On Geometry and Dynamics of Exotic Rationally Integrable Planar Dual Billiards

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Planar dual billiards generalize usual billiards on the plane and on surfaces of constant curvature and outer planar billiards. They were introduced by S.Tabachnikov [3]. A planar *dual billiard* is a planar curve γ equipped with a family $(\sigma_P)|_{P\in\gamma}$ of projective involutions of the projective lines L_P tangent to γ at P that fix P. A dual billiard is called *rationally integrable,* if there exists a non-constant rational function $R(x,y)$ of two variables, called *first integral,* whose restriction to each tangent line L_P is σ_P-invariant. In the previous author's paper [1] it was shown that rationally integrable dual billiards exist only on conics punctured at k points, $0 \leq k \leq 4$. Their classification given there includes standard examples with quadratic integrals, defined by conical pencils, and an infinite family of exotic examples with minimal degree of integrals being any even number greater than two. The complexified involution family there is rational with at most four simple poles. For each integrable example on a conic γ, the *dual billiard map* is a birational map acting on the two-dimensional phase space: the complex algebraic surface consisting of pairs (Q,P), where $Q \in \mathbb{CP}^2$, $P \in \gamma$ and the line QP is tangent to γ at P. Namely, for every $Q \notin \gamma$ and $P \in \gamma$ at which the involution σ_P is well-defined, the pair (Q,P) is sent to the pair (Q^*,P^*), where $Q^* = \sigma_P(Q)$, Q^*P^* is the tangent line to γ different from Q^*P, and P^* is their tangency point. The phase space is fibered by invariant fibers, each being a ramified double cover over a level curve of the integral.

It is well-known that for a standard integrable dual billiard defined by a generic conical pencil, a generic invariant fiber is a complex torus and the dual billiard map acts there as a translation [2].

In the talk we present the recent author's results on the structure of the exotic rationally integrable dual billiards. For each example we find the type of a generic level curve of the integral and of the corresponding fiber. We show that a generic level curve is rational in most of examples and elliptic in two examples. In the rational case we find explicit rational parametrizations of generic level curves. We present formulas for holomorphic differentials on the elliptic level curves. We describe the dynamics of the dual billiard map along invariant fibers in the phase space.

The research is supported by the MSHE "Priority 2030" strategic academic leadership program.

References

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