# Interaction between two quantized fields losses included

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# Summary

How to consider mirror losses.

The interaction of two fields.

Nonunitary transformations leading to non-Hermitian Hamiltonians.

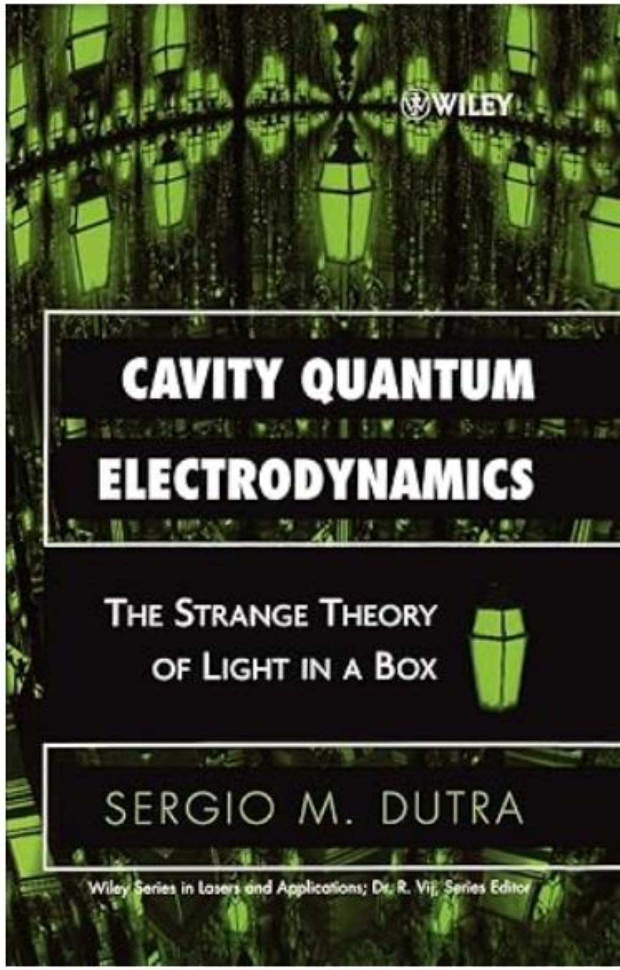
Realizing such nonunitary transformations in waveguide arrays.

How to make your own non-Hermitian Hamiltonian.

# La ecuación maestra (que incluye pérdidas es)

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H, \hat{\rho}] + \hat{L}\hat{\rho}$$

$$\boxed{\hat{L}\hat{\rho} = 2\gamma\hat{a}\hat{\rho}\hat{a}^{+} - \gamma\hat{a}^{+}\hat{a}\hat{\rho} - \gamma\hat{\rho}\hat{a}^{+}\hat{a},}$$




S.M. Dutra obtiene esta ecuación usando un principio de correspondencia (lo que es válido para estados coherentes –cuasiclásicos- es válido para todos los campos. Eur. J. Phys. 18, 194 (1997).

# Cavidades reales: pérdidas llevan a ecuaciones maestras

$$E(x,t) = E(x,0)\mathrm{e}^{-\gamma t}$$

$$|\alpha\rangle\langle\alpha| \to |\alpha \mathrm{e}^{-\gamma t}\rangle\langle\alpha \mathrm{e}^{-\gamma t}|.$$

$$|\alpha \mathrm{e}^{-\gamma\delta t}\rangle\langle\alpha \mathrm{e}^{-\gamma\delta t}| \approx |\alpha\rangle\langle\alpha| + \gamma\delta t\,(2\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger - \hat{n}|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|\hat{n}),$$

$$\boxed{\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} \equiv \hat{\mathscr{L}}\hat{\rho} = 2\gamma\hat{a}\hat{\rho}\hat{a}^\dagger - \gamma\hat{a}^\dagger\hat{a}\hat{\rho} - \gamma\hat{\rho}\hat{a}^\dagger\hat{a}.}$$

