Two Approaches to Average Stochastic Perturbations of Integrable Systems

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In symplectic space $(\mathbf{R}^{2n}_{x,y}, dx \wedge dy)$ we consider a Birkhoff-integrable Hamiltonian system — its Hamiltonian H depends only on the actions $I_j = (x_j^2 + y_j^2)/2$. Introducing complex coordinates $v_j = x_j + iy_j$ we write this system in the convenient complex form:

$$\frac{\partial}{\partial t} v_k(t) = i\nabla_k H(I) v_k, \quad k = 1, \dots, n, \quad I = I(v), \quad v(t) \in \mathbf{C}^n. \tag{1}$$

Note that $I_j = \frac{1}{2}|v_j|^2$. Study of small perturbations of this system for large values of time, either globally in $\mathbf{R}^n$, or locally near an equilibrium, is a classical problem of dynamical systems starting the end on XVIII century. There are two main settings:

A) the perturbation is a small smooth vector field,

B) the perturbation is a small Hamiltonian field.

We will consider a third case and examine an ε-small stochastic perturbation of the integrable system (1) for for $0 \le t \le T\varepsilon^{-1}$:

$$\frac{\partial}{\partial t} v_k(t) = i\nabla_k H(I) v_k + \varepsilon P_k(v) + \sqrt{\varepsilon} \sum_{j=1}^{n_1} B_{kj}(v) \dot\beta_j^c(t),$$
$$k = 1, \dots, n, \qquad v(0) = v^0,$$

where $\{\beta_k^c(t), 1 \le k \le n\}$ are standard independent complex Wiener processes. That is, $\beta_k^c(t) = \beta_k^+(t) + i\beta_k^-(t)$, and $\beta_k^\pm(t)$ are standard independent real Wiener processes. Passing to the slow time $\tau = \varepsilon t$ and using vector notation we re-write this system as

$$\dot v(\tau) = i\varepsilon^{-1} diag\{\nabla_k H(I)\} v + P(v) + B(v)\dot\beta^c(\tau), \quad v(0) = v^0, \tag{2}$$

where $v(\tau) \in \mathbf{C}^n$ is a complex $n \times n_1$ matrix and $0 \le \tau \le T$. We impose some mild regularity assumptions on coefficients of this equation and assume that the matrix $B(v)$ is non-degenerate: its rank is n for all v (so $n_1 \ge n$).

A solution of this equation will be denoted $v^\varepsilon(\tau)$. The goal is to study the behaviour of the vector of actions $I^\varepsilon(\tau) = (I_1^\varepsilon, \dots, I_n^\varepsilon)(\tau)$, where $I_j^\varepsilon(\tau) = I\big(v_j^\varepsilon(\tau)\big)$, when $\varepsilon \to 0$, on time-intervals $0 \le \tau \le T$. Following the works [1, 2] we will discuss two approaches to do that.

In the first approach we introduce in $\mathbf{C}^n = \{v = (v_1, \dots, v_n)\}$ the usual action-angle coordinates (I, φ), where $I = (I_1, \dots, I_n) \in \mathbf{R}_+^n$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$: $I_j = \frac{1}{2}|v_j|^2$, $\varphi_j = Arg\, v_j$, $\quad j = 1, \dots, n$ (so $v_j = \sqrt{2I_j}\exp i\varphi_j$). Then v-equation (2) may be re-written in terms of these coordinates. The equations for actions are:

$$\dot I_k^\varepsilon(\tau) = \Re(\bar v_k^\varepsilon\, P_k(v^\varepsilon)) + \sum_l |B_{kl}(v^\varepsilon)|^2 + \sum_{j=1}^{n_1} \Re\big(\bar v_k^\varepsilon B_{kj}(v^\varepsilon)\big)\dot\beta_j(\tau), \qquad (3)$$

where $k = 1, \dots, n$ and $\beta_1(\tau), \dots, \beta_{n_1}(\tau)$ are independent standard real Wiener processes. The φ- equations are:

$$\dot\varphi_k^\varepsilon(\tau) =$$
$$= \varepsilon^{-1}\nabla_k H(I^\varepsilon) + \text{term of order one with strong singularity at } \{I_k^\varepsilon = 0\},$$

$1 \le k \le n$. We have got a *slow-fast system*, where the actions I_k^ε's are *slow variables* and the angles φ_k^ε's are *fast variables.* Then, by the usual logic of averaging in fast-slow systems, we in Eq. (3) write the coefficients of the equations as functions of actions and and angles, and next average them in angles (the matrix-function $B(I, \varphi) = \{B_{kl}\}(I, \varphi)$ has to be averaged by the rules of stochastic averaging). This provides us with averaged equations for actions. For this approach it is shown that the averaged equations for actions describes the limiting dynamics of actions of solutions $v^\varepsilon(\tau)$ as $\varepsilon \to 0$, in sense of distribution, if in (2) $P(v)$ is C^2-smooth and B is a constant matrix. It is very likely that these restrictions are necessary, see [2].

In the second approach we build an effective equation for Eq. (3), following the following procedure: in the Eq. (3) we

– dropp its first term,

– suitably average the other two terms with respect to the action of the torus $\mathbf{T^n} = \{\omega\}$ on $\mathbf{C^n}$ by means of the rotation-operators $\Phi_\omega = diag\left(e^{i\omega_1}, \dots, e^{i\omega_n}\right)$
(again, the matrix-function $B(v) = \{B_{kl}\}(v)$ has to be averaged by the rules of stochastic averaging). Then, without any restrictions on equation (3), we prove that as $\varepsilon \to 0$ the actions of solutions for the effective equation approximate those of solutions $v^\varepsilon(\tau)$, in sense of distributions.

Moreover, it turns out that if the first few moments of the norms $|v^\varepsilon(\tau)|$ are bounded uniformly in $\varepsilon > 0$ and $\tau \ge 0$, then this approximation is uniform in time. We provide easy sufficient conditions which imply these uniform bounds for the moments.

At the end of the talk we will compare these results with what is known for classical perturbative questions A) and B), given at the beginning of this note.

References

[1] Huang G., Kuksin S.B., Piatnitski A., Averaging for stochastic perturbations of integrable systems, *J. Dyn. Diff. Eq.* 2025, vol. 37, pp. 1053–1105.

[2] Guo J., Kuksin S.B., Liu Z., On the averaging theorems for stochastic perturbations of conservative linear systems, arXiv:2504.04379 (2025).