Billiards and Families of Quadrics Associated with Integrable Geodesic Flows for Geodesically Equavalent Metrics

Vladislav Kibkalo[1,2], Andrey Konyaev[1,2]

[1] *Lomonosov Moscow State University, Moscow, Russia*
[2] *Moscow Center for Fundamental and Applied Mathematics, Moscow, Russia*

An operator field on a smooth manifold M^n is called a Nijenhuis operator field if its Nijenhuis torsion vanishes $\mathcal{N}_L(\xi,\eta) = [L\xi, L\eta] + L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi, L\eta] = 0$.

Such operators are connected to various geometrical and algebraical constructions, arise in integrable hierarchies and finite-dimensional systems. Recent papers by A. Bolsinov, A. Konyaev and V. Matveev are dedicated to systematic study of their properties and applications [1].

In the talk we apply Nijenhuis geometry to integrability problem of geodesic flows and billiards on a plane. For the Euclidean metric case, well-known Bikhoff conjecture states (skipping some details) that a billiard is integrable if its domain is bounded by confocal quadrics. Several recent papers contain its proof under some additional assumptions, e.g. on the shape of the domain, properties of trajectories or first integral, see [2, 3]. Integrable billiards in confocal domains and their integrable generalizations (both classical and new one, like Vedyushkina billiard books or Fomenko billiards with slipping) turns out to form a "wide" class in the class of nondegenerate integrable systems [4] in the sense from the topological point of view, i.e. properties of their Liouville foliations, see also [5].

To describe new integrable geodesic flows and billiards in 2-dimensional case, we apply recent results [6] obtained from the Nijenhuis geometry and originated from the theory of projectively equivalent metrics, classical subject dated back to Dini.

Recall that two metrics $\tilde{g}, g$ on M^n are called *geodesically equivalent* if they have the same geodesics considered as unparametized curves. Thus operator $L = (\det\tilde{g}/\det g)^{1/(n+1)}\,\tilde{g}^{ik}g_{kj}$ is called *geodesically compatible* with both g and $\tilde{g}$. Condition $\mathcal{N}_L = 0$ is necessary but not sufficient.

Theorem 1. *Each pair of geodesically compatible flat metric g and operator L on R^2 belongs ti one of 15 classes. In appropriate flat coordinates metric g has one of the following forms $dx^2+dy^2, dx^2-dy^2, -dx^2-dy^2, dxdy$ and operator L depends on one or several real parameters.*

The obtained classification of geodesically compatible pairs (g, L) generates classification of integrable geodesic flows with Hamiltonian $H = \frac{1}{2}g^{ij}p_ip_j$ and following first integral F

$$F = \frac{1}{2}K^i_q g^{qj}p_ip_j, \qquad K^i_q := L^i_q - \operatorname{tr} L\, \delta^i_q.$$

Theorem 2. *Geodesic flow of flat metric generated by a geodesically compatible pair (g, L) belongs to one of eight families I-IV and Ia-IVa of pairs (H, F). In the chosen flat coordinates (x, y), function H is described in Theorem 1 and quadratic integral F depends on real parameters.*

Separating coordinates for these systems can be constructed from analysis of eigenvalues λ_1, λ_2 of the operator L. Equation $\chi_l(\lambda) = 0$ determine a quadric on x, y which coefficients depend on λ and parameters of the family. For all values of λ except a finite number of them, such a quadric is nondegenerate for families I-IV and is a parabola for families $Ia - IVa$. Note that these family of quadrics is orthogonal with respect both quadratic forms H, F.

