Topology of 4-Manifolds that Admit Non-Singular Flows with Saddle Orbits of the Same Index

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This paper studies regular topological flows f^t defined on closed topological manifolds M^n. The chain recurrent set of such a flow consists of a finite number of topologically hyperbolic fixed points and periodic orbits. Like their smooth analogues — Morse – Smale flows [3] — regular flows possess a continuous Morse-Bott function that decreases outside the chain recurrent set and is constant on the chain components of the flow. This circumstance leads to a close connection between such flows and the topology of the carrying manifold. In particular, the ambient manifold for non-singular flows (regular flows without fixed points), by the Poincare – Hopf formula, has a zero Euler characteristic. The latter property is a criterion for a manifold M^n to admit a non-singular flow in all dimensions except dimension $n = 3$ [1, 2]. Thus, in higher dimensions, any odd-dimensional manifold admits a non-singular flow, and the list of even-dimensional manifolds is quite broad; at the very least, it includes all manifolds of the form $M^{n-1} \times S^1$, where M^{n-1} is any closed $(n-1)$-manifold.

A surprising result of the present paper is the proof of the fact that in dimension 4, all this variety of carrying spaces can only be achieved if the flow has saddle orbits of different Morse indices. Specifically, for dimensional reasons, the Morse index of a saddle orbit of a non-singular flow $f^t : M^4 \to M^4$ can only take two values, 1 or 2. We prove that non-singular 4-flows with saddle orbits of the same Morse index exist only on skew or direct products of the 3-sphere and the circle, i.e., $M^4 \cong S^3 \tilde{\times} S^1$ or $M^4 \cong S^3 \times S^1$.

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References

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