Dynamics of a Rotating Test Body in the Schwarzschild Metric

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Consider a test body that moves in space-time with a given metric $g_{\alpha\beta}$. Let p^α be the momentum of the test body and let $S^{\alpha\beta}$ be its tensor of angular momentum. The motion of this body in pole-dipole approximation is described by the Mathisson—Papapetrou equations [1, 2]:

$$\frac{Dp^\alpha}{d\tau} = -\frac{1}{2} R^\alpha{}_{\lambda\mu\nu} u^\lambda S^{\mu\nu},$$
$$\frac{DS^{\alpha\beta}}{d\tau} = p^\alpha u^\beta - u^\alpha p^\beta,$$

where $R^\alpha{}_{\mu\nu\lambda}$ is the tensor of curvature and the covariant derivative is introduced:

$$\frac{Dp^\alpha}{d\tau} = \frac{dp^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} p^\mu u^\nu, \quad \frac{DS^{\alpha\beta}}{d\tau} = \frac{dS^{\alpha\beta}}{d\tau} + \Gamma^\alpha{}_{\mu\nu} S^{\mu\beta} u^\nu + \Gamma^\beta{}_{\mu\nu} S^{\alpha\mu} u^\nu,$$

here $\Gamma^\alpha{}_{\mu\nu}$ is the Christoffel symbol and u^α is the velocity of the point in the body that can be defined from the Tulczyjew condition [3] $S^{\alpha\beta} p_\beta = 0$.

In the general case, this problem reduces to investigating a Hamiltonian system with six degrees of freedom. For the Schwarzschild metric, this system has additional integrals of motion and admits reduction by three or four degrees of freedom. The resulting reduced system possesses invariant manifolds on which it can be reduced to explicit quadratures. In this paper, a qualitative analysis of the resulting quadratures is carried out.

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References

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[2] Papapetrou A., Spinning test-particles in general relativity. I, Proceedings of the Royal Society of London. *Series A. Mathematical and Physical Sciences*, 1951, vol. 209, no. 1097, pp. 248–258.

[3] Tulczyjew W., Motion of multipole particles in general relativity theory, *Acta Phys. Pol.*, 1959, vol. 18, no. 393, p. 94.