_{1}^{1}\rangle,\lvert\varsigma_{2}^{1}\rangle,\lvert\varsigma_{2}^{7}\rangle,\lvert\varsigma_{3}^{7}\rangle\}$ | 9 | $\{\lvert\varsigma_{1}^{1}\rangle,\lvert\varsigma_{3}^{1}\rangle,\lvert\varsigma_{0}^{4}\rangle,\lvert\varsigma_{1}^{4}\rangle\}$ |
| 10 | $\{\lvert\varsigma_{1}^{1}\rangle,\lvert\varsigma_{3}^{1}\rangle,\lvert\varsigma_{0}^{5}\rangle,\lvert\varsigma_{1}^{5}\rangle\}$ | 11 | $\{\lvert\varsigma_{2}^{1}\rangle,\lvert\varsigma_{3}^{1}\rangle,\lvert\varsigma_{0}^{2}\rangle,\lvert\varsigma_{1}^{2}\rangle\}$ | 12 | $\{\lvert\varsigma_{2}^{1}\rangle,\lvert\varsigma_{3}^{1}\rangle,\lvert\varsigma_{0}^{3}\rangle,\lvert\varsigma_{1}^{3}\rangle\}$ |
| 13 | $\{\lvert\varsigma_{0}^{2}\rangle,\lvert\varsigma_{1}^{2}\rangle,\lvert\varsigma_{2}^{3}\rangle,\lvert\varsigma_{3}^{3}\rangle\}$ | 14 | $\{\lvert\varsigma_{0}^{2}\rangle,\lvert\varsigma_{2}^{2}\rangle,\lvert\varsigma_{2}^{8}\rangle,\lvert\varsigma_{3}^{8}\rangle\}$ | 15 | $\{\lvert\varsigma_{0}^{2}\rangle,\lvert\varsigma_{2}^{2}\rangle,\lvert\varsigma_{2}^{10}\rangle,\lvert\varsigma_{3}^{10}\rangle\}$ |
| 16 | $\{\lvert\varsigma_{0}^{2}\rangle,\lvert\varsigma_{3}^{2}\rangle,\lvert\varsigma_{2}^{9}\rangle,\lvert\varsigma_{3}^{9}\rangle\}$ | 17 | $\{\lvert\varsigma_{0}^{2}\rangle,\lvert\varsigma_{3}^{2}\rangle,\lvert\varsigma_{2}^{11}\rangle,\lvert\varsigma_{3}^{11}\rangle\}$ | 18 | $\{\lvert\varsigma_{1}^{2}\rangle,\lvert\varsigma_{2}^{2}\rangle,\lvert\varsigma_{0}^{9}\rangle,\lvert\varsigma_{1}^{9}\rangle\}$ |
| 19 | $\{\lvert\varsigma_{1}^{2}\rangle,\lvert\varsigma_{2}^{2}\rangle,\lvert\varsigma_{0}^{11}\rangle,\lvert\varsigma_{1}^{11}\rangle\}$ | 20 | $\{\lvert\varsigma_{1}^{2}\rangle,\lvert\varsigma_{3}^{2}\rangle,\lvert\varsigma_{0}^{8}\rangle,\lvert\varsigma_{1}^{8}\rangle\}$ | 21 | $\{\lvert\varsigma_{1}^{2}\rangle,\lvert\varsigma_{3}^{2}\rangle,\lvert\varsigma_{0}^{10}\rangle,\lvert\varsigma_{1}^{10}\rangle\}$ |
| 22 | $\{\lvert\varsigma_{2}^{2}\rangle,\lvert\varsigma_{3}^{2}\rangle,\lvert\varsigma_{0}^{3}\rangle,\lvert\varsigma_{1}^{3}\rangle\}$ | 23 | $\{\lvert\varsigma_{0}^{3}\rangle,\lvert\varsigma_{2}^{3}\rangle,\lvert\varsigma_{2}^{12}\rangle,\lvert\varsigma_{3}^{12}\rangle\}$ | 24 | $\{\lvert\varsigma_{0}^{3}\rangle,\lvert\varsigma_{2}^{3}\rangle,\lvert\varsigma_{2}^{15}\rangle,\lvert\varsigma_{3}^{15}\rangle\}$ |
| 25 | $\{\lvert\varsigma_{0}^{3}\rangle,\lvert\varsigma_{3}^{3}\rangle,\lvert\varsigma_{2}^{13}\rangle,\lvert\varsigma_{3}^{13}\rangle\}$ | 26 | $\{\lvert\varsigma_{0}^{3}\rangle,\lvert\varsigma_{3}^{3}\rangle,\lvert\varsigma_{2}^{14}\rangle,\lvert\varsigma_{3}^{14}\rangle\}$ | 27 | $\{\lvert\varsigma_{1}^{3}\rangle,\lvert\varsigma_{2}^{3}\rangle,\lvert\varsigma_{0}^{13}\rangle,\lvert\varsigma_{1}^{13}\rangle\}$ |
