"Källén function & around"

I. Krasil'schick Seminar "Geometry of Differential Equations"

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Plan of the talk

§1 Definition and properties

§2 Hero(n) & his formula

§3 G. Källén & his function; a bit of physics

§4 Two-valued (formal) groups & Källén function

§5 2-Bessel, kernels, Catalan ... & Källén function

§1 Definitions & properties

The "Källen function" - what is it?
Almost nobody knows. But, in fact, almost everybody knows:

$$\boxed{\lambda(x,y,z) := x^2+y^2+z^2-2xy-2yz-2zx}$$

$x,y,z \in \mathbb{k} (= \mathbb{R}, \mathbb{C})$

Basic properties:

- $\lambda(x,y,z) \in \mathbb{k}[x,y,z]$;
- λ is Σ_3-invariant: $\lambda(x,y,z) = \lambda(z,x,y) = \ldots$ etc;
- λ is a quadratic in each of variables:

$$\lambda(x,y,z) = \boxed{z^2 - 2(x+y)z + (x-y)^2} = x^2 - 2(y+z)x + (y-z)^2 = y^2 - 2(z+x)y + (z-x)^2;$$

(NB: It is explicitly simpliest 2-value algebraic formal group law)

- The discriminants of $\lambda(x,y,z)$ are splitted:

$$\Delta_x = 4(y+z)^2 - 4(y-z)^2 = 4y \cdot 4z$$

$$\Delta_y = 4x \cdot 4z \, ; \, \Delta_z = 4x \cdot 4y \, ;$$

- $\lambda(x,y,z)$ itself can be written in a discriminant form:

$$\lambda(x,y,z) = (x+y-z)^2 - 4yz = \Delta_t P,$$

where
$$P(t) = z t^2 + (x+y-z)t + y =$$
$$= x t^2 + (y+z-x)t + z =$$
$$= y t^2 + (x+z-y)t + x;$$

- $\lambda(x,y,z)$ is "factorized" or "reducible" under the "2-Kummer lift":

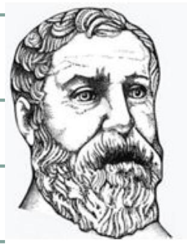
$K : k[x,y,z] \to k[u,v,w]$ such that
$$x = u^2, \; y = v^2, \; z = w^2 :$$

$$\lambda(u^2,v^2,z^2) = u^4 + v^4 + w^4 - 2(u^2v^2 + v^2w^2 + w^2u^2) =$$
$$= (w^2 - (u+v)^2)((u-v)^2 - w^2) =$$
$$= (u+v+w)(u+v-w)(u-v+w)(-u+v+w);$$

$$\lambda(x,y,z) = (z - (\sqrt{x}+\sqrt{y})^2)((\sqrt{x}-\sqrt{y})^2 - z);$$

(NB: generalized shift operators)

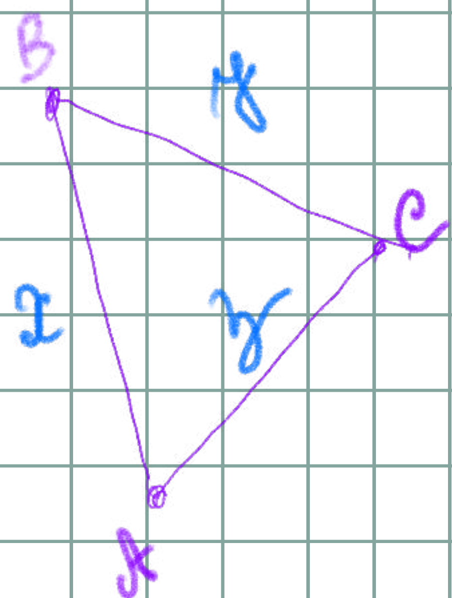
§2 Heron & his formula



Hero of Alexandria

The formula is credited to Hero (or Heron) of Alexandria, who was a Greek Engineer and Mathematician in 10 – 70 AD.

Amongst other things, he developed the ***Aeolipile***, the first known steam engine, but it was treated as a toy!