$$\hat{\rho}(t) = \exp[(\hat{J} + \hat{L})t]\hat{\rho}(0).$$

$$\hat{\rho}(t) = \exp(\hat{L}t)\exp(f(t)\hat{J})\hat{\rho}(0).$$

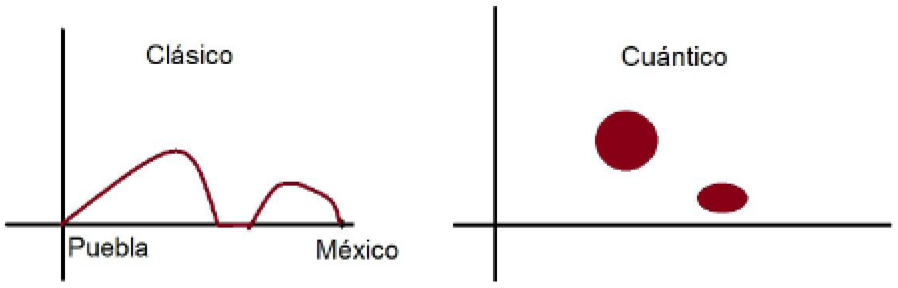
$$f(t) = \frac{1 - \mathrm{e}^{-2\gamma t}}{2\gamma}.$$

## Quasiprobability distribution functions

A classical probability density P(q,p) may be written as

$$P(q,p) = \int \delta(q-q')\delta(p-p')P(q',p')dq'dp'$$

$$P(q,p) = \frac{1}{4\pi^2}\int e^{iu(p-p')}e^{iv(q-q')}P(q',p')dudvdq'dp'$$
$$= \frac{1}{4\pi^2}\int e^{iup}e^{ivq}\left(\int P(q',p')e^{-iq'v}e^{-ip'u}dq'dp'\right)dudv$$




$$< e^{-i(q'v+p'u)} >$$

$$Tr\{\hat{\rho}e^{-i(u\hat{p}+v\hat{q})}\}$$

Correspondence principle.