| 28 | $\{\lvert\varsigma_{1}^{3}\rangle,\lvert\varsigma_{2}^{3}\rangle,\lvert\varsigma_{0}^{14}\rangle,\lvert\varsigma_{1}^{14}\rangle\}$ | 29 | $\{\lvert\varsigma_{1}^{3}\rangle,\lvert\varsigma_{3}^{3}\rangle,\lvert\varsigma_{0}^{12}\rangle,\lvert\varsigma_{1}^{12}\rangle\}$ | 30 | $\{\lvert\varsigma_{1}^{3}\rangle,\lvert\varsigma_{3}^{3}\rangle,\lvert\varsigma_{0}^{15}\rangle,\lvert\varsigma_{1}^{15}\rangle\}$ |
| 31 | $\{\lvert\varsigma_{0}^{4}\rangle,\lvert\varsigma_{1}^{4}\rangle,\lvert\varsigma_{2}^{5}\rangle,\lvert\varsigma_{3}^{5}\rangle\}$ | 32 | $\{\lvert\varsigma_{0}^{4}\rangle,\lvert\varsigma_{2}^{4}\rangle,\lvert\varsigma_{1}^{8}\rangle,\lvert\varsigma_{3}^{8}\rangle\}$ | 33 | $\{\lvert\varsigma_{0}^{4}\rangle,\lvert\varsigma_{2}^{4}\rangle,\lvert\varsigma_{1}^{12}\rangle,\lvert\varsigma_{3}^{12}\rangle\}$ |
| 34 | $\{\lvert\varsigma_{0}^{4}\rangle,\lvert\varsigma_{3}^{4}\rangle,\lvert\varsigma_{1}^{9}\rangle,\lvert\varsigma_{3}^{9}\rangle\}$ | 35 | $\{\lvert\varsigma_{0}^{4}\rangle,\lvert\varsigma_{3}^{4}\rangle,\lvert\varsigma_{1}^{13}\rangle,\lvert\varsigma_{3}^{13}\rangle\}$ | 36 | $\{\lvert\varsigma_{1}^{4}\rangle,\lvert\varsigma_{2}^{4}\rangle,\lvert\varsigma_{0}^{9}\rangle,\lvert\varsigma_{2}^{9}\rangle\}$ |
| 37 | $\{\lvert\varsigma_{1}^{4}\rangle,\lvert\varsigma_{2}^{4}\rangle,\lvert\varsigma_{0}^{13}\rangle,\lvert\varsigma_{2}^{13}\rangle\}$ | 38 | $\{\lvert\varsigma_{1}^{4}\rangle,\lvert\varsigma_{3}^{4}\rangle,\lvert\varsigma_{0}^{8}\rangle,\lvert\varsigma_{2}^{8}\rangle\}$ | 39 | $\{\lvert\varsigma_{1}^{4}\rangle,\lvert\varsigma_{3}^{4}\rangle,\lvert\varsigma_{0}^{12}\rangle,\lvert\varsigma_{2}^{12}\rangle\}$ |
| 40 | $\{\lvert\varsigma_{2}^{4}\rangle,\lvert\varsigma_{3}^{4}\rangle,\lvert\varsigma_{0}^{5}\rangle,\lvert\varsigma_{1}^{5}\rangle\}$ | 41 | $\{\lvert\varsigma_{0}^{5}\rangle,\lvert\varsigma_{2}^{5}\rangle,\lvert\varsigma_{1}^{10}\rangle,\lvert\varsigma_{3}^{10}\rangle\}$ | 42 | $\{\lvert\varsigma_{0}^{5}\rangle,\lvert\varsigma_{2}^{5}\rangle,\lvert\varsigma_{1}^{15}\rangle,\lvert\varsigma_{3}^{15}\rangle\}$ |
| 43 | $\{\lvert\varsigma_{0}^{5}\rangle,\lvert\varsigma_{3}^{5}\rangle,\lvert\varsigma_{1}^{11}\rangle,\lvert\varsigma_{3}^{11}\rangle\}$ | 44 | $\{\lvert\varsigma_{0}^{5}\rangle,\lvert\varsigma_{3}^{5}\rangle,\lvert\varsigma_{1}^{14}\rangle,\lvert\varsigma_{3}^{14}\rangle\}$ | 45 | $\{\lvert\varsigma_{1}^{5}\rangle,\lvert\varsigma_{2}^{5}\rangle,\lvert\varsigma_{0}^{11}\rangle,\lvert\varsigma_{2}^{11}\rangle\}$ |

