Invariant reduction for PDEs. III: Poisson brackets

Kostya Druzhkov

PIMS Postdoctoral fellow

Department of Mathematics and Statistics
University of Saskatchewan

Geometry of Differential Equations, IUM
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A few words on the history of symmetry reduction

Integrable systems: O. I. Bogoyavlenskii and S. P. Novikov (1976) approach via conservation laws $\Rightarrow$ a generalization by O. I. Mokhov (1984)

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Group analysis (symmetry methods): $\leftarrow$ [A. Sjöberg approach (2007) $\Rightarrow$ a generalization by A. H. Bokhari et al (2010)] $\Rightarrow$ a generalization by S. C. Anco and M. L. Gandarias (2020)

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> "*. . . as there is no direct relationship between the Poisson bracket in the functional space $u(x)$ and the Poisson bracket in the finite-dimensional space (p, q) for an Euler–Lagrange type equation $\delta I/\delta u = 0$.*"

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Invariant reduction mechanism in a nutshell

For a system of PDEs

$$\mathcal{E}: \quad F^i = 0\,, \quad D_{x^k}(F^i) = 0\,, \quad \ldots \tag{1}$$

and its evolutionary symmetry $X = E_\varphi|_\mathcal{E}$

$$E_\varphi = \varphi^i \partial_{u^i} + D_{x^k}(\varphi^i)\partial_{u^i_{x^k}} + \ldots = D_\alpha(\varphi^i)\partial_{u^i_\alpha} \tag{2}$$

there is a mechanism of reduction of X-invariant cohomology to the subsystem describing X-invariant solutions

$$\mathcal{E}_X: \quad F^i = 0\,, \quad \varphi^j = 0\,, \quad D_{x^k}(F^i) = 0\,, \quad D_{x^k}(\varphi^j) = 0\,, \quad \ldots \tag{3}$$

The mechanism is based on the observation

$$X|_{\mathcal{E}_X} = 0 \qquad \Rightarrow \qquad \mathcal{L}_X|_{\mathcal{E}_X} = 0 \tag{4}$$

and reduces a “horizontal degree” by one,

$$\mathcal{L}_X\omega = \partial\vartheta \qquad \Rightarrow \qquad 0 = \partial\big(\vartheta|_{\mathcal{E}_X}\big) \tag{5}$$

Jets: notation

Let $\pi\colon E^{n+m} \to M^n$ be a locally trivial smooth vector bundle. Denote by

- $x = (x^1, \ldots, x^n)$ coordinates in $U \subset M$ (independent variables),
- $u = (u^1, \ldots, u^m)$ coordinates along the fibers (dependent variables).
- u^i_α adapted coordinates along the fibers of $\pi_\infty\colon J^\infty(\pi) \to M$ over U.

Here $\alpha = \alpha_1 x^1 + \ldots \alpha_n x^n = \alpha_i x^i$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

$$\pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi)\,, \qquad \pi_k\colon J^k(\pi) \to M \tag{6}$$

Functions and differential forms on $J^\infty(\pi)$:

$$\mathcal{F}(\pi) = \bigcup_{k\geqslant 0} \pi^*_{\infty,k}\, C^\infty(J^k(\pi))\,, \qquad \Lambda^*(\pi) = \bigcup_{k\geqslant 0} \pi^*_{\infty,k}\, \Lambda^*(J^k(\pi)) \tag{7}$$

The Cartan distribution on $J^\infty(\pi)$ is spanned by the total derivatives

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha+x^k}\partial_{u^i_\alpha}\,, \qquad k = 1, \ldots, n \tag{8}$$

Dual description: the ideal of Cartan (contact) forms

$$\mathcal{C}\Lambda^*(\pi) \subset \Lambda^*(\pi)\,, \quad \omega^\alpha_i \theta^i_\alpha \in \mathcal{C}\Lambda^1(\pi)\,, \quad \theta^i_\alpha = du^i_\alpha - u^i_{\alpha+x^k}dx^k \tag{9}$$