With $\rho=|\psi\rangle\langle\psi|$

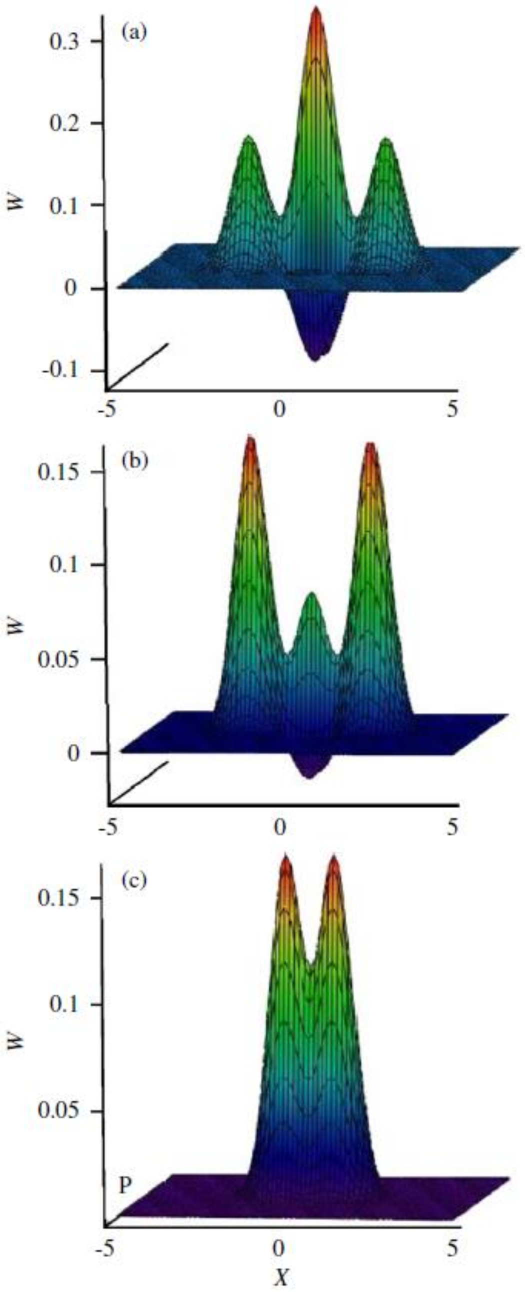


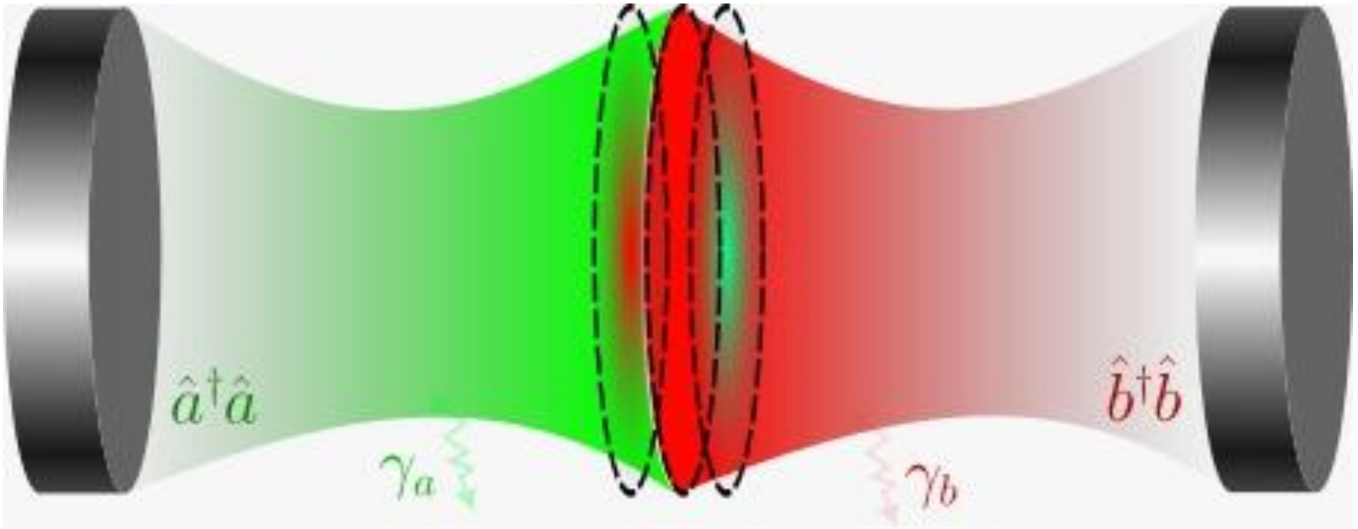

Fig. 8. Wigner function $\beta = -\alpha = 2$ for (a) $\gamma t = 0$, (b) $\gamma t = .1$ and (c) $\gamma t = .5$.

19. W. K. Lai, V. Buzek, and P. L. Knight, Phys. Rev. A **43**, 6323 (1991).

$$\hat{H}_{\text{eff}} = -\mathrm{i}\left[\frac{\gamma}{2}\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right) + \frac{\Delta}{2}\left(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}\right)\right] + g\left(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}\right),$$

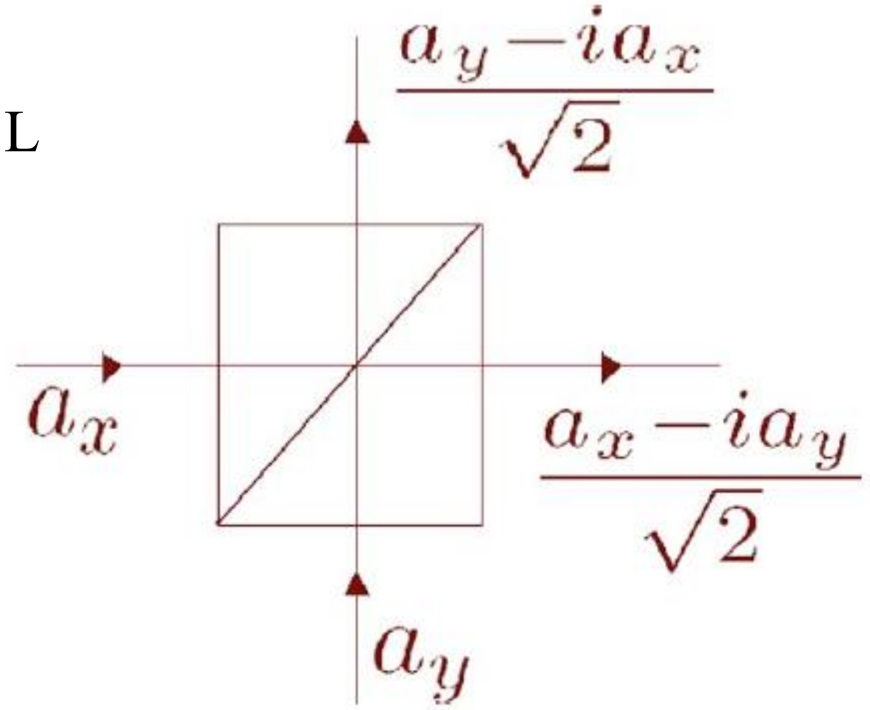
$$\hat{N} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}, \quad \hat{J}_y = \frac{\mathrm{i}}{2}\left(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger\right),$$