• $\boxed{S_{\Delta ABC} = \sqrt{p(p-x)(p-y)(p-z)}}$

$$p = x+y+z/2$$

$$\Downarrow$$

• $\sqrt{\lambda(x^2, y^2, z^2)} = 4S_{\Delta ABC}$

$$\Downarrow$$

• & determinant formulas:

$S_{\Delta ABC} = \sqrt{pabc}$, where

$$a := \frac{x+y-z}{2},\ b = \frac{x-y+z}{2},\ c = \frac{-x+y+z}{2}$$

$$\Downarrow$$

$$S_{\Delta ABC} = \sqrt{\frac{1}{2}(a+b+c)\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix}} \Rightarrow$$

$$\Rightarrow \boxed{\lambda(x^2, y^2, z^2) = \frac{(x+y+z)}{2}\begin{vmatrix} 0 & x+y-z & x-y+z \\ x+y-z & 0 & -x+y+z \\ x-y+z & -x+y+z & 0 \end{vmatrix}}$$

Cayley-Menger determinants

Let σ - n-simplex in $\mathbb{R}^n$ with

n vertices $v_1, \dots, v_{n+1}$

$$\|\beta_{ij}\| := \| |v_i - v_j|^2 \| \in \mathrm{Mat}_{n+1}(\mathbb{R}^n)$$

Consider

$M_n \in \mathrm{Mat}_{n+2}(\mathbb{R}^n)$ such that

$$M_n := \begin{Vmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & \beta_{ij} & \\ 1 & & & \end{Vmatrix}$$

$$\boxed{\det M_n = 2^n (n!)^2 (-1)^{n+1} \mathrm{Vol}_n^2(\sigma)}$$

For $n=2$

$$\boxed{\lambda(x^2,y^2,z^2) = -\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x^2 & y^2 \\ 1 & x^2 & 0 & z^2 \\ 1 & y^2 & z^2 & 0 \end{vmatrix}}$$

For $n=3$ (Thetrahedron volume V)

$$288 V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix}$$

$$d_{ij}^2 = |v_i - v_j|^2 ; \quad 1 \le i,j \le 4$$

§ 3 Källén & his function

Gunnar Källén
(1926 – 1968)

• CERN (Niel Bohr Institute 1952–57)

• Källen – Lehman spectral representation
(for 2-point particle interaction as a sum of free propagators (casual Green fun's

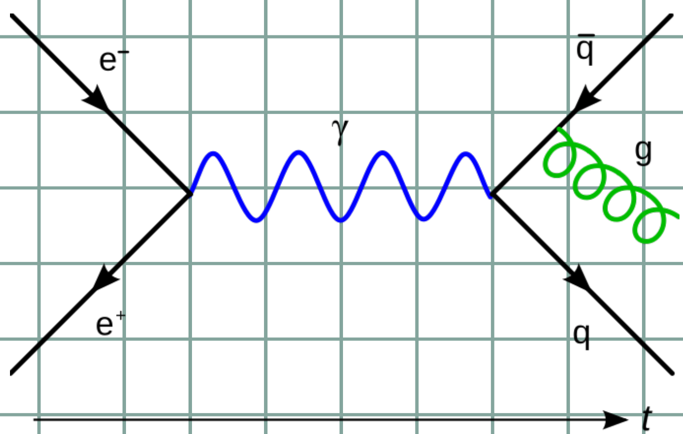
$$\Delta(p) = \int_0^\infty \rho(\mu^2)\,\Delta(p,\mu^2)\,d\mu^2$$

$\rho(\mu^2)$ – spectral density function

$$\Delta(p, \mu^2) = \frac{1}{p^2 - \mu^2 + i\epsilon}\,.$$

free-particle propagator

Feynman diagram



electron e^- and positron e^+ annihilates ⇒ photon γ that becomes quark (q) - anti-quark ($\bar{q}$) pair, and after that, quark radiates a gluon g.

- Källén function λ

Appears as the Kinematic function

The states of a single free particle in Relativistic QM are characterised by their 4-momentum (Spin neglected)

- $P = (p^\mu)$ basic 4-vector of 4-momentum

(*) - $p = mu = m\gamma(v)(c, \underline{v})$ rest mass m times 4-velocity

Here $\gamma(v) = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ has a central significance in special relativity.

Alternatively,

(*) $p = \left(\dfrac{E}{c}, \underline{p}\right)$ $\begin{cases} E \text{ - energy} \\ \underline{p} \text{ - 3-momentum} \\ c \end{cases}$

NB - highly nontrivial: energy-matter equivalence

$\Downarrow$

$$\begin{cases} E = m\gamma(v)c^2 \\ \bar{p} = m\gamma(v)\underline{v} \end{cases} \Rightarrow$$

inertial mass $m\gamma(v)$ replaces rest mass m in this formulation

The Lorentz invariant length:

$$p^2 = m^2 u^2 = m^2 c^2 =$$
$$= (E/c)^2 - \underline{p}^2,\ p\text{ - time-like 4-vector}$$

The invariant mass of 2-particle system (particles a & b with 4-momenta $p_a = (E_a, \underline{p}_a)$ and $p_b = (E_b, \underline{p}_b)$ collide), is the total 4-momentum square

$$s = (p_a + p_b)^2 = (E_a + E_b)^2 - (\underline{p}_a + \underline{p}_b)^2$$
$$= m_a^2 + m_b^2 + 2(E_a E_b - \underline{p}_a \cdot \underline{p}_b)$$