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Horizontal forms:

$$\Lambda_h^k(\pi) = \Lambda^k(\pi)/\mathcal{C}\Lambda^k(\pi)\,, \qquad d_h\colon \Lambda_h^k(\pi) \to \Lambda_h^{k+1}(\pi) \tag{10}$$

$$\Lambda_h^k(\pi) \simeq \mathcal{F}(\pi)\cdot \pi_\infty^*(\Lambda^k(M)) \ni \xi_{j_1\ldots j_k}\, dx^{j_1}\wedge \ldots \wedge dx^{j_k} \tag{11}$$

$$d_h(\xi_{j_1\ldots j_k}\, dx^{j_1}\wedge \ldots \wedge dx^{j_k}) = dx^i \wedge D_{x^i}(\xi_{j_1\ldots j_k})\, dx^{j_1}\wedge \ldots \wedge dx^{j_k} \tag{12}$$

Symmetries of $J^\infty(\pi) \simeq$ elements of the $\mathcal{F}(\pi)$-module of characteristics

$$\varkappa(\pi) = \bigcup_{k\geqslant 0} \Gamma(\pi_k^*(\pi)) \qquad \varkappa(\pi) \ni \varphi \quad \Rightarrow \quad E_\varphi = D_\alpha(\varphi^i)\partial_{u^i_\alpha} \tag{13}$$

Let ζ be a locally trivial smooth vector bundle over the same base M, $\mathrm{rank}\,\zeta = \mathrm{rank}\,\pi = m$. Consider the following $\mathcal{F}(\pi)$-modules

$$P(\pi) = \bigcup_{k\geqslant 0} \Gamma(\pi_k^*(\zeta))\,, \qquad \widehat{P}(\pi) = \mathrm{Hom}_{\mathcal{F}(\pi)}(P(\pi), \Lambda_h^n(\pi)) \tag{14}$$

A section $F \in P(\pi)$ defines the corresponding differential equation

$$F = 0 \qquad \Leftrightarrow \qquad F^i(x, u_\alpha) = 0 \tag{15}$$

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The infinite prolongation (the set of formal solutions) $\mathcal{E} \subset J^\infty(\pi)$

$$\mathcal{E}: \qquad D_\alpha(F^i) = 0 \qquad |\alpha| \geqslant 0 \tag{16}$$

is endowed with its Cartan distribution $\mathcal{C}$ and $\mathcal{C}\Lambda^*(\mathcal{E}) \subset \Lambda^*(\mathcal{E})$,

$$\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)/I_\mathcal{E}\,, \quad \Lambda^i(\mathcal{E}) = \Lambda^i(\pi)/(I_\mathcal{E} \cdot \Lambda^i(\pi) + dI_\mathcal{E} \wedge \Lambda^{i-1}(\pi))\,, \tag{17}$$

where $I_\mathcal{E} = \{f \in \mathcal{F}(\pi) : \ f|_\mathcal{E} = 0\}$.

Let $l_F(\varphi) = E_\varphi(F)$. Denote $l_\mathcal{E} = l_F|_\mathcal{E}$.

Regularity (and ℓ-normality) conditions

- $\pi_\mathcal{E}(\mathcal{E}) = M$, where $\pi_\mathcal{E} = \pi_\infty|_\mathcal{E}$.
- The differentials dF^i_r are independent for any $r \in J^\infty(\pi)$ s.t. $F(r) = 0$.
- $f|_\mathcal{E} = 0$ iff $f = \square(F)$ for some total differential operator $\square = \square^\alpha_j D_\alpha$.
- $H^i_{dR}(\mathcal{E}) = 0$ for $i > 0$.
- $A \circ l_\mathcal{E} = 0 \quad \Rightarrow \quad A = 0$ (ℓ-normality).