$$\hat{J}_x = \frac{1}{2}\left(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}\right), \quad \hat{J}_z = \frac{1}{2}\left(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}\right),$$



$$B = e^{-i\theta(a_y^\dagger a_x + a_x^\dagger a_y)}$$

$$e^{sA} B e^{-sA} = B + s[A,B] + \frac{s^2}{2!}[A,[A,B]] + \frac{s^3}{3!}[A,[A,[A,B]]] + \ldots$$




$$B a_x B^\dagger = \cos(\theta) a_x + i \sin(\theta) a_y,$$

$$B a_y B^\dagger = \cos(\theta) a_y + i \sin(\theta) a_x$$

# Interaction between two-classical fields

$$\dot{\varrho} = -i[H, \varrho] + \mathcal{L}\varrho, \qquad H = \omega\hat{n} + g(ae^{i\omega t} + a^\dagger e^{-i\omega t})$$

$$T = e^{i\omega\hat{n}t}, \qquad D(g/\omega) = e^{\frac{g}{\omega}(a^\dagger - a)}$$

$$\frac{\partial\rho}{\partial t} = \kappa(N_r + 1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \kappa N_r(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger)$$

$$J_1\rho = 2\gamma a\rho a^\dagger$$
$$J_2\rho = 2\beta a^\dagger\rho a$$

$$L\rho = -(\gamma + \beta)(a^\dagger a\rho + \rho a^\dagger a)$$
$$\frac{\partial\rho}{\partial t} = (J_1 + J_2 + L - 2\beta)\rho$$

$$[J_2, J_1]\rho = \frac{4\beta\gamma}{\beta+\gamma} L\rho - 4\beta\gamma\rho$$

$$[J_1, L]\rho = -2(\beta+\gamma) J_1 \rho$$

$$[J_2, L]\rho = 2(\beta+\gamma) J_2 \rho.$$

$$\rho(t) = \exp(-2\beta t) \exp[f_3(t)] \exp[f_2(t) J_2] \exp[f_0(t) L] \exp[f_1 J_1] \rho(0).$$

$$f_0 = \frac{\gamma-\beta}{\gamma+\beta} t + \frac{1}{\gamma+\beta} \ln\left(\frac{\beta e^{2t(\beta-\gamma)} - \gamma}{\beta-\gamma}\right)$$

$$f_1 = f_2 = \frac{1}{2} \frac{e^{2t(\beta-\gamma)} - 1}{\beta e^{2t(\beta-\gamma)} - \gamma}$$

$$f_3 = 2\beta t - \ln\left(\frac{\beta e^{2t(\beta-\gamma)} - \gamma}{\beta-\gamma}\right).$$

$$\rho(t) = \frac{\exp\left[N(t)|\tilde{\alpha}(t)|^2\right]}{1+N(t)} \sum_{n=0}^{\infty} \left(\frac{N(t)}{1+N(t)}\right)^n \frac{1}{n!} (a^\dagger)^n |\tilde{\alpha}(t)\rangle\langle\tilde{\alpha}(t)| a^n.$$

$$\tilde{\alpha}(t) = \frac{\alpha e^{-\kappa t}}{1+N(t)}$$

$$N(t) = N_r\left(1 - e^{-2\kappa t}\right)$$

$$\rho(t) = \frac{\exp\left[N(t)|\tilde{\alpha}(t)|^2\right]}{1+N(t)} \sum_{n=0}^{\infty} \left(\frac{N(t)}{1+N(t)}\right)^n \frac{1}{n!} \sum_{k=0}^{n} \sum_{m=0}^{n} \binom{n}{k}\binom{n}{m} \sqrt{k!}\sqrt{m!}$$
$$\times \tilde{\alpha}^*(t)^{n-k}\, \tilde{\alpha}(t)^{n-m}\, |\tilde{\alpha}(t), k\rangle\langle\tilde{\alpha}(t), m|$$