| | | | | | |
|---|---|---|---|---|---|
| 46 | $\{\lvert\varsigma_{1}^{5}\rangle,\lvert\varsigma_{2}^{5}\rangle,\lvert\varsigma_{0}^{14}\rangle,\lvert\varsigma_{2}^{14}\rangle\}$ | 47 | $\{\lvert\varsigma_{1}^{5}\rangle,\lvert\varsigma_{3}^{5}\rangle,\lvert\varsigma_{0}^{10}\rangle,\lvert\varsigma_{2}^{10}\rangle\}$ | 48 | $\{\lvert\varsigma_{1}^{5}\rangle,\lvert\varsigma_{3}^{5}\rangle,\lvert\varsigma_{0}^{15}\rangle,\lvert\varsigma_{2}^{15}\rangle\}$ |
| 49 | $\{\lvert\varsigma_{0}^{6}\rangle,\lvert\varsigma_{1}^{6}\rangle,\lvert\varsigma_{2}^{7}\rangle,\lvert\varsigma_{3}^{7}\rangle\}$ | 50 | $\{\lvert\varsigma_{0}^{6}\rangle,\lvert\varsigma_{2}^{6}\rangle,\lvert\varsigma_{1}^{8}\rangle,\lvert\varsigma_{2}^{8}\rangle\}$ | 51 | $\{\lvert\varsigma_{0}^{6}\rangle,\lvert\varsigma_{2}^{6}\rangle,\lvert\varsigma_{1}^{14}\rangle,\lvert\varsigma_{2}^{14}\rangle\}$ |
| 52 | $\{\lvert\varsigma_{0}^{6}\rangle,\lvert\varsigma_{3}^{6}\rangle,\lvert\varsigma_{1}^{9}\rangle,\lvert\varsigma_{2}^{9}\rangle\}$ | 53 | $\{\lvert\varsigma_{0}^{6}\rangle,\lvert\varsigma_{3}^{6}\rangle,\lvert\varsigma_{1}^{15}\rangle,\lvert\varsigma_{2}^{15}\rangle\}$ | 54 | $\{\lvert\varsigma_{1}^{6}\rangle,\lvert\varsigma_{2}^{6}\rangle,\lvert\varsigma_{0}^{9}\rangle,\lvert\varsigma_{3}^{9}\rangle\}$ |
| 55 | $\{\lvert\varsigma_{1}^{6}\rangle,\lvert\varsigma_{2}^{6}\rangle,\lvert\varsigma_{0}^{15}\rangle,\lvert\varsigma_{3}^{15}\rangle\}$ | 56 | $\{\lvert\varsigma_{1}^{6}\rangle,\lvert\varsigma_{3}^{6}\rangle,\lvert\varsigma_{0}^{8}\rangle,\lvert\varsigma_{3}^{8}\rangle\}$ | 57 | $\{\lvert\varsigma_{1}^{6}\rangle,\lvert\varsigma_{3}^{6}\rangle,\lvert\varsigma_{0}^{14}\rangle,\lvert\varsigma_{3}^{14}\rangle\}$ |
| 58 | $\{\lvert\varsigma_{2}^{6}\rangle,\lvert\varsigma_{3}^{6}\rangle,\lvert\varsigma_{0}^{7}\rangle,\lvert\varsigma_{1}^{7}\rangle\}$ | 59 | $\{\lvert\varsigma_{0}^{7}\rangle,\lvert\varsigma_{2}^{7}\rangle,\lvert\varsigma_{1}^{10}\rangle,\lvert\varsigma_{2}^{10}\rangle\}$ | 60 | $\{\lvert\varsigma_{0}^{7}\rangle,\lvert\varsigma_{2}^{7}\rangle,\lvert\varsigma_{0}^{13}\rangle,\lvert\varsigma_{3}^{13}\rangle\}$ |
