Cusps of Caustics by Reflection, Results and Conjectures

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The "Last Geometric Statement of Jacobi" asserts that the conjugate locus of a non-umbilic point on a triaxial ellipsoid has exactly four cusps. Proved only in this century, this result conjecturally holds for the loci of the 2nd, 3rd, etc., conjugate points as well.

I shall discuss a billiard version of this problem: the surface is replaced by a plane oval, and the conjugate loci is replaced by the 1st, 2nd,... caustics by reflection with the radiant point located inside the oval. For every oval, the caustics by reflection have at least four cusps, and if the oval is an ellipse then, conjecturally, this number is exactly four. I shall present a partial result in this direction.

This problem has many extensions, for example, to Finsler billiards associated with a projective Finsler metric (the subject of Hilbert's 4th Problem) and to magnetic billiards. I shall explain why each caustic by reflection in the former case has at least four cusps. The four cusp result in the case of magnetic billiards follows from still another 4-point theorem, a conjectural strengthening of the classical 4-vertex theorem of Mukhopadhyaya.

References

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