Quantum Semiclass. Opt. 10 (1998) 671–674. L. M Arévalo-Aguilar and H. Moya-Cessa, Solution to the master equation for a quantized cavity mode

# Where is the cheating?

$$< e^{-i(q'v+p'u)} >$$

$$Tr\{\hat{\rho} e^{-i(u\hat{p}+v\hat{q})}\}$$

$$F(\alpha,s) = \frac{2}{\pi(1-s)} \sum_{k=0}^{\infty} \left(\frac{s+1}{s-1}\right)^k \langle \alpha,k|\rho|\alpha,k\rangle \quad (1)$$

with $s$ the quasiprobability function's parameter that indicates which is the relevant distribution ($s = -1$ Husimi [12], $s = 0$ Wigner [13] and $s = 1$ Glauber-Sudarshan [14, 15] distribution functions), $\rho$ the density matrix and the states $|\alpha,k\rangle$ are the so-called displaced number states [16].

H Moya-Cessa, PL Knight
Physical Review A 48 (3), 2479 (1993).

JOURNAL OF MODERN OPTICS, 1995, VOL. 42, NO. 8, 1741–1754

**Generation and properties of superpositions of displaced Fock states**

HÉCTOR MOYA-CESSA†

$$\rho(t) = \frac{\exp\left[N(t)|\tilde{\alpha}(t)|^2\right]}{1+N(t)} \sum_{n=0}^{\infty} \left(\frac{N(t)}{1+N(t)}\right)^n \frac{1}{n!} (a^\dagger)^n |\tilde{\alpha}(t)\rangle \langle\tilde{\alpha}(t)| a^n .$$

$$\boxed{\begin{gathered} Q(\beta) = \frac{1}{\pi}\langle\beta|\rho|\beta\rangle = \frac{e^{\frac{N(t)}{1+N(t)}|\beta(t)|^2}}{\pi[1+N(t)]} |\langle\beta|\tilde{\alpha}(t)\rangle|^2 \\ \langle a^n a^{\dagger k}\rangle = \int d^2\beta\, Q(\beta)\beta^{*k}\beta^n \end{gathered}}$$

Letter
Vol. 48, No. 20 / 15 October 2023 / *Optics Letters* 5435
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# Exact solution for the interaction of two decaying quantized fields

L. Hernández-Sánchez, I. Ramos-Prieto,* F. Soto-Eguibar, and H. M. Moya-Cessa

$$\frac{d\hat{\rho}}{dz} = -ig\hat{S}\hat{\rho} + \gamma_a \mathcal{L}[\hat{a}]\hat{\rho} + \gamma_b \mathcal{L}[\hat{b}]\hat{\rho}, \quad (1)$$

where $\hat{S}\hat{\rho} = [\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}, \hat{\rho}]$ determines the interaction of the two fields or light modes, while the second term of the right-hand

$$\hat{\rho} = \exp\left[-\chi\left(\hat{J}_a + \hat{J}_b\right)\right]\hat{\varrho}, \quad \textbf{(2)}$$

to obtain

$$\frac{d\hat{\varrho}}{dz} = -\mathrm{i}g\hat{S}\hat{\varrho} + \gamma_a\left[(1-2\chi)\hat{J}_a - \hat{L}_a\right]\hat{\varrho} + \gamma_b\left[(1-2\chi)\hat{J}_b - \hat{L}_b\right]\hat{\varrho}, \quad \textbf{(3)}$$

and by setting $\chi = 1/2$, the above equation reads

$$\frac{d\hat{\varrho}}{dz} = -\mathrm{i}\left(\hat{H}_{\text{eff}}\hat{\varrho} - \hat{\varrho}\hat{H}_{\text{eff}}^{\dagger}\right), \quad \textbf{(4)}$$

