On the Isomorphism of the Problems of the Rubber Rolling of Bodies of Revolution and on the Dynamics of a Rubber Torus

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Consider the rolling motion of a heavy rigid body of revolution on a plane. Assume that the body rolls on the plane without slipping (the velocity of the point of contact is zero) and without spinning (the projection of the angular velocity of the body onto the vertical is zero) and that it moves with only one point, P, in contact with the supporting plane.

The equations of motion are reduced to a conformally Hamiltonian system with two degrees of freedom with the Hamiltonian

$$H = \frac{1}{2}\left(\frac{J(\vartheta)^2}{B(\vartheta)}p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}\right) + U(\vartheta)$$

and the equations

$$\dot{\vartheta} = \frac{1}{J(\vartheta)}\frac{\partial H}{\partial p_\vartheta}, \quad \dot{\varphi} = \frac{1}{J(\vartheta)}\frac{\partial H}{\partial p_\varphi}, \quad \dot{p}_\vartheta = -\frac{1}{J(\vartheta)}\frac{\partial H}{\partial \vartheta}, \quad \dot{p}_\varphi = -\frac{1}{J(\vartheta)}\frac{\partial H}{\partial \varphi}. \tag{1}$$

Here the following notation is used:

$$J(\vartheta) = \sqrt{i_1\cos^2\vartheta + i_3\sin^2\vartheta + m(\chi_1(\vartheta)\sin^2\vartheta + \chi_2(\vartheta)\cos\vartheta)^2},$$
$$B(\vartheta) = i_1 + m(\chi_1(\vartheta)^2\sin^2\vartheta + \chi_2(\vartheta)^2),$$

the potential energy $U = -mg(\boldsymbol{r}, \boldsymbol{\gamma})$ is expressed in terms of $\chi_1(\vartheta)$, $\chi_2(\vartheta)$ as follows:

$$U(\vartheta) = -mg(\chi_1(\vartheta)\sin^2\vartheta + \chi_2(\vartheta)\cos\vartheta),$$

and $\chi_1(\vartheta)$, $\chi_2(\vartheta)$ are arbitrary functions that define the form of the body of revolution and are related by

$$\frac{d\chi_2(\vartheta)}{d\vartheta} = -\sin\vartheta\chi_1(\vartheta) - \frac{\sin^2\vartheta}{\cos\vartheta}\frac{d\chi_1(\vartheta)}{d\vartheta}.$$

In the case of a body of revolution the variable φ is cyclic, and hence p_φ is an integral of motion. Thus, equations (1) admit two integrals of motion and an invariant measure (since they are conformally Hamiltonian). In addition, it is easy to show that the integral manifolds of the system under consideration are bounded. Consequently, by the Euler – Jacobi theorem (see, e.g., [1]), in nonsingular cases these manifolds are two-dimensional tori, and the system (1) can be reduced to quadratures.

This paper addresses the question of the existence of different axisymmetric bodies whose foliations on invariant manifolds are identical.

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References

[1] Bolsinov A. V., Borisov A. V., Mamaev I. S., Topology and stability of integrable systems, *Russian Math. Surveys*, 2010, vol. 65, no. 2, pp. 259–318.