ℓ-normal equations: notation and regularity conditions

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The Vinogradov $\mathcal{C}$-spectral sequence

Vinogradov's $\mathcal{C}$-spectral sequence $(E_r^{p,q}(\mathcal{E}), d_r^{p,q})$ originates from

$$\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}^2\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}^3\Lambda^\bullet(\mathcal{E}) \supset \dots \tag{18}$$

Here all $d_r^{p,q} \colon E_r^{p,q}(\mathcal{E}) \to E_r^{p+r,\,q+1-r}(\mathcal{E})$ are induced by d,

$$E_0^{p,q}(\mathcal{E}) = \frac{\mathcal{C}^p\Lambda^{p+q}(\mathcal{E})}{\mathcal{C}^{p+1}\Lambda^{p+q}(\mathcal{E})}, \qquad d_0^{p,q} \colon E_0^{p,q}(\mathcal{E}) \to E_0^{p,q+1}(\mathcal{E}) \tag{19}$$

$$E_1^{p,q}(\mathcal{E}) = \ker d_0^{p,q} / \operatorname{im} d_0^{p,q-1}, \qquad d_1^{p,q} \colon E_1^{p,q}(\mathcal{E}) \to E_1^{p+1,q}(\mathcal{E}) \tag{20}$$

Using $\pi_\mathcal{E} = \pi_\infty|_\mathcal{E} \colon \mathcal{E} \to M$, we identify

$$E_0^{p,q}(\mathcal{E}) \simeq \mathcal{C}^p\Lambda^p(\mathcal{E}) \wedge \pi_\mathcal{E}^*(\Lambda^q(M)), \qquad d_0 = dx^k \wedge \mathcal{L}_{\overline{D}_{x^k}} \tag{21}$$

Variational k-forms of $\mathcal{E}$ are elements of

$$E_1^{k,n-1}(\mathcal{E}) = \ker d_0^{k,n-1} / \operatorname{im} d_0^{k,n-2} \tag{22}$$

Presymplectic structures of $\mathcal{E}$ = d_1-closed variational 2-forms.
Conservation laws of $\mathcal{E}$ = variational 0-forms.

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An evolutionary symmetry of $\mathcal{E} \subset J^\infty(\pi)$ is

the restriction $X = E_\varphi|_\mathcal{E}$ of an evolutionary vector field E_φ such that

$$E_\varphi(F)|_\mathcal{E} = 0 \tag{23}$$

If $\pi_{\infty,0}(\mathcal{E}) = J^0(\pi)$, any symmetry is equivalent to some $E_\varphi|_\mathcal{E}$ (or to $\varphi|_\mathcal{E}$).

$$\varkappa(\mathcal{E}) = \varkappa(\pi)/I_\mathcal{E} \cdot \varkappa(\pi)\,, \qquad \widehat{P}(\mathcal{E}) = \widehat{P}(\pi)/I_\mathcal{E} \cdot \widehat{P}(\pi) \tag{24}$$

Let $\nabla\colon \widehat{P}(\mathcal{E}) \to \varkappa(\mathcal{E})$ be a total differential operator such that

$$l_\mathcal{E} \circ \nabla - \nabla^* \circ l_\mathcal{E}^* = 0 \tag{25}$$

There are an extension $\nabla_e\colon \widehat{P}(\pi) \to \varkappa(\pi)$, $\nabla = \nabla_e|_\mathcal{E}$ and an operator $\Delta\colon P(\pi) \times \widehat{P}(\pi) \to P(\pi)$ such that

$$l_F \circ \nabla_e - \nabla_e^* \circ l_F^* = \Delta(F, \cdot) \tag{26}$$

Symmetries and Hamiltonian operators

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For the operator Δ from $l_F \circ \nabla_e - \nabla_e^* \circ l_F^* = \Delta(F, \cdot)$, we put

$$\Delta_\psi(\cdot) \overset{def}{=} \Delta(\cdot, \psi) \tag{27}$$

For $\nabla_e = {\nabla_e}^{ij\,\alpha} D_\alpha$ and $\chi \in \varkappa(\pi)$, we also put $E_\chi(\nabla_e) = E_\chi({\nabla_e}^{ij\,\alpha}) D_\alpha$.