| 61 | $\{\lvert\varsigma_{0}^{7}\rangle,\lvert\varsigma_{3}^{7}\rangle,\lvert\varsigma_{1}^{11}\rangle,\lvert\varsigma_{2}^{11}\rangle\}$ | 62 | $\{\lvert\varsigma_{0}^{7}\rangle,\lvert\varsigma_{3}^{7}\rangle,\lvert\varsigma_{0}^{12}\rangle,\lvert\varsigma_{3}^{12}\rangle\}$ | 63 | $\{\lvert\varsigma_{1}^{7}\rangle,\lvert\varsigma_{2}^{7}\rangle,\lvert\varsigma_{0}^{11}\rangle,\lvert\varsigma_{3}^{11}\rangle\}$ |
| 64 | $\{\lvert\varsigma_{1}^{7}\rangle,\lvert\varsigma_{2}^{7}\rangle,\lvert\varsigma_{1}^{12}\rangle,\lvert\varsigma_{2}^{12}\rangle\}$ | 65 | $\{\lvert\varsigma_{1}^{7}\rangle,\lvert\varsigma_{3}^{7}\rangle,\lvert\varsigma_{0}^{10}\rangle,\lvert\varsigma_{3}^{10}\rangle\}$ | 66 | $\{\lvert\varsigma_{1}^{7}\rangle,\lvert\varsigma_{3}^{7}\rangle,\lvert\varsigma_{1}^{13}\rangle,\lvert\varsigma_{2}^{13}\rangle\}$ |
| 67 | $\{\lvert\varsigma_{0}^{8}\rangle,\lvert\varsigma_{1}^{8}\rangle,\lvert\varsigma_{2}^{10}\rangle,\lvert\varsigma_{3}^{10}\rangle\}$ | 68 | $\{\lvert\varsigma_{0}^{8}\rangle,\lvert\varsigma_{2}^{8}\rangle,\lvert\varsigma_{1}^{12}\rangle,\lvert\varsigma_{3}^{12}\rangle\}$ | 69 | $\{\lvert\varsigma_{0}^{8}\rangle,\lvert\varsigma_{3}^{8}\rangle,\lvert\varsigma_{1}^{14}\rangle,\lvert\varsigma_{2}^{14}\rangle\}$ |
| 70 | $\{\lvert\varsigma_{1}^{8}\rangle,\lvert\varsigma_{2}^{8}\rangle,\lvert\varsigma_{0}^{14}\rangle,\lvert\varsigma_{3}^{14}\rangle\}$ | 71 | $\{\lvert\varsigma_{1}^{8}\rangle,\lvert\varsigma_{3}^{8}\rangle,\lvert\varsigma_{0}^{12}\rangle,\lvert\varsigma_{2}^{12}\rangle\}$ | 72 | $\{\lvert\varsigma_{2}^{8}\rangle,\lvert\varsigma_{3}^{8}\rangle,\lvert\varsigma_{0}^{10}\rangle,\lvert\varsigma_{1}^{10}\rangle\}$ |
| 73 | $\{\lvert\varsigma_{0}^{9}\rangle,\lvert\varsigma_{1}^{9}\rangle,\lvert\varsigma_{2}^{11}\rangle,\lvert\varsigma_{3}^{11}\rangle\}$ | 74 | $\{\lvert\varsigma_{0}^{9}\rangle,\lvert\varsigma_{2}^{9}\rangle,\lvert\varsigma_{1}^{13}\rangle,\lvert\varsigma_{3}^{13}\rangle\}$ | 75 | $\{\lvert\varsigma_{0}^{9}\rangle,\lvert\varsigma_{3}^{9}\rangle,\lvert\varsigma_{1}^{15}\rangle,\lvert\varsigma_{2}^{15}\rangle\}$ |