where $\hat{H}_{\text{eff}} = -\mathrm{i}\gamma_a\hat{a}^{\dagger}\hat{a} - \mathrm{i}\gamma_b\hat{b}^{\dagger}\hat{b} + g\left(\hat{a}b^{\dagger} + \hat{a}^{\dagger}\hat{b}\right)$ is an effective non-Hermitian Hamiltonian. It is important to emphasize

$$[\hat{L}, \hat{J}]\rho = 2\gamma\hat{J}\rho,$$

# Dirac notation

$$|0> = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \hat{r}_{0}, \quad |1> = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \hat{r}_{1}, \quad |2> = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \hat{r}_{2}, \quad |3> = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \hat{r}_{3},$$

$$< j| = \begin{pmatrix}0 & \cdots & j & 0\end{pmatrix} = \hat{r}_j^T$$

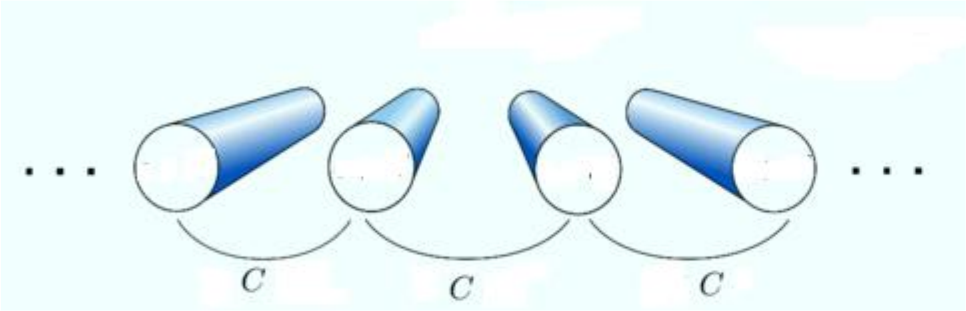
$$< j|k> = \hat{r}_j^T \hat{r}_k = \delta_{jk}$$

$$\vec{x} = \sum_{n=0}^{3} c_n \hat{r}_n \rightarrow \quad |x> = \sum_{n=0}^{3} c_n |n>$$

$$V_{01} = |0\rangle\langle 1| = \hat{r}_0 \hat{r}_1^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sum_{j=0}^{2} V_{j,j+1} = \sum_{j=0}^{2} |j\rangle\langle j+1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

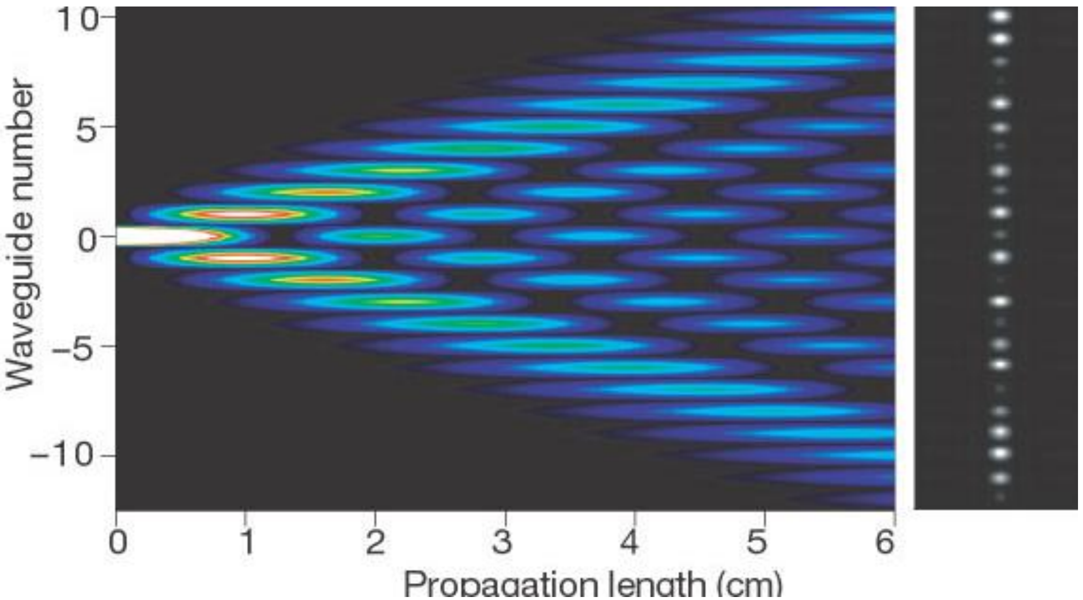
# Infinite waveguides array




$$\boxed{\frac{dE_j}{dz} + ik_z\left(E_{j+1} + E_{j-1}\right) = 0}$$

# Discrete Diffraction

A beam injected into one of the waveguides in the array spreads to the rest of them by wave coupling.