The operator ∇ is a (local) Hamiltonian operator of $\mathcal{E}$ if

for any $\psi_1, \psi_2 \in \widehat{P}(\pi)$, the expression

$$E_{\nabla_e(\psi_1)}(\nabla_e)(\psi_2) - E_{\nabla_e(\psi_2)}(\nabla_e)(\psi_1) - \nabla_e(\Delta^*_{\psi_1}(\psi_2)) \tag{28}$$

vanishes on $\mathcal{E}$.

A cons. law $\xi_1 \in E_1^{0,n-1}(\mathcal{E}) \Rightarrow$ its cosymmetry $\psi_1|_{\mathcal{E}} = \bar{\psi}_1 \in \ker l_{\mathcal{E}}^* \Rightarrow$

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The Poisson bracket of $\xi_1, \xi_2 \in E_1^{0,n-1}(\mathcal{E})$ is

$$\{\xi_1, \xi_2\}_\nabla = \mathcal{L}_{E_{\nabla(\bar{\psi}_1)}} \xi_2 \tag{30}$$

Invariant reduction (classes of differential forms, cohomology)?

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Invariant reduction (classes of differential forms, cohomology)?

The degree-shifted cotangent covering

In geometric terms, the cotangent equation $\mathcal{E}^*$ is

the Euler-Lagrange equation for

$$L = \langle p, F \rangle = p_i F^i dx^1 \wedge \ldots \wedge dx^n \tag{31}$$

Here $p = (p_1, \ldots, p_m)$ are coordinates along the fibers of the densitized dual η to the bundle ζ, $\hat{\pi} = \pi \oplus \eta$,

- $p_{i\,\alpha}$ are odd variables of degree 1
- $\mathcal{E}^*$ is assumed to be ℓ-normal

$\mathcal{E}^* \subset J^\infty(\hat{\pi})$ is given by the infinite prolongation of the system

$$l_F^*(p) = 0\,, \qquad F = 0\,. \tag{32}$$

The algebra $\Lambda^*(\mathcal{E})$ is a direct summand (of internal degree 0) in $\Lambda^*(\mathcal{E}^*)$.
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Elements of the graded-commutative geometry of $J^\infty(\widehat{\pi})$

The algebra $\Lambda^*(\hat{\pi})$ is bigraded, with the bigrading assigned as follows:

$$x^i(0,0)\,,\quad u^i_\alpha(0,0)\,,\quad p_{i\,\alpha}(1,0)\,,\quad dx^i(0,1)\,,\quad du^i_\alpha(0,1)\,,\quad dp_{i\,\alpha}(1,1) \tag{34}$$

The second component of the bidegree is the differential form degree.

Signs in algebraic expressions $\Leftarrow$ Inner product of the bigradings

$$p_i\,du^j = du^j\,p_i\,,\qquad du^i \wedge dp_j = -dp_j \wedge du^i\,, \tag{35}$$

$$p_i\,dp_j = -dp_j\,p_i\,,\qquad dp_i \wedge dp_j = dp_j \wedge dp_i \tag{36}$$

For graded derivations X, Y of $\mathcal{F}(\hat{\pi})$ and any differential form $\omega \in \Lambda^*(\hat{\pi})$:

$$\mathcal{L}_X\,\omega = d(X \lrcorner\, \omega) + X \lrcorner\, d\omega, \tag{37}$$

$$\mathcal{L}_X(Y \lrcorner\, \omega) = [X, Y] \lrcorner\, \omega + (-1)^{|X|\cdot|Y|}\, Y \lrcorner\, (\mathcal{L}_X \omega) \tag{38}$$

For $X = X^i_\alpha \partial_{u^i_\alpha} + X_{i\,\alpha}\partial_{p_{i\,\alpha}}$ and $\omega = du^i_\alpha\,\omega^\alpha_i + dp_{i\,\alpha}\,\omega^{i\,\alpha}$, one has

$$X \lrcorner\, \omega = X^i_\alpha \omega^\alpha_i + X_{i\,\alpha}\omega^{i\,\alpha} \tag{39}$$