($c = 1$ and take CM frame further)

Taking target system condition $p_b^T = 0$, $E_b^T = m_b$ obtain

$$p_a^T = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b}$$

The first appearance Källén function / kinematic funct. ("Particle Kinematic" ch. I - VI, X) by Eero Byckling K. Kajantie Jyväskylä, 1971

We write this result in the form

$$p_a^T = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b} \tag{9.2}$$

Here appears the kinematical function

$$\lambda(x,y,z) = (x-y-z)^2 - 4yz \tag{9.3}$$

$$= x^2+y^2+z^2-2xy-2yz-2zx \tag{9.4}$$

$$= \left[x-(\sqrt{y}+\sqrt{z})^2\right]\left[x-(\sqrt{y}-\sqrt{z})^2\right] \tag{9.5}$$

$$= (\sqrt{x}-\sqrt{y}-\sqrt{z})(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}-\sqrt{y}+\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z}) \tag{9.6}$$

$$= x^2-2(y+z)x+(y-z)^2 \; . \tag{9.7}$$

Example 6 Källen considers the kinematic function in the context of 3-particle connecting vertices with 3 different masses: top quark q_t
bottom quark q_b
W-boson

$q_t(p_t, m_t, E_t)$, $q_b(p_b, m_b, E_b)$
$W(p_W, m_W, E_W)$ in the CM frame

$$q_t(p_t) \longrightarrow q_b(p_b) + W^+(p_W)$$

CM

$$\boxed{p_b + p_W = 0 \qquad E_b + E_W = m_t}$$

and ("on-shell" condition)

$$m_b^2 = E_b^2 - p_b^2 \,, \quad m_W^2 = E_W^2 - p_W^2$$

and using CM obtain

$$m_W^2 = (m_t - E_b)^2 - p_W^2 =$$

$$= m_t^2 + E_b^2 - 2m_t E_b - p_b^2 \quad \text{(CM)}$$

$$\boxed{\begin{aligned} E_b &= \frac{m_t^2 + m_b^2 - m_W^2}{2m_t} \\ E_W &= \frac{m_t^2 - m_b^2 + m_W^2}{2m_t} \\ |p_W| = |p_b| &= \frac{\sqrt{\lambda(m_t^2; m_b^2, m_W^2)}}{2m_t} \end{aligned}}$$

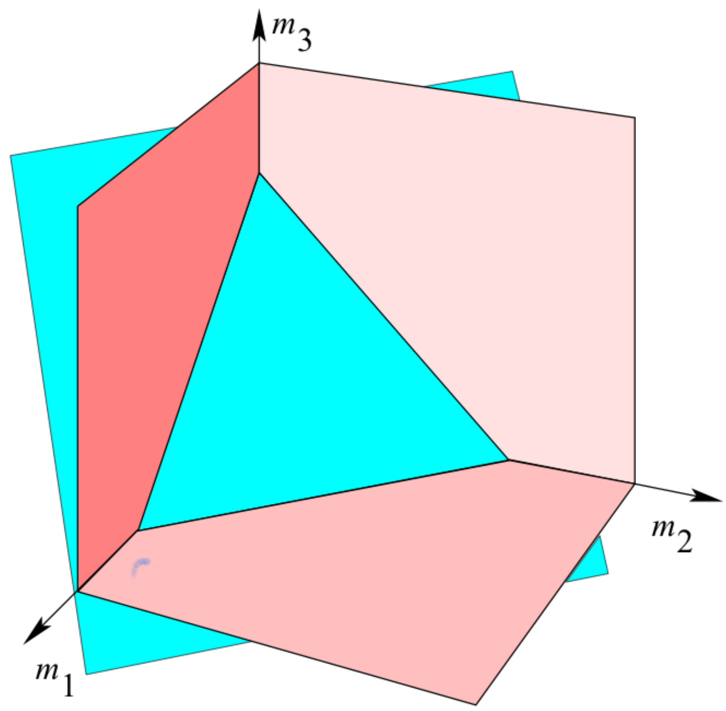
(The example due to L. Kaldamäe - S. Groote hep-ph 1404, 7714)

$$\lambda(m_t^2; m_b^2, m_W^2) = 0$$

if any one of the thesholds relations take place:

$$\left.\begin{aligned} m_1^2 &= (m_2 \pm m_3)^2 \\ m_2^2 &= (m_3 \pm m_1)^2 \\ m_3^2 &= (m_1 \pm m_2)^2 \end{aligned}\right\}$$