This phenomenon has been referred to as **DISCRETE DIFFRACTION**

$$E_j(z) = (i)^j J_j\left(2k_z z\right)$$

$$V = \sum_{n=-\infty}^{\infty} | n >< n+1 |,$$

$$V^{\dagger} = \sum_{n=-\infty}^{\infty} | n+1 >< n |,$$

$$< n | m > = \delta_{n,m}$$

$$V_{01} = | 0 >< 1 | = \hat{r}_0 \hat{r}_1^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sum_{j=0}^{2} V_{j,j+1} = \sum_{j=0}^{2} | j >< j+1 | = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

## Schrödinger-like equation

$$-i \frac{d | E >}{dZ} = k_z (V + V^{\dagger}) | E >,$$

$$| E > = \sum_{j=-\infty}^{\infty} E_j | j >$$

$$\frac{dE_j}{dz} + ik_z \left( E_{j+1} + E_{j-1} \right) = 0$$

$$-i\frac{d|E>}{dz} = k_z(V+V^\dagger)|E>,$$

$$|E> = e^{ik_z(V+V^\dagger)z}|E(0)>$$

$$= e^{k_z(iV-\frac{1}{iV})z}|m>$$

$$= \sum_{n=-\infty}^{\infty} i^n J_n(2k_z)V^n|m>$$

$$<j|E> = <j|\sum_{n=-\infty}^{\infty} i^n J_n(2k_z)|m-n>$$

$$V = \sum_{n=-\infty}^{\infty}|n><n+1|,$$

$$V^\dagger = \sum_{n=-\infty}^{\infty}|n+1><n|,$$

$$<n|m> = \delta_{n,m}$$

$$V^{-1} = V^\dagger$$

$$\boxed{E_j(z) = (i)^j J_j\left(2k_z z\right)}$$

**IOP** Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **57** (2024) 285201 (22pp) https://doi.org/10.1088/1751-8121/ad5942

# Non-Hermitian propagation in equally-spaced waveguide arrays

**Ivan A Bocanegra-Garay[1,2,*] and Héctor M Moya-Cessa[1]**

$$i\frac{\partial|\Psi(t)\rangle}{\partial t}=\left[\frac{\hat{p}^2}{2}+V(x)\right]|\Psi(t)\rangle. \tag{1}$$

By doing

$$|\Psi(t)\rangle=e^{\alpha x}|\Phi(t)\rangle,\qquad \alpha\in\mathbb{R}, \tag{2}$$

the non-Hermitian Schrödinger equation [36]

$$i\frac{\partial|\Phi(t)\rangle}{\partial t}=\left[\frac{(\hat{p}+i\alpha)^2}{2}+V(x)\right]|\Phi(t)\rangle, \tag{3}$$

is reached. In fact, non-unitary transformations naturally lead to non-Hermitian dynamics [37].

[37] Villegas-Martínez B M, Soto-Eguibar F, Hojman S A, Asenjo F A and Moya-Cessa H M 2024 Non-Hermitian dynamics from Hermitian systems: do-it-yourself *Mod. Phys. Lett.* B **38** 2450178

$$\mathrm{i}\dot{c}_n(z) = \alpha\left[c_{n-1}(z) + c_{n+1}(z)\right], \tag{4}$$

where $\alpha \propto \mathrm{e}^{-d} \in \mathbb{R}$ is the coupling constant, $d$ is the distance between adjacent waveguides and the dot denotes derivative with respect to $z$. Equation (4) has an associated equation of the Schrödinger type (a partner equation [8], see also [22])

$$\mathrm{i}\frac{\partial|\psi(z)\rangle}{\partial z} = H|\psi(z)\rangle, \tag{5}$$