Elements of the graded-commutative geometry of $J^\infty(\widehat{\pi})$

The algebra $\Lambda^*(\hat{\pi})$ is bigraded, with the bigrading assigned as follows:

$$x^i(0,0)\,,\quad u^i_\alpha(0,0)\,,\quad p_{i\,\alpha}(1,0)\,,\quad dx^i(0,1)\,,\quad du^i_\alpha(0,1)\,,\quad dp_{i\,\alpha}(1,1) \tag{34}$$

The second component of the bidegree is the differential form degree.

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The main structure on $\mathcal{E}^*$: the canonical variational 1-form

The Green formula on $J^\infty(\pi)$:

$$\langle l_F(\chi), \psi\rangle - \langle \chi, l_F^*(\psi)\rangle = d_h\, \omega_{\chi,\psi} \qquad \forall \chi \in \varkappa(\pi), \psi \in \widehat{P}(\pi) \tag{40}$$

For $\chi \in \varkappa(\pi)$, one can consider the evolutionary vector field on $J^\infty(\hat{\pi})$

$$E_{(\chi,\,0)} = D_\alpha(\chi^i)\partial_{u^i_\alpha} \tag{41}$$

There exists a Cartan n-form $\omega_L \in \mathcal{C}\Lambda^1(\hat{\pi}) \wedge \hat{\pi}^*_\infty(\Lambda^{n-1}(M))$ such that

$$\langle l_F(\chi), p\rangle - \langle \chi, l_F^*(p)\rangle = d_h(E_{(\chi,\,\dots)} \lrcorner\, \omega_L) \qquad \forall \chi \in \varkappa(\pi) \tag{42}$$

Here ω_L is linear in $p_{i\,\alpha}$ and doesn't involve the forms $dp_{i\,\alpha} - p_{i\,\alpha+x^k}dx^k$.

The canonical variational 1-form (of internal degree 1)

The restriction $\omega_L|_{\mathcal{E}^*}$ represents the canonical variational 1-form $\rho \in E_1^{1,n-1}(\mathcal{E}^*)$. The corresponding presymplectic structure is

$$\Omega = d_1\rho \in E_1^{2,\,n-1}(\mathcal{E}^*) \tag{43}$$

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In fact, the form ρ is produced by the Lagrangian $L = p_i F^i dx^1 \wedge \ldots \wedge dx^n$.

However,

as we will see below, the form ρ is important itself.

Let $X = E_\varphi|_\mathcal{E}$ be an evolutionary symmetry of $\mathcal{E}$. There exists a total differential operator $\Phi \colon P(\pi) \to P(\pi)$ such that

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$\chi = \phi \quad + \quad$ integration by parts in

$$\langle l_F(\varphi), p\rangle - \langle \varphi, l_F^*(p)\rangle = d_h(E_{(\varphi,\ldots)} \lrcorner\, \omega_L) \tag{45}$$

show that $(-\varphi, \Phi^*(p))$ is a characteristic of a conservation law of $\mathcal{E}^* \Rightarrow$ the following evolutionary field restricts to a symmetry of $\mathcal{E}^*$

$$E_{(\varphi, -\Phi^*(p))} = \varphi^i \partial_{u^i} - \Phi^*(p)_i \partial_{p_i} + \ldots \tag{46}$$

Denote by $\mathcal{X}$ this restriction = the lift of X. The conservation law with the characteristic $(-\varphi, \Phi^*(p))$ is $\mathcal{X} \lrcorner\, \rho$.

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$$\mathcal{L}_{\mathcal{X}} \rho = 0 \tag{47}$$

In other words,

the canonical variational 1-form ρ is $\mathcal{X}$-invariant for any $X \Rightarrow$ invariant reduction of ρ under $\mathcal{X}$ for any X.

One can show that the lift preserves commutators.

