Shilnikov Criteria in the Extended Shimizu – Morioka System

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The Shimizu – Morioka model

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - \lambda y - xz, \\ \dot{z} = -\alpha z + x^2 \end{cases} \tag{1}$$

is a classical example of a system possessing the Lorenz attractor. The study of this system is important since it appears as a normal form for certain classes of codimension-3 bifurcations of equilibrium states and periodic orbits. Like the Lorenz system, system (1) has $\mathbb{Z}2$-symmetry $(-x, -y, z) \mapsto (x, y, z)$. In contrast to the Lorenz system, system (1) has a very complicated boundary of the Lorenz attractor existence region. The boundary contains two important codimension-2 points [1] where conditions of Shilnikov criteria [2] are met. The pair of homoclinic loops of a neutral saddle is observed at the first Shilnikov point, while vanishing of the so-called separatrix value A along the pair of homoclinic loops occurs at the second point. It is important to note that the Lorenz attractor existence region lies on one side from the neutral saddle curve in system (1).

In this work we study the extension of the Shimizu – Morioka system [3]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - \lambda y - xz - Bx^3, \\ \dot{z} = -\alpha z + x^2. \end{cases} \tag{2}$$

System (2) also possesses the same symmetry and can be viewed as a normal form for certain classes of codimension-3 bifurcations of equilibrium states and periodic orbits. We show that for $B = 0.2$ the neutral saddle curve intersects the Lorenz attractor existance region and, as result, two additional homoclinic curves, give rise to 2 more bifurcation points satisfying the conditions of the first Shilnikov criterion. At the first point, the separatrix value lies in the range $0 < A < 1$, so only the region with the Lorenz attractor emerges. At the second point, where $1 < A < 2$, two regions arise simultaneously: one contains the Lorenz attractor, the other contains the Rovella attractor [4].

The Rovella attractor is not pseudohyperbolic, since the corresponding saddle is contracting. We also show that as the parameter B increases, the points satisfying the second Shilnikov criterion, which give rise to the so-called Shilnikov flames, are gradually replaced by points corresponding to the first criterion.

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References

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