On Global Resonances in Area-Preserving Maps Leading to Infinitely Many Elliptic Periodic Points and Poincaré Problems in Celestial Mechanics

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We consider analytic (or C^r, $r \geq 3$) two-dimensional area-preserving maps that have a quadratic homoclinic tangency of invariant manifolds of a saddle fixed point. For such maps, we specify *homoclinic resonance conditions* under which the map has, in a small neighborhood of the orbit of homoclinic tangency, *a countable set of generic (KAM-stable) elliptic points of all successive periods* $k_0, k_0 + 1, \ldots,$ *starting from some* k_0.

The set of such trajectories forms a specific cluster of an (infinitely) large number of generic elliptic points. This cluster is not structurally stable as a whole even with respect to conservative perturbations. However, if we mean "any given finite number of points", then it becomes stable with respect to this property. Therefore, we can ask about a possibility of observing this phenomenon. The question of where this can be observed, as seems to us, can receive an extremely unexpected answer.

The fact is that the problem under consideration, if we consider it in retrospect, is directly related to those problems of celestial mechanics that H. Poincaré considered in his famous "New Methods of Celestial Mechanics" [1], and in particular, to his planar circular restricted problem of three bodies. (Note that this problem is cardinally simpler than the general problem of three bodies: instead of nine degrees of freedom, it has only two, and moreover can be reduced to the consideration of a two-dimensional area-preserving map). In this problem, each elliptic periodic orbit can be compared with some cosmic body, not necessarily large, it can be a stone, a piece of ice, etc. Individually, such a body can be "invisible", but when many of them gather in one place, then this can already be an observable cosmic object. Objects of this type are well known, the most famous being the ring of Saturn.

It seems clear that some global resonance must be the cause of such a phenomenon. What its nature is, if so many small bodies are involved in this resonance, is not very clear. But in light of this work, we can assume that this is something like the homoclinic resonance we have considered, which can arise in the problem of celestial mechanics that Poincaré once considered.

Note that the considered resonance condition has a *bifurcation codimension equal 2*, since it includes only two "equality-type" assumptions: that the homoclinic tangency is quadratic and that some invariant of the homoclinic structure is zero. Both of the latter conditions are effectively verifiable, since they involve calculating only certain coefficients at constant, linear, and quadratic terms in the Taylor series of the map at the homoclinic point.

General remarks.

1) *Mathematical.*

(A) We have found effectively verifiable conditions under which two-dimensional area-preserving maps have infinitely many generic elliptic periodic points.

(B) We have shown that such points form a certain cluster in which all its points are ordered by their orbits with periods $k_0, k_0 + 1, \ldots$

(C) All points are concentrated in one region: they run for a period along a neighborhood of the homoclinic orbit.

(D) *Proof outline.* The global resonance is manifested in the fact that all first-return maps T_k near a homoclinic point, where $k = k_0, k_0 + 1, \ldots,$ are the return times to its neighborhood, in some (rescaled) coordinates have the same form of the quadratic Hénon map

$$\bar{x} = y,\ \bar{y} = M - x - y^2 \ ,$$

up to asymptotically small as $k \to \infty$ terms. It is well known that this conservative Hénon map at $-1 < M < 3$ has an elliptic fixed point, which is KAM-stable at $M \neq 0; 3/5$ (i.e., except for 1:4 and 1:3 resonances). Automatically, the initial map under consideration will have infinitely many elliptic periodic orbits of all consecutive periods, starting with k_0.

(E) *Other examples*. It is interesting that such type global resonance, only of *bifurcation codimension 1*, takes place in the case of symmetric cubic homoclinic tangencies in the case of two-dimensional reversible maps. Here, the condition of global resonance holds automatically at the moment of tangency and all the first-return maps Tk in some (rescaled) coordinates will be asymptotically close to the cubic Hénon map

$$\bar{x} = y,\ \bar{y} = My - x + \alpha y^3 \ ,$$

where $\alpha = \pm 1$ depending on the type of tangency. This map has always the fixed point $O(x = 0, y = 0)$ that is generic elliptic for $-2 < M < 2$ and $M \neq \{-1; 0\}$.

2) *Physical (celestial mechanics).* It seems that, following the approach of Poincaré, who combined Hamiltonian dynamics and Celestial mechanics,

we have proposed some very simple mechanism for the emergence of global resonances involving a large number of small cosmic bodies in one big cluster.

This is joint work with M. Gonchenko. Some results on this topic was published in [2, 3].

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References

[1] Poincaré J. H., Selected works, Moscow: Nauka, 1971–1974 (in Russian).

[2] Gonchenko S. V., On two-dimensional area-preserving maps with homoclinic tangencies, *Dokl. RAN*, 2001, vol. 378, no. 6, pp. 727–732.

[3] Gonchenko S. V., Gonchenko M. S., On cascades of elliptic periodic points in two-dimensional symplectic maps with homoclinic tangencies, *Regul. Chaotic Dyn.*, 2009, vol. 14, no. 1, pp. 116–136.