Integrable Cases of a Heavy Rigid Body with a Gyrostat and Contact Magnetic Geodesic and Sub-Riemannian Flows on $V_{n,2}$

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The rank-two Stiefel variety $V_{n,2} = SO(n)/SO(n-2)$ is the variety of ordered sets of two orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2$ in the Euclidean space $(\mathbf{R}^n, \langle\cdot,\cdot\rangle)$. It can be seen also as the unit sphere bundle T_1S^{n-1} with respect to the standard round sphere metric

$$T_1S^{n-1} = \{\mathbf{e}_2 \in T_{\mathbf{e}_1}S^{n-1} \,|\langle \mathbf{e}_2, \mathbf{e}_2\rangle = 1,\, \mathbf{e}_1 \in S^{n-1}\},$$
$$S^{n-1} = \{\mathbf{e}_1 \in \mathbf{R}^n \,|\, \langle \mathbf{e}_1, \mathbf{e}_1\rangle = 1\}.$$

Therefore, it carries the *standard contact form*, the restriction of the Liouville 1-form from $TS^{n-1} \cong T^*S^{n-1}$ to the unit tangent bundle $\alpha = -\mathbf{e}_2 d\mathbf{e}_1|_{V_{n,2}} = -\sum_{i=1}^n e_2^i de_1^i|_{V_{n,2}}$.

Let $\mathcal{H} = \ker\alpha \subset TV_{n,2}$ be the *standard contact distribution*. The fact that $(V_{n,2}, \alpha)$ is a contact manifold is equivalent to the non-degeneracy of the closed two form $\omega_{mag} = d\alpha = d\mathbf{e}_1 \wedge d\mathbf{e}_2|_{V_{n,2}} = \sum_{i=1}^n de_1^i \wedge de_2^j|_{V_{n,2}}$ restricted to $\mathcal{H}$, or, to the condition $\alpha \wedge \omega_{mag}^{n-2} \neq 0$. We refer to ω_{mag} as the *standard contact magnetic form* on $V_{n,2}$.

Thus, we can study the following three natural problems on the rank-two Stiefel variety:

- Magnetic geodesic flows with respect to the magnetic force defined by $\eta\,\omega_{mag}$. Here η is a real parameter representing the strength of the magnetic field, and for $\eta = 0$ we have the usual geodesic flows.
- Sub-Riemannian magnetic geodesic flows, with the sub-Riemannian structures defined on $\mathcal{H}$ and other $SO(n)$-invariant bracket generating distributions.
- Natural mechanical systems with influence of the magnetic field defined by $\eta\,\omega_{mag}$.

We prove the integrability of magnetic geodesic flows of $SO(n)$-invariant Riemannian metrics on the rank two Stefel variety $V_{n,2}$ with respect to the magnetic field $\eta\, d\alpha$, where α is the standard contact form on $V_{n,2}$ and

η is a real parameter. Also, we prove the integrability of magnetic sub-Riemannian geodesic flows for $SO(n)$-invariant sub-Riemannian structures on $V_{n,2}$. All statements in the limit $\eta = 0$ imply the integrability of the problems without the influence of the magnetic field. We also consider integrable pendulum-type natural mechanical systems with the kinetic energy defined by $SO(n) \times SO(2)$-invariant Riemannian metrics. For $n = 3$, using the isomorphism $V_{3,2} \cong SO(3)$, the obtained integrable magnetic models reduce to integrable cases of a motion of a heavy rigid body with a gyrostat around a fixed point: Zhukovskiy – Volterra gyrostat, the Lagrange top with a gyrostat, and the Kowalevski top with a gyrostat. As a by-product we obtain the Lax presentations for the Lagrange gyrostat and the Kowalevski gyrostat in the fixed reference frame (dual Lax representations).

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