NB 3-Gaudin model branching points (N. Amburg I. Tolstukhin)



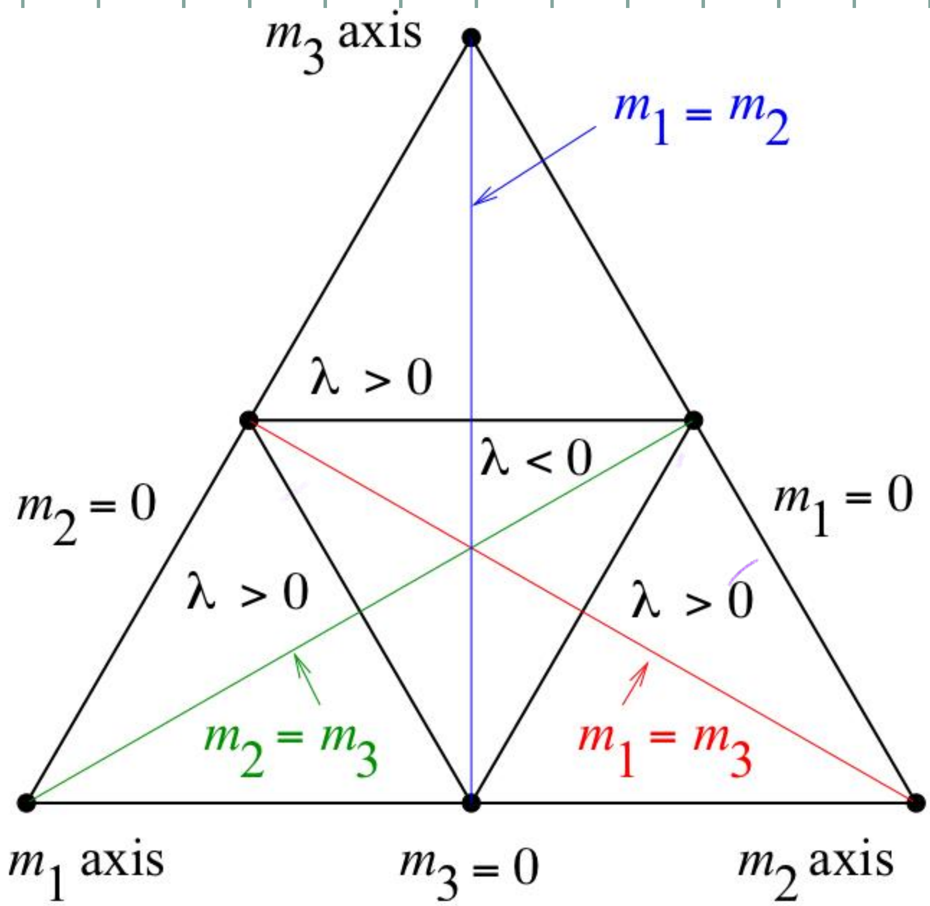
Källén triangle in "mass phase space" (Kaldamäe Groote)

1st octant $(m_1, m_2, m_3 > 0)$.

Zeros of $\lambda(m_1, m_2, m_3) = 0$ locate on three plates spanned by each two of three diagonals

$m_1 = m_2$, $m_2 = m_3$, $m_3 = m_1$

Källén triangle & treshold conditions



§4 Two-valued (formal) groups & Källén function

Definition Commutative 1-dimensional 2-valued formal group over a commutative ring R is called 2-valued formal series $\mathcal{F}(x,y)$ given by the equation

$$\mathcal{F}^2(x,y) - \Theta_1(x,y)\mathcal{F}(x,y) + \Theta_2(x,y) = 0,$$

$$\Theta_i(x,y) \in R[[x,y]], \quad i=1,2 \quad \text{s.t.}$$

- $\mathcal{F}(x,0)$ is 2-valued series
  $$\mathcal{F}^2 - 2x\mathcal{F} + x^2 = 0; \qquad 1B$$
- $\mathcal{F}(\mathcal{F}(x,y),z) = \mathcal{F}(x,\mathcal{F}(y,z)); \qquad (**)$

  class. subst. alg. fun. in alg. fun.
- $\mathcal{F}(x,y) = \mathcal{F}(y,x)$.

The simplest example: let $f(x,y)$ – formal commutative group over R, then

$$\mathcal{F}^2 - 2f(x,y)\mathcal{F} + f^2(x,y) = 0$$

$$(\mathcal{F} - f(x,y))^2$$

- $\mathcal{F}^2 - 2(x+y)\mathcal{F} + (x^2 + \Lambda xy + y^2) = 0.$ –

Elementary 2-valued formal law!.