Similarly, as the Green formula shows,

for the operator Δ_p from

$$l_F \circ \nabla_e - \nabla_e^* \circ l_F^* = \Delta(F, \cdot)\,, \qquad \Delta_p(\cdot) \overset{def}{=} \Delta(\cdot, p)\,, \tag{48}$$

the restriction of the evolutionary vector field with the characteristic

$$\left(\nabla_e(p), -\frac{1}{2}\Delta_p^*(p)\right) \tag{49}$$

is a degree-1 symmetry of $\mathcal{E}^*$ ($\Leftarrow$ odd parity of $p_{i\,\alpha}$). We denote it by s_∇.

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The Noether theorem relates s_∇ to the degree-2 conservation law $\mathcal{H}_\nabla$,

$$\mathcal{H}_\nabla = -\frac{1}{2} s_\nabla \lrcorner\, \rho\,, \qquad s_\nabla \lrcorner\, \Omega = d_1 \mathcal{H}_\nabla \tag{50}$$

$\Rightarrow$ s_∇ depends only on ∇ (not on ∇_e). Moreover, the variational bivector $\nabla + \Box \circ l_{\mathcal{E}}^*$ determines the same s_∇ (here $\Box\colon \widehat{\varkappa}(\mathcal{E}) \to \varkappa(\mathcal{E})$ and $\Box^* = \Box$).

If ∇ is a Hamiltonian operator, then s_∇ is a cohomological vector field, i.e.,

$$[s_\nabla, s_\nabla] = 0 \tag{51}$$

Here $[s_\nabla, s_\nabla] = 2 s_\nabla \circ s_\nabla$ since s_∇ is odd.

If $X = E_\varphi|_{\mathcal{E}}$ is a symmetry of $\mathcal{E}$,

we say that ∇ is X-invariant if

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Action of variational bivectors in terms of odd symmetries

Let us recall that the system $\mathcal{E}^*_{\mathcal{X}}$ is determined by

$$l_F^*(p) = 0\,, \qquad F = 0\,, \tag{54}$$

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The variational bivector represented by $\nabla\colon \widehat{P}(\mathcal{E}) \to \varkappa(\mathcal{E})$,

$$l_{\mathcal{E}} \circ \nabla - \nabla^* \circ l_{\mathcal{E}}^* = 0 \tag{56}$$

maps $\{\text{cosymmetries of } \mathcal{E}\} = \ker l_{\mathcal{E}}^* \simeq E_1^{1,n-1}(\mathcal{E})$ to $\{\text{symmetries of } \mathcal{E}\}$.

More specifically, one can show that

for $\nu \in E_1^{1,n-1}(\mathcal{E}) \subset E_1^{1,n-1}(\mathcal{E}^*)$ and the corresponding symmetry $X_{\nabla(\nu)}$,

$$d_1(s_\nabla \lrcorner\, \nu) = \mathcal{X}_{\nabla(\nu)} \lrcorner\, \Omega \tag{57}$$

This can be taken as an equivalent definition of $\mathcal{X}_{\nabla(\nu)}$ and hence of $X_{\nabla(\nu)}$
$\Rightarrow$ We don't need ∇ itself anymore.

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Let us recall the invariant reduction mechanism

Let X be an evolutionary symmetry of $\mathcal{E} \subset J^\infty(\pi) \Rightarrow \mathcal{E}_X \subset \mathcal{E}$ is given by

$$F = 0\,, \qquad \varphi = 0 \tag{58}$$

where $X = E_\varphi|_\mathcal{E}$, while $F = 0$ determines $\mathcal{E}$.

