The Dynamics of an Elastic String under the Action of Dry Friction

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The presence of dry friction fundamentally complicates the analysis of mechanical systems. This complication is a consequence of the non-smooth nature of the forces involved: dry friction is not a continuous function of the velocity, it has a gap at points where relative motion ceases, i.e. velocity is zero. Unlike viscosity, Coulomb friction introduces a force that changes direction instantaneously upon reversal of motion but has magnitude determined by normal forces and dry friction coefficient only when sliding occurs. This mathematical ambiguity – "dry friction to be undetermined" – represents a critical point where standard assumptions about the well-posedness and smoothness of the equations governing mechanical systems break down.

This inherent discontinuity requires special analytical tools beyond classical theory. One significant approach is Filippov's theory of differential inclusions [5, 6]. This theory replaces the Newtonian equation $\dot{x}(t) = f(t, x(t))$ with a condition where the right-hand side is a measurable function. Then the derivative $\dot{x}(t)$ becomes an element of a set defined by f such that values of this function f on null sets, particularly at points of zero velocity, are insignificant. So it effectively capture the non-uniqueness or ambiguity in the friction force direction.

Alternatively, subdifferential operators offer another powerful mathematical concept for modeling dry friction [2, 4]. This tool allows describing dissipative effects without explicitly invoking Coulomb's law with its directional ambiguity. Systems driven by such subdifferential terms have been successfully analyzed to determine qualitative features of their long-term evolution.

While these generalized tools are analytically successful, much applied work relies on approximating the discontinuous friction force by a continuous function [1, 7]. While different approximations may introduce their own specific modeling errors [9], this approach facilitates numerical simulation as it inherently smooths out the sharp transitions present in dry friction.

This work addresses the core mathematical problem presented by dry friction, specifically targeting its impact on infinite-dimensional systems. We consider an oscillating string fixed on the rough table with a general case of dry friction coefficient. This mechanical problem can be formally expressed as a wave equation with a discontinuous term:

$$u_{tt}(t,x) = u_{xx}(t,x) + f(t,x) + \\ + \sigma(t,x,u,u_x,u_t)\left(-\chi_{\{u_t\neq 0\}}(t,x)\frac{u_t(t,x)}{|u_t(t,x)|} + \chi_{\{u_t=0\}}(t,x)\lambda(t,x)\right),$$

where u represents a displacement of string points, f is an external force acting on the string, σ is a nonnegative function that describes dry friction coefficient, χ is a characteristic function of the set ($\chi_B(t,x) = 1$ if $(t,x) \in B$ and 0 otherwise), and λ, $|\lambda| \leq 1$, is a static dry friction force. To solve it we suggest a functional differential interpretation of this problem to extend powerful inclusion theory mentioned above.

The primary contribution is a solution definition and a comprehensive existence proof for solutions governed by the suggested model. The solution in the sense of the suggested definition can be obtained as a passage to the limit in some preliminary approximations that replace the original problem with a sequence of smooth PDE. These smooth non-linear hyperbolic problems can be studied using methods discussed in [8]. Building upon these results important properties such as the uniqueness of the solution and Lipschitz continuity of the corresponding semiflow can be proved. This all establishes that the considered dry friction problem is well-posed.

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