Energy Dissipation in Weakly Damped Hamiltonian Chains

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We consider a Hamiltonian chain of $N \geq 2$ rotators (in general nonlinear) in which the first rotator is damped. We are interested in the dissipation rate of total energy of the chain when the energy is large.

This problem is motivated by nonequilibrium statistical mechanics of crystals where Hamiltonian chains of interacting particles, in which the first and last particles are driven by a stochastic perturbation and damped, are classical models. A fundamental question is whether such a chain is mixing, i.e. is it true that distributions of all solutions converge to a unique stationary measure when time goes to infinity? The chain of rotators gives an example when the mixing property is expected but is very hard to prove: despite many attempts undertaken in the last 20 years, it is established only when $N \leq 4$ [1–3]. The main difficulty is to get good control of the energy decay rate provided by the damping when the energy is large.

We show that time derivative of the total energy H is bounded by $-H^{-2N-3}$ once H is sufficiently large. This upper bound coincides with that obtained earlier in paper [4] on a different time scale, in which the authors also give numerical evidence that this estimate is optimal (so the energy indeed decays very slowly when $H \gg 1$, which explains the difficulty in proving the mixing property). The method employed in [4] is based on a KAM-like procedure, is technically complicated, and works only for very special initial conditions. This does not allow to apply it for proving the mixing property for the (stochastically driven) chains of significant length.

On the contrary, our proof is simple, short and holds for arbitrary initial conditions. We adopt completely different approach going back to Malisoff and Mazenec [5], relying on a method that allows under mild assumptions to explicitly construct a strict Lyapunov function once a non-strict one is given (in our case the latter is the Hamiltonian).

Unfortunately, the obtained Lyapunov function provides sufficiently good control only in absence of the stochastic perturbation. Constructing a Lyapunov function that controls the energy decay in the stochastically driven chain (and hence allows to prove the mixing property) is still an open problem.

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References

[1] Hairer M., Mattingly J.C., Slow energy dissipation in anharmonic oscillator chains, *Comm. Pure Appl. Math.*, 2009, vol. 62, pp. 999–1032.

[2] Cuneo N., Eckmann J.-P., Poquet C., Non-equilibrium steady state and subgeometric ergodicity for a chain of three coupled rotors, *Nonlinearity*, 2015, vol. 28, pp. 2397–2421.

[3] Cuneo N., Eckmann J.-P., Non-equilibrium steady states for chains of four rotors, *Comm. Math. Phys.*, 2016, vol. 345, pp. 185–221.

[4] Cuneo N., Eckmann J.-P., Wayne C. E. Energy dissipation in Hamiltonian chains of rotators, *Nonlinearity*, 2017, vol. 30, R81.

[5] Malisoff M., Mazenec F., *Construction of strict Lyapunov functions*, London: Springer, 2009.