Suppose $\omega \in E_0^{p,q}(\mathcal{E})$ represents an X-invariant element of $E_1^{p,q}(\mathcal{E})$, where $q \geqslant 1$. Then for some $\vartheta \in E_0^{p,q-1}(\mathcal{E})$

$$\mathcal{L}_X\omega = d_0\vartheta \qquad \Rightarrow \qquad d_0\vartheta|_{\mathcal{E}_X} = 0 \tag{59}$$

We denote by $\mathcal{R}_X^{p,q}$ the homomorphism $E_1^{p,q}(\mathcal{E})^X \to E_1^{p,q-1}(\mathcal{E}_X)$, provided it is well-defined. The graded-commutative case (fixed internal degree):

$$(\mathcal{E}, X) \mapsto (\mathcal{E}^*, \mathcal{X}) \tag{60}$$

Below we apply the reduction to invariant elements of $E_1^{*,n-1}$ of the ℓ-normal systems $\mathcal{E}$ and $\mathcal{E}^* \Rightarrow$ it is well-defined. In particular, we get

$$\mathcal{R}_\mathcal{X}(\rho) \in E_1^{1,n-2}(\mathcal{E}^*_\mathcal{X}) \tag{61}$$

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Reduction of formulas

Theorem 1 (fixed internal degree)

Let $\mathcal{E}$ be an infinitely prolonged system of differential equations, and let X be its evolutionary symmetry. Suppose that the invariant reduction is well-defined for $E_1^{p,q}(\mathcal{E}^*)^{\mathcal{X}}$ and $E_1^{p+1,q}(\mathcal{E}^*)^{\mathcal{X}}$. Then on $E_1^{p,q}(\mathcal{E}^*)^{\mathcal{X}}$,

$$\mathcal{R}_{\mathcal{X}}^{p+1,q} \circ d_1 = -d_1 \circ \mathcal{R}_{\mathcal{X}}^{p,q} \tag{62}$$

Theorem 2 (fixed internal degree)

Suppose that X and X_1 are commuting evolutionary symmetries of an infinitely prolonged system $\mathcal{E}$. If the invariant reduction is well-defined for $E_1^{p,q}(\mathcal{E}^*)^{\mathcal{X}}$ and $E_1^{p-1,q}(\mathcal{E}^*)^{\mathcal{X}}$, then on $E_1^{p,q}(\mathcal{E}^*)^{\mathcal{X}}$,

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Now we can reduce the main formula, where $\nu \in E_1^{1,n-1}(\mathcal{E}) \subset E_1^{1,n-1}(\mathcal{E}^*)$

$$d_1(s_\nabla \lrcorner \nu) = \mathcal{X}_{\nabla(\nu)} \lrcorner \Omega \tag{64}$$

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How to interpret the reduced structures?

If all ingredients of $d_1(s_\nabla \lrcorner \nu) = \mathcal{X}_{\nabla(\nu)} \lrcorner \Omega$ are $\mathcal{X}$-invariant, then

$$d_1(s_\nabla|_{\mathcal{E}^*_\mathcal{X}} \lrcorner \mathcal{R}_X(\nu)) = -\mathcal{X}_{\nabla(\nu)}|_{\mathcal{E}^*_\mathcal{X}} \lrcorner \mathcal{R}_\mathcal{X}(\Omega) \tag{65}$$

Perhaps one can similarly define the action of the reduction of ∇ on any $\mu \in E_1^{1,n-2}(\mathcal{E}_X) \subset E_1^{1,n-2}(\mathcal{E}^*_\mathcal{X})$ by

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However, computationally, this formula is not particularly nice and its properties (existence and uniqueness of ?) are not easy to study.

One can potentially overcome these challenges using

a compatibility complex for the linearization of $\mathcal{E}^*_\mathcal{X}$ and the k-line theorem.

But when $\mathcal{E}_X$ is a finite-dimensional smooth manifold, there is a better way through the naive notion of its cotangent equation given in terms of the intrinsic geometry...