<u>Corollary</u>: there are two 2-valued formal groups over R with no zero divisors:

- $\mathcal{F}^2 - 2(x+y)\mathcal{F} + (x-y)^2 = 0$;
- $\mathcal{F}^2 - 2(x+y)\mathcal{F} + (x+y)^2 = 0$.

$$\boxed{\Lambda = \pm 2}$$ - Associativity condition!! For

$X_1 = \mathcal{F}^{(1)}(x,y)$, $X_2 = \mathcal{F}^{(2)}(x,y)$

$Y_1 = \mathcal{F}^{(1)}(y,z)$, $Y_2 = \mathcal{F}^{(2)}(y,z)$

$(**) \Rightarrow \Theta_2(x, Y_1)\Theta_2(x, Y_2) = \Theta_2(X_1, z)\Theta_2(X_2, z)$

$$\Rightarrow \boxed{(\Lambda - 2)^2(\Lambda + 2) = 0.}$$

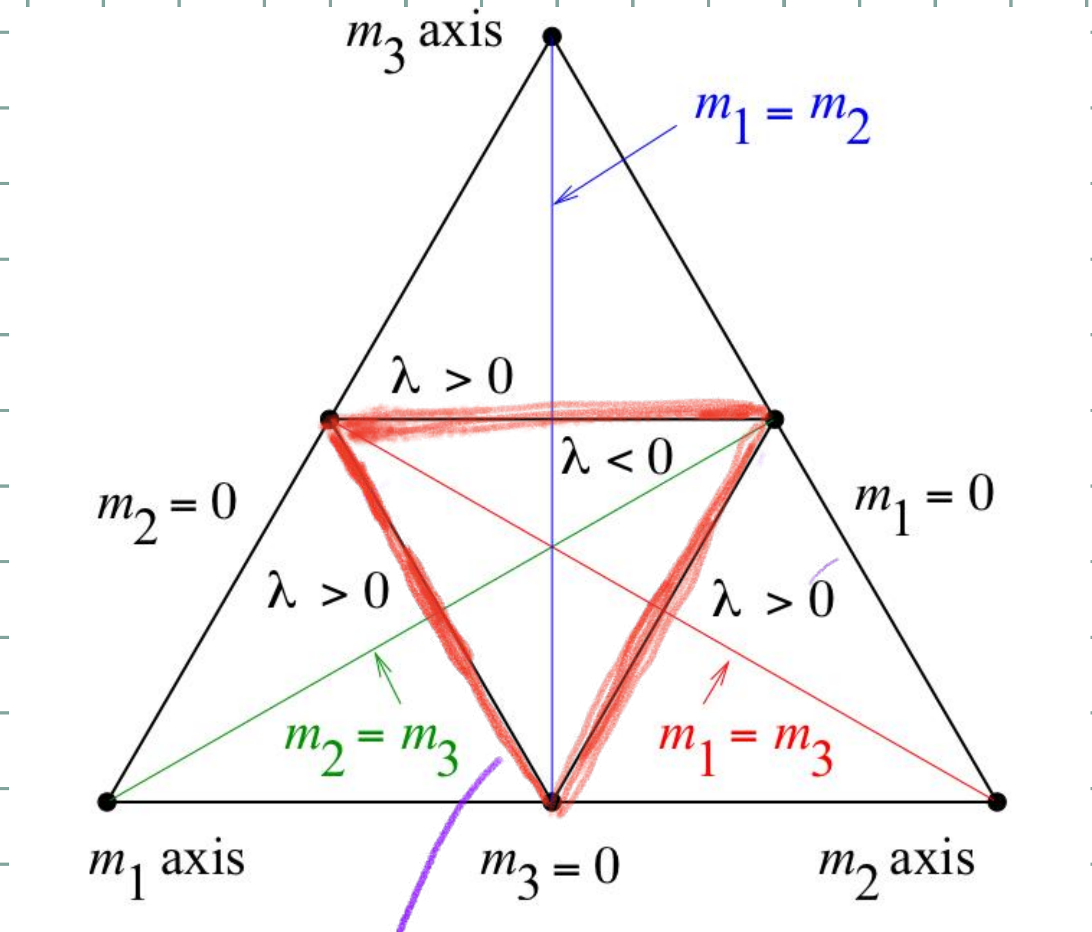
(NB $(**)$ - the only valuable condition)

$\Rightarrow$ Unicity condition (Uniqueness of λ from ± 2)

Let us consider the 2-valued formal group equation: first type:

$z = F(x,y)$, $z^2 - 2(x+y)z + (x-y)^2 = 0$

Another words, $\boxed{\lambda(x,y,z) = 0}$ - two-valued formal group law.