| 76 | $\{\lvert\varsigma_{1}^{9}\rangle,\lvert\varsigma_{2}^{9}\rangle,\lvert\varsigma_{0}^{15}\rangle,\lvert\varsigma_{3}^{15}\rangle\}$ | 77 | $\{\lvert\varsigma_{1}^{9}\rangle,\lvert\varsigma_{3}^{9}\rangle,\lvert\varsigma_{0}^{13}\rangle,\lvert\varsigma_{2}^{13}\rangle\}$ | 78 | $\{\lvert\varsigma_{2}^{9}\rangle,\lvert\varsigma_{3}^{9}\rangle,\lvert\varsigma_{0}^{11}\rangle,\lvert\varsigma_{1}^{11}\rangle\}$ |
| 79 | $\{\lvert\varsigma_{0}^{10}\rangle,\lvert\varsigma_{2}^{10}\rangle,\lvert\varsigma_{1}^{15}\rangle,\lvert\varsigma_{3}^{15}\rangle\}$ | 80 | $\{\lvert\varsigma_{0}^{10}\rangle,\lvert\varsigma_{3}^{10}\rangle,\lvert\varsigma_{0}^{13}\rangle,\lvert\varsigma_{3}^{13}\rangle\}$ | 81 | $\{\lvert\varsigma_{1}^{10}\rangle,\lvert\varsigma_{2}^{10}\rangle,\lvert\varsigma_{1}^{13}\rangle,\lvert\varsigma_{2}^{13}\rangle\}$ |
| 82 | $\{\lvert\varsigma_{1}^{10}\rangle,\lvert\varsigma_{3}^{10}\rangle,\lvert\varsigma_{0}^{15}\rangle,\lvert\varsigma_{2}^{15}\rangle\}$ | 83 | $\{\lvert\varsigma_{0}^{11}\rangle,\lvert\varsigma_{2}^{11}\rangle,\lvert\varsigma_{1}^{14}\rangle,\lvert\varsigma_{3}^{14}\rangle\}$ | 84 | $\{\lvert\varsigma_{0}^{11}\rangle,\lvert\varsigma_{3}^{11}\rangle,\lvert\varsigma_{0}^{12}\rangle,\lvert\varsigma_{3}^{12}\rangle\}$ |
| 85 | $\{\lvert\varsigma_{1}^{11}\rangle,\lvert\varsigma_{2}^{11}\rangle,\lvert\varsigma_{1}^{12}\rangle,\lvert\varsigma_{2}^{12}\rangle\}$ | 86 | $\{\lvert\varsigma_{1}^{11}\rangle,\lvert\varsigma_{3}^{11}\rangle,\lvert\varsigma_{0}^{14}\rangle,\lvert\varsigma_{2}^{14}\rangle\}$ | 87 | $\{\lvert\varsigma_{0}^{12}\rangle,\lvert\varsigma_{1}^{12}\rangle,\lvert\varsigma_{2}^{15}\rangle,\lvert\varsigma_{3}^{15}\rangle\}$ |
| 88 | $\{\lvert\varsigma_{2}^{12}\rangle,\lvert\varsigma_{3}^{12}\rangle,\lvert\varsigma_{0}^{15}\rangle,\lvert\varsigma_{1}^{15}\rangle\}$ | 89 | $\{\lvert\varsigma_{0}^{13}\rangle,\lvert\varsigma_{1}^{13}\rangle,\lvert\varsigma_{2}^{14}\rangle,\lvert\varsigma_{3}^{14}\rangle\}$ | 90 | $\{\lvert\varsigma_{2}^{13}\rangle,\lvert\varsigma_{3}^{13}\rangle,\lvert\varsigma_{0}^{14}\rangle,\lvert\varsigma_{1}^{14}\rangle\}$ |

