Metric Geometry and Forced Oscillations in Mechanical Systems

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Let M be a Riemannian manifold with metric T. Equivalently, we can say that a Lagrangian mechanical system with configuration space M and kinetic energy T is given. The kinetic energy is a positive definite quadratic form on the generalized velocities:

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\dot{q}_i\dot{q}_j, \quad a_{ij}(q) = a_{ji}(q) \text{ for all } q \in M, 1 \leqslant i, j \leqslant n.$$

Let Q be a generalized force (possibly depending on the generalized velocity $\dot{q}$: $Q = Q(q, \dot{q})$; $Q(q, \dot{q})$ is a covector, for any given q and $\dot{q}$). If we consider the generalized force to be non-autonomous, we should add the time dependence: $Q = Q(q, \dot{q}, t)$.

The Lagrange equations can be written in the usual way:

$$[T] = Q,$$

where $[\,\cdot\,]$ denotes the Lagrangian derivative.

Let Q now be a τ-periodic in t generalized force, i.e., for any q, $\dot{q}$ and $t \in \mathbb{R}$, the following is satisfied

$$Q(q, \dot{q}, t) = Q(q, \dot{q}, t + \tau).$$

We will study the existence of τ-periodic solutions in the system $[T] = Q$.

Even in the simplest case of a harmonic oscillator when $M = \mathbb{R}$, $T = \dot{q}^2/2$ and $Q = -q + \sin t$ there may be no forced oscillations in the system. Moreover, in this case all solutions are unbounded.

If M is a closed manifold and the generalized forces are small (in the sense that the parameter ε is present in the system and the generalized forces have the form εQ), then for any field Q there exists a τ-periodic solution provided that $\varepsilon > 0$ is small and the Euler characteristic $\chi(M)$ is not zero. The proof of this statement follows from the existence of a positive injectivity radius for the exponential mapping of the geodesic flow.

In the talk, we discuss connections between the behavior of geodesics (i. e., solutions of the equation $[T] = 0$) and the existence of forced oscillations for the equation $[T] = Q$. The main method we use is to consider some modified system that is obtained from $[T] = Q$ by adding dissipative terms defined only for sufficiently large generalized velocities. Under some additional weak assumptions, this allows us to prove the existence of a forced oscillation in the modified system. Further we use the property of the system $[T] = Q$ that for large velocities the trajectories of its solutions are close to those of the system $[T] = 0$ (provided that the right-hand side grows slower than quadratically with the growth of the generalized velocity). This allows us to show that the found solution of the modified system cannot pass through the points where we introduced friction, i. e., the corresponding periodic solution exists in the original system as well.