— treshold associativity $\Rightarrow$ algebraic 2-valued group laws

Threshold conditions:

$m_1 = (m_2 \pm m_3)^2$, $m_2 = (m_1 \pm m_3)^2$ $m_3 = (m_1 \pm m_2)^2$

What is the correspondence 2-valued group?

Reminder: Two-valued group - A set X with 2-valued multiplication law

$* : X \times X \longrightarrow Sym^2 X$

with a "unit" (neutral) element $e \in X$,
with an "inverse" $x \to x^{-1}$,
which satisfies the following axioms:

- Associativity: $\forall x, y, z \in X$ the 4-elemented sets
  $(x*y)*z = x*(y*z)$ coincide.
- Strong unity: $\forall x \in X$
  $e*x = x*e = [x, x]$ (Nb!)
- Existence of inverse: $\forall x \in X$ each of two sets $x * x^{-1}$ and $x^{-1} * x$ contains e.

Our (toy) example:

Consider $(\mathbb{C}, +)$ with order 2 automorphism

$z \to \varepsilon z$, $\varepsilon^2 = 1$

2-valued group structure on $\mathbb{C}/\mathbb{Z}_2$:

$x, y \in \mathbb{C}$ $\quad x * y = [z_1, z_2] = [(\sqrt{x}+\sqrt{y})^2, (\sqrt{x}-\sqrt{y})^2]$

The unity e is 0, the inverse

$z \to z^{-1}$ is itself z.

Take $S : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$(z_1, z_2) \longrightarrow \sigma_1(z_1, z_2), \sigma_2(z_1, z_2))$$

to get the polynomial $F(x, y, z)$:

$$F(x, y, z) = z^2 - 2(x+y)z + (x-y)^2 =$$

$$= (z + x + y)^2 - 4(xy + yz + zx) =$$

$$= \Delta_t P(t, x, y, z) \text{ with}$$

$$\boxed{P(t, x, y, z) = t^2 + (x+y+z)t + xy + yz + zx}$$

Two roots z_1, z_2 of $F(x, y, z) = 0$

defines 2-valued multiplication

$$z_{1,2} = x+y \pm \sqrt{(x+y)^2 - (x-y)^2} \Rightarrow$$

$$(z_1, z_2) = \left(x+y+2\sqrt{xy},\ x+y-2\sqrt{xy}\right) =$$

$$= \left((\sqrt{x}+\sqrt{y})^2, (\sqrt{x}-\sqrt{y})^2\right) = x^{-1} * y^{-1} = x * y$$

$$F(x,y,z) = \left(z - \left((x)^{-1} * (y)^{-1}\right)_1\right)\left(z - \left((x)^{-1} * (y)^{-1}\right)_2\right) =$$

$$= \left(z - (\sqrt{x}+\sqrt{y})^2\right)\left(z - (\sqrt{x}-\sqrt{y})^2\right)$$

This is exact reincarnation of the Källén function!

$$F(x,y,z) = \lambda(x,y,z) = (z-x-y)^2 - 4xy$$

$$= x^2+y^2+z^2-2xy-2yz-2zx.$$

Moral: Zero locus of the Källén function — 2-valued algebraic symmetric group law.

§ 5 2-Bessel, Kernels, Catalan and Källén ...

"classical" Bessel function of order 0:

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x^2}{4}\right)^n$ solves standard

Bessel ($\nu = 0$) equation:

$$\boxed{u f'' + f' + u f = 0}$$

- Change of variables:

$x = 2i\sqrt{u}$, $J_0(2i\sqrt{u}) = I_0(2\sqrt{u}) \Rightarrow$

$J^{(2)}(u) := I_0(2\sqrt{u}) = \sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} = {}_0F_1\left(\begin{matrix} - \\ 1 \end{matrix} \middle| u\right)$

- unique holomorphic solution of

2-Kloosterman / 2-quantum equation

$$(u f)'' - u f = 0 \Leftrightarrow \boxed{\theta^2 f = u f} \quad (\ast\ast\ast)$$

$\left(\theta := u \frac{d}{du} \right)$

The function $J^{(2)}(u)$ is the

1-eigenfunction of Buchstaber-Kholodov shift operator

$d_1 := u \frac{d^2}{du^2} + \frac{d}{du}$

Cauchy problem for 2-valued formal groups (Sturm-Liouville) case

V. M. Buchstaber (1975): If the formal 2-valued law is given by

$$z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0$$

then the canonical shift operator has the form $d_1 = \frac{1}{2}\varphi_1 \frac{d}{dx} + \frac{1}{8}\varphi_2(x)\frac{d^2}{dx^2}$

$$\varphi_1 := \frac{\partial \Theta_1}{\partial y}\Big|_{y=0} \qquad \varphi_2 = \frac{\partial \sigma}{\partial y}\Big|_{y=0},$$

and $\sigma = \Theta_1^2 - 4\Theta_2$.