# Structure of 105 bases for 60 states

Some properties of 60 stabilizer states and 105 bases mentioned here already were considered in relation with contextuality in quantum mechanics and have some analogs with configurations discussed earlier.

*M. Waegell and P. K. Aravind*, Parity proofs of the Kochen-Specker theorem based on 60 complex rays in four dimensions, `arXiv:1109.1299`, J. Phys. A **44**, 505303 (2011).

# Structure of 105 bases for 60 states

Some properties of 60 stabilizer states and 105 bases mentioned here already were considered in relation with contextuality in quantum mechanics and have some analogs with configurations discussed earlier.

*M. Waegell and P. K. Aravind*, Parity proofs of the Kochen-Specker theorem based on 60 complex rays in four dimensions, `arXiv:1109.1299`, J. Phys. A **44**, 505303 (2011).

**Problem with definition of contextual closure group**

Unlike previous configurations, 105 bases are not homogeneous with respect to intersections: only bases from the second set can have single state in common, but non-overlapping bases from the first set always have two common states with some bases from the second set. Due to such property, a group that acts transitively on all 105 bases cannot exist.

# Stabilizer circuits

## Pauli group

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

## Clifford gates and stabilizer group ($\mathcal{G}$)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad H = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathsf{C}_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

For two qubits Clifford gates generate group $\mathcal{G}$ with 92160 elements corresponding to 11520 different quantum gates (up to phase multiplier).

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## Rational subgroup of stabilizer group ($\acute{\mathcal{G}}$)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \acute{H} = \mathrm{e}^{\pi \mathrm{i}/4} H = \frac{1+\mathrm{i}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathsf{C}_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Rational subgroup $\acute{\mathcal{G}}$ has 46080 elements corresponding to *the same* 11520 gates.

# Constructions of ‘rational’ stabilizer group

## Rational stabilizer group $\acute{\mathcal{G}}_2$

It is more convenient to work with group $\acute{\mathcal{G}}$ generated with ‘rational’ Hadamard gate $\acute{H}$, because action of such group on $(1,0,0,0)$ produces 240 vectors from complexification of $E_8$ root system. Group with standard Hadamard gate $H$ would duplicate number of vectors by pairs with irrational phase multiplier $(1+\mathrm{i})/\sqrt{2}$. Anyway there are only 60 different states after normalization.

## Generation by complex reflections

Group $\acute{\mathcal{G}}_2$ can also be generated by second order complex reflections with respect to 60 stabilizer states $R\langle{}^2\varsigma_j^{\mathsf{k}}\rangle$. The minimal number of such reflections necessary for generation of a whole group is five, for example $R\langle{}^2\varsigma_0^{\mathsf{k}}\rangle$ with $\mathsf{k} = 1, 2, 3, 4, 8$.

# Problem with ‘contextual closure’

In previous examples, *any basis* could be generated from computational basis by groups of symmetries of configuration, but some subgroup could generate only a ‘non-contextual’ set with non-overlapping bases.

The two-qubit stabilizer group acts transitively both on first set with 15 bases and on the second set with 90 bases, but it is not possible to transform basis from the first set into the second one using gate from stabilizer group.
For 60 stabilizer states, *only 15 non-overlapping (‘non-contextual’) bases* can be generated by Clifford gates.

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For 60 stabilizer states, *only 15 non-overlapping ('non-contextual') bases* can be generated by Clifford gates.