where the Hamiltonian operator

$$H = \alpha\left(V^\dagger + V\right), \tag{6}$$

for a pair of well-defined conjugate ladder operators $V$ and $V^\dagger$, is Hermitian, namely $H = H^\dagger$. An abstract Hilbert space $\mathcal{H}$ is assumed, where each $c_n(z)$ is associated to an abstract vector $|n\rangle \in \mathcal{H}$. The partner equations (4) and (5) are connected by

$$|\psi(z)\rangle = \sum_n c_n(z)\,|n\rangle, \tag{7}$$

$$|\psi(z)\rangle = \mathrm{e}^{-\mathrm{i}Hz}|\psi(0)\rangle,$$

$$|\psi(0)\rangle = \mathrm{e}^{-\gamma\hat{n}}|\phi(0)\rangle, \qquad \hat{n}|k\rangle = k|k\rangle,\ |k\rangle$$

$$\mathrm{i}\frac{\partial|\phi(z)\rangle}{\partial z} = \tilde{H}|\phi(z)\rangle,$$

$$\tilde{H} = \alpha\left(\mathrm{e}^{\gamma}V^{\dagger} + \mathrm{e}^{-\gamma}V\right) = k_1V^{\dagger} + k_2V,$$

Modern Physics Letters B
Vol. 38, No. 22 (2024) 2450178 (15 pages)

DOI: 10.1142/S0217984924501781

# Non-Hermitian dynamics from Hermitian systems: Do-it-yourself

Braulio M. Villegas-Martínez [*,†,‡], Francisco Soto-Eguibar [*,§]
and Héctor M. Moya-Cessa [*,¶]

Sergio A. Hojman Felipe A. Asenjo

# 4. The Complex Deformed Harmonic Oscillator

A very well-known $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian, is the complex deformed harmonic oscillator given by

$$\hat{H} = \hat{p}^2 + \hat{x}^2(i\hat{x})^\epsilon, \tag{13}$$

where $\epsilon$ is a non-negative real number.[10,12,22]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$$|\Psi\rangle = \hat{S}_r^{-1}|\psi\rangle, \text{with } \hat{S}_r = \exp[\tfrac{r}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})]$$

$$\hat{\mathcal{H}} = \hat{S}_r \hat{H} \hat{S}_r^{-1}.$$

$$\begin{aligned}
\hat{S}_r \hat{x} \hat{S}_r^{-1} &= \exp\Big[\frac{r}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\Big]\hat{x}\exp\Big[-\frac{r}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\Big] \\
&= \exp(-ir)\hat{x}, \\
\hat{S}_r \hat{p} \hat{S}_r^{-1} &= \exp\Big[\frac{r}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\Big]\hat{p}\exp\Big[-\frac{r}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\Big] \\
&= \exp(ir)\hat{p},
\end{aligned}$$

$$\hat{\mathcal{H}} = \exp(2ir)\hat{p}^2 + \exp\Big[-ir(2+\epsilon) + i\frac{\pi}{2}\epsilon\Big]\hat{x}^{2+\epsilon}.$$

As $r$ is an arbitrary real constant, we choose it such that

$$\exp(2ir) = \exp\Big[-ir(2+\epsilon) + i\frac{\pi}{2}\epsilon\Big],$$

$$\hat{\mathcal{H}}_k = \exp\left(i\pi\frac{\epsilon+4k}{4+\epsilon}\right)(\hat{p}^2 + \hat{x}^{2+\epsilon}),$$

$$\hat{H} = \hat{p}^2 + V(i\hat{x}) \tag{22}$$

can be transformed with the non-unitary squeezing transformation $\hat{S} = \exp[\frac{\pi}{4} \times (\hat{x}\hat{p} + \hat{p}\hat{x})]$ (which is the same one used in the case of the complex deformed harmonic oscillator with $r = \pi/2$) to

$$\hat{\mathcal{H}} = \exp(i\pi)\hat{p}^2 + V(i\hat{x}e^{-i\frac{\pi}{2}}), \tag{23}$$

which becomes the Hermitian Hamiltonian

$$\hat{\mathcal{H}} = -\hat{p}^2 + V(\hat{x}). \tag{24}$$

# Conclusions

It was shown how to obtain non-Hermitian Hamiltonians via non-unitary transformations.

Such transformations were applied in several cases, among them the interaction between two fields.