$\Theta_1 = 2(x+y) \Rightarrow \varphi_1 = 2x$

$\Theta_2 = (x-y)^2 \Rightarrow \sigma = 4(x+y)^2 - 4(x-y)^2 = 4x\,4y$

$\Rightarrow \varphi_2 = \frac{1}{8}\cdot 16x = 2x \Rightarrow$

$$d_1 = x\frac{d}{dx}$$

Reflections / Speculations

The 1st Lie theorem garanties

$\forall \xi \in \mathfrak{g} \rightsquigarrow$ 1-parametric subgroup

$\mathfrak{g} = Lie(G)$, $\exp(t\xi) \subset G$

In the Bessel case the order 2 DE (because of $x\frac{d^2}{dx^2}$) only one regular solution from two specifies

B. M. Levitan / V.M. Buchstaber:

The role of 2-valued group theory?

It provides a specificity of one (analytic) solution from 2 possible

$$\left(T_x^y J_0\right)(x) = J_0(y) J_0(x)$$

Generalized Shift operators

In classical analysis: shift operator

$(T_y f)(x) = f(x+y)$ has a crucial property: $e^{\lambda x}$ - are eigenfunctions:

$$T_y(e^{\lambda x}) = e^{\lambda(x+y)} = \underbrace{e^{\lambda y}}_{\text{eigenvalue}} \cdot e^{\lambda x}.$$

Delsarte - Levitan: Different family

$\varphi_\lambda(x)$ are eigenfunctions of "generalized shift operators" $\Rightarrow$ exponential functions are replaced by Bessel, Legendre etc.

The generalized shift operator T^y is defined by the requirement that the generalized eigenfunctions $\varphi_\lambda(x)$ are also eigenfunctions of T^y:

$$\boxed{T^y(\varphi_\lambda)(x) = \varphi_\lambda(y)\,\varphi_\lambda(x)}$$

The canonical example:

Differential operator: $\mathcal{L}_0 := \frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)$.

Eigenfunctions: $\varphi_0(x) = J_0(x)$
$(\lambda = 1)$

Generalized shift operator (Delsarte's Bessel)

$$(T^y f)(x) = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2})}\int_0^{\pi} f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)d\theta$$

Product formula (Sonin - Gegenbauer)

$$J_0(x)J_0(y) = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2})}\int_0^{\pi} J_0\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)d\theta$$

$$\Downarrow$$

Integral representation – multiplicative kernels

Exponents:

$$T^y[e^{\lambda x}] = e^{\lambda x}\cdot e^{\lambda y} = \int_{\mathbb{R}}\delta(z-x-y)\,e^{\lambda z}dz$$

$$\Rightarrow K(x,y|z) = \delta(z-x-y).$$

$$e^{\lambda(x+y)} = \frac{1}{2\pi i} \oint \frac{e^{\lambda z}\, dz}{z-(x+y)}$$

Cauchy integral: $|z-(x+y)| = \varepsilon$, $K = \frac{1}{z-(x+y)}$

Bessel kernel (Sonin - Gegenbauer)

$$T^y\left(J_0(x)\right) = J_0(y)\, J_0(x) = \frac{1}{2\pi} \int_{x-y}^{x+y} \frac{J_0(z)\, z\, dz}{\sqrt{\lambda(x^2, y^2, z^2)}}$$

Again, Källén function!

Remark: The same kernel (the same Källén function takes place for the second (non-analytic) solution of the "quantum" 2-Bessel ⁎⁎⁎ - Macdonald function $K_\nu(z)$, $\nu \in \mathbb{C}$.

The only difference is - the integration path and replacem.

$\cos\theta \longmapsto \cosh\Theta$:

$$K_0(x)K_0(y) = \int_0^\infty K_0\left(\sqrt{x^2+y^2+2xy\cosh\theta}\right) d\theta$$

with the constraint:

$$\left|\arg\left(\sqrt{x^2+y^2+2xy\cosh\Theta}\right)\right| < \pi/2$$

Then

$$K_0(x)K_0(y) = \int_0^\infty \frac{K_0(z)\, z\, dz}{\sqrt{\lambda(x^2,y^2,z^2)}} \;;$$

Remark: Both formulas are not only classically known in analysis and special functions but they are a (small) part of product formulas for spherical functions related to Gelfand pair (G,K) where $G = U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F})$

$$K = U_n(\mathbb{F})$$

The kernel enters as the measure

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\, \nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta)$$

(Gelfand-Berezin, Proc. MMS, 1956)