Some gate *outside of stabilizer group* is necessary for construction of all 105 bases. However, any 'contextual closure group' constructed in such a way is infinite,

*e.g.*, see *G. Nebe, E. M. Rains, and N. J. A. Sloane*, The invariants of the Clifford groups, `arXiv:math/0001038`, Designs, Codes and Cryptography **24**, 99–122 (2001).

# Extra gate for contextual bases

## Controlled phase gate

An example of gate for transition between bases form different sets is *controlled phase* gate:

$$\mathsf{C}_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix} = R\langle {}^4\varsigma_3^1 \rangle.$$

It is possible to transform a basis from the first set into a basis from the second one using this gate:

$$\mathsf{C}_S : \left\{ |\varsigma_0^2\rangle, |\varsigma_1^2\rangle, |\varsigma_2^2\rangle, |\varsigma_3^2\rangle \right\} \mapsto \left\{ |\varsigma_0^2\rangle, |\varsigma_1^2\rangle, |\varsigma_2^3\rangle, |\varsigma_3^3\rangle \right\}.$$

# Clifford+CS group

## Construction of ‘contextual bases’

For two qubits Clifford gates together with $\mathsf{C}_S$ can be used for generation of all 105 bases discussed above, but group generated in such a way is infinite and only finite number of transformations should be used for such a purpose.

# Clifford+CS group

For two qubits Clifford gates together with $C_S$ can be used for generation of all 105 bases discussed above, but group generated in such a way is infinite and only finite number of transformations should be used for such a purpose.

This group (also known as Clifford+CS) is universal for quantum computation.

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## Optimal two-qubit circuits for universal fault-tolerant quantum computation

Andrew N. Glaudell, Neil J. Ross and Jacob M. Taylor

We study two-qubit circuits over the Clifford+CS gate set, which consists of the Clifford gates together with the controlled-phase gate CS = diag(1, 1, 1, *i*). The Clifford+CS gate set is universal for quantum computation and its elements can be implemented fault-tolerantly in most error-correcting schemes through magic state distillation. Since non-Clifford gates are typically more expensive to perform in a fault-tolerant manner, it is often desirable to construct circuits that use few CS gates. In the present paper, we introduce an efficient and optimal synthesis algorithm for two-qubit Clifford+CS operators. Our algorithm inputs a Clifford+CS operator $U$ and outputs a Clifford+CS circuit for $U$, which uses the least possible number of CS gates. Because the algorithm is deterministic, the circuit it associates to a Clifford+CS operator can be viewed as a normal form for that operator. We give an explicit description of these normal forms and use this description to derive a worst-case lower bound of $5\log_2(\frac{1}{\epsilon}) + O(1)$ on the number of CS gates required to $\epsilon$-approximate elements of SU(4). Our work leverages a wide variety of mathematical tools that may find further applications in the study of fault-tolerant quantum circuits.

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# Complexification of $E_8$ lattice and 'magic'

240 $E_8$ roots used for construction of the configuration with 60 states correspond to the 'first shell' of $E_8$ lattice with norm 2. The 'second shell' of $E_8$ lattice with 2160 vectors of norm 4 after complexification and normalization produces 540 states: the 60 stabilizer states (again) and 480 'maximally magic' states found recently due to analysis of so-called *stabilizer Rényi entropy (SRE)*.

*M. Ohta and K. Sakurai*, Extremal magic states from symmetric lattices, `arXiv:2506.11725` (2025).

*Q. Liu, I. Low, and Z. Yin*, Maximal magic for two-qubit states, `arXiv:2502.17550` (2025).

Here, the term 'magic' or 'non-stabilizerness' of quantum states is used to characterize their difference from states obtained using Clifford circuits.

# Conclusion

- Structures and symmetries of $E_8$ lattice can be useful for applications in quantum information science.
- Four different configurations derived from $E_8$ were considered.
- Structure of symmetry groups of such configurations can be used for characterization of quantum contextuality.
- Considered configurations can be useful for quantum key distribution.
- Problem of contextuality in such protocols has interesting relation with analysis of resources for universal and fault-tolerant quantum computations.

# Thank you for attention!