Family I corresponds to a confocal family on Euclidean plane $2H = p_1^2 + p_2^2$ for parameters $a \neq b$. For $a = b$, they are concentric circles and radii: instead of a levels of a constant eigenvalue $\lambda_2 = a$, second separating coordinate is determined by eigenvector of nonconstant eigenvalue $\lambda_1 = a - x^2 - y^2$. Family II determines family of quadrics confocal for pseudo-Euclidean metric. Billiards in such domains were studied by V. Dragovic and M. Radnovic [7]. Other nondegenerate families III and IV determine families of hyperbolas not familiar to us from literature. Hamiltonian has the form $H = p_1p_2$ and first integrals are the following for $K = \pm 1, a, b \in \mathbb{R}$:

$$2F_{III} = -Kx^2p_1^2 - 2(Kxy - a)p_1p_2 + (1 - Ky^2)p_2^2;$$

$$2F_{IV} = -(a + Kx^2)p_1^2 - 2(Kxy - b)p_1p_2 + (a - Ky^2)p_2^2.$$

Note that foci of the obtained hyperbolas belong to another hyperbola $y^2 - x^2 = 2/K$ for case III and $y^2 - x^2 = 2a/K$ for the case IV. It will be interesting to compare it with the well-known confocal property of quadrics and its pseudo-Euclidean analog.

The studied class of integrable geodesic flows can be generalized by adding appropriate potential fields U and V to H and F such that $\tilde{H} = H + U(x, y)$ and $\tilde{F} = F + V(x, y)$ Poisson commute. The obtained integrability condition on potentials develop the well-known Kozlov equation [8] for billiards in an ellipse to other flat metrics I-IV, Ia-IVa.

$$(a-b)V_{xy}+3(yV_x-xV_y)+(y^2-x^2)V_xy+xy(V_{xx}-V_{yy})=0.$$

Theorem 3. *For a geodesic flow (H, F) from families I-IV, Ia-IVa from Theorems 1-2, its integrable perturbations by analytic potentials can be described as following, where s_1, s_2 are functions on a real variable and λ_1, λ_2 are separating variables of the initial geodesic flow:*

$$U=\frac{s_2(\lambda_2)-s_1(\lambda_1)}{\lambda_2-\lambda_1}, \qquad \mathrm{d}V=K^*\mathrm{d}U \text{ for } K^i_j=L^i_j-(\operatorname{tr}L)\,\delta^i_j.$$

The last point of this approach states that for each rectangular (in λ_1, λ_2 coordinates) domain such a geodesic flow (with or without potential field) generates an integrable billiard. First integrals $\tilde{H}$ and $\tilde{F}$ preserves after reflection from coordinate lines because their forms are diagonal one (in these coordinates) and thus value on a vector does not depend on signs of the latter components.

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References

[1] Bolsinov A. V., Konyaev A. Yu., Matveev V. S., Nijenhuis geometry, *Adv. Math.*, 2022, vol. 394, 108001.

[2] Glutsyuk A. A., On polynomially integrable Birkhoff billiards on surfaces of constant curvature, *J. Eur. Math. Soc.*, 2021, vol. 23, no. 3, pp. 995–1049.

[3] Bialy M., Mironov A. E., The Birkhoff-Poritsky conjecture for centrally-symmetric billiard tables, *Annals of Math.*, 2022, vol. 196, no. 1, pp. 389–413.

[4] Fomenko A. T., Vedyushkina V. V., Billiards and integrable systems, *Russ. Math. Surv.*, 2023, vol. 78, no. 5, pp. 881–954.

[5] Bolsinov A. V., Fomenko A. T., *Integrable Hamiltonian systems. Geometry, topology, classification*, New York: Chapman & Hall/CRC, Boca Raton, FL, 2004.

[6] Bolsinov A. V., Konyaev A. Yu., Matveev V. S., Applications of Nijenhuis geometry II: maximal pencils of multi-Hamiltonian structures of hydrodynamic type, *Nonlinearity*, 2021, vol. 34, no. 8, pp. 5136–5162.

[7] Dragović V., Radnović M., Topological invariants for elliptical billiardsand geodesicsonellipsoids in the Minkowski space, *J.Math.Sci.*, 2017, vol. 223, no. 6, pp. 686–694.

[8] Kozlov V. V., Some integrable generalizations of the Jacobi problem on geodesics on an ellipsoid, *J. Appl. Math. Mech.*, 1995, vol. 59, no. 1, pp. 1–7.