Duplication formulae

It is easy to check that the "Kernel function" $K(x,y,z) = \frac{1}{\sqrt{\lambda(x,y,z)}}$ in its turn satisfies the 1st order DE:

$$\begin{cases} \lambda(x,y,z)\,\dfrac{dK}{dx} + (z-x-y)K = 0 \\ K(0,y,z) = \dfrac{1}{z-y} \end{cases}$$

(the kernel of $e^{\lambda x}$)

The "2-Bessel" function $J^{(2)}(x) = I_0(2\sqrt{x})$, can be written (as the degenerate hypergeometric ${}_0F_1(\,-\,;1;x)$ as

$$J^{(2)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \Longrightarrow$$

$$J^{(2)}(x)\,J^{(2)}(y) = \sum_{m,n} \binom{m+n}{n}^2 \frac{x^n y^m}{(m+n)!^2} =$$

$$= \frac{1}{2\pi i}\oint K(x,y,z)\,J^{(2)}(z)\,\frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \oint H\left(\frac{x}{z}, \frac{y}{z}\right) J^{(2)}(z) \frac{dz}{z},$$

$$H\left(\frac{x}{z}, \frac{y}{z}\right) = \sum_{n,m} \binom{n+m}{n}^2 \left(\frac{x}{z}\right)^n \left(\frac{y}{z}\right)^m$$

Take $z = 1$

$$H(x, y) = \frac{1}{\sqrt{\lambda(x, y, 1)}}$$

Let's study the behavior of the function $K(x, y, 1)$ on the "diagonal"

$x = y$, $K(x, x, 1) = \frac{1}{\sqrt{\lambda(x, x, 1)}} =$

$$\sqrt{\lambda(x, x, 1)} = \sqrt{1 - 4x} \Rightarrow$$

$$\left(J^{(2)}(x)\right)^2 = \frac{1}{2\pi i} \oint K\left(\frac{x}{z}, \frac{x}{z}, 1\right) J^{(2)}(z) \frac{dz}{z}$$

$$\left[y^{(2)}(x)\right]^2 = \frac{1}{2\pi i} \oint \frac{1}{\sqrt{1-4\left(\frac{x}{z}\right)}} \, y^{(2)}(z) \, \frac{dz}{z}$$

$\Rightarrow$ the analogue of "Clausen duplication"

$$\left[y^{(2)}(x)\right]^2 = \left(\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}\right)^2 = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \cdot C_n$$

$$C_n = \sum_{k=0}^{n} \binom{n}{k}^2 \Rightarrow \left(\text{Taylor for } \frac{1}{\sqrt{1-4x}}\right)$$

$$K(x,x,1) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} \binom{n}{k}^2\right)}_{\text{central binomial coefficients}} x^n$$

$$C_n = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

$\Rightarrow$ relation with world-known

Catalan numbers:

$$C_n = \frac{1}{(n+1)}\binom{2n}{n} = \frac{c_n}{n+1},$$

Let $C(x) := \sum_{n\geq 0} C_n x^n$ - Catalan GF

Proposition $\left(xC(x)\right)' = \frac{1}{\sqrt{\lambda(x,x,1)}} \Rightarrow$

$$\boxed{C(x) = \frac{1-\sqrt{\lambda(x,x,1)}}{2x}}$$

Out of the "diagonal":

$$\lambda(x,y,1) = (1-x-y)^2 - 4xy = 0$$

$${}_2F_1\left(\begin{matrix}a_1\ a_2\\ b_{11}\end{matrix}\middle| x\right),\ {}_2F_1\left(\begin{matrix}a_1\ a_2\\ b_{12}\end{matrix}\middle| y\right) -$$

"usual" Gauss ergeometry

$$\lambda(x,y,1) = F_C^{2,2}(\bar{a}, B, x, y) = F_4\left(\begin{matrix}\bar{a}\\ \bar{b}\end{matrix}\middle| x, y\right)$$

$$= \sum_{(m,n)\in\mathbb{N}^2} \frac{(a_1)_{n+m}\,(a_2)_{n+m}}{(b_{11})_n\, n!\,(b_{12})_m\, m!}\, x^n y^m$$

$b_{1,1} = b_{12} = 1$, $a_1 = \emptyset$ $a_2 = \emptyset$

" degeneration" to the ${}_0F_1(\overline{}, 1 | x, y)$ functions

Appell hypergeometric

$\lambda(x, y, 1)$ - "finite" ($\mathbb{C}^2$-part $xy\,\lambda(x, y, 1) = 0$)

of singular locus for the Appell rank 4 system

That's all, folks!

(to the end of this day)

No rooms for Källén & dilogarithm, for Källén & and 3-Gaudin branching points (Amburg - Tolstukhin)