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The naive cotangent equation (finite-dimensional case)

Let $\pi_{\mathcal{S}} : \mathcal{S}^{n+N} \to M^n$ be a surjective submersion + a flat connection $\mathcal{C}$. Denote by $D^v(\mathcal{S})$ the module of $\pi_{\mathcal{S}}$-vertical derivations. The decomposition

$$D(\mathcal{S}) = \mathcal{C}D(\mathcal{S}) \oplus D^v(\mathcal{S}) \tag{67}$$

of the module $D(\mathcal{S})$ of derivations on $\mathcal{S} \Rightarrow \mathcal{C}D(\mathcal{S})$ acts on $D^v(\mathcal{S})$ by means of the vertical components of the corresponding Lie derivatives.

$$D^v(\mathcal{S}) \quad \Leftrightarrow \quad \text{fiberwise linear functions on} \quad \tau^*_{\mathcal{S}} : \mathfrak{T}^*\mathcal{S} \to \mathcal{S} \tag{68}$$

Here $\tau^*_{\mathcal{S}} \circ \pi_{\mathcal{S}} : \mathfrak{T}^*\mathcal{S} \to M$ is endowed with the corresponding flat connection.

For $x = (x^1, \ldots, x^n)$ on M and $(x, v) = (x^1, \ldots, x^n, v^1, \ldots, v^N)$ on $\mathcal{S}$

s.t. $\pi_{\mathcal{S}}(x, v) = x$, denote $\varpi_j = \partial_{v^j} \Rightarrow$ for Cartan forms $\gamma^j = dv^j - \Gamma^j_k dx^k$,

$$\mathrm{End}(D^v(\mathcal{S})) \ni \mathbb{I} \quad \simeq \quad \rho_c = \gamma^j \varpi_j \in E_1^{1,0}(\mathfrak{T}^*[1]\mathcal{S}) \tag{69}$$

Kernel distribution of $d\rho_c \in E_1^{2,0}(\mathfrak{T}^*[1]\mathcal{S})$ = Cartan distribution of $\mathfrak{T}^*[1]\mathcal{S}$. Informally, $\rho_c = \mathfrak{T}^*[1]\mathcal{S}$, and [the coefficient of the Cartan form γ^j] $\Leftrightarrow \partial_{v^j}$

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The main idea

We want to take $\mathcal{E}_X$ as $\mathcal{S}$ and

to identify $\mathcal{R}_{\mathfrak{X}}(\rho) \in E_1^{1,n-2}(\mathcal{E}^*_{\mathfrak{X}})$ with the form $\rho_c \in E_1^{1,0}(\mathfrak{T}^*[1]\mathcal{E}_X)$ $\Rightarrow$
to identify $\mathcal{E}^*_{\mathfrak{X}}$ and $\mathfrak{T}^*[1]\mathcal{E}_X$ $\Rightarrow$
to recognize $p_{i\,\alpha}|_{\mathcal{E}^*_{\mathfrak{X}}}$ as $\pi_{\mathcal{E}_X}$-vertical derivations on $\mathcal{E}_X$.

If $\mathfrak{X}$ is a single symmetry (not an appropriate algebra), then $n = 2$.

An isomorphism of bundles $\mathcal{E}^*_{\mathfrak{X}}$ and $\mathfrak{T}^*[1]\mathcal{E}_X$ over $\mathcal{E}_X$ that relates the forms $\mathcal{R}_{\mathfrak{X}}(\rho) \in E_1^{1,n-2}(\mathcal{E}^*_{\mathfrak{X}})$ and $\rho_c \in E_1^{1,0}(\mathfrak{T}^*[1]\mathcal{E}_X)$ is unique (if it exists).

Since $d\mathcal{R}_{\mathfrak{X}}(\rho) = d\rho_c$, such an isomorphism relates the Cartan distributions of $\mathcal{E}^*_{\mathfrak{X}}$ and $\mathfrak{T}^*[1]\mathcal{E}_X$.

(these distributions have the same rank $n \Rightarrow$ they both can be identified as the kernel distributions of the respective forms).

The main idea

We want to take $\mathcal{E}_X$ as $\mathcal{S}$ and

to identify $\mathcal{R}_{\mathfrak{X}}(\rho) \in E_1^{1,n-2}(\mathcal{E}^*_{\mathfrak{X}})$ with the form $\rho_c \in E_1^{1,0}(\mathfrak{T}^*[1]\mathcal{E}_X)$ $\quad\Rightarrow$
to identify $\mathcal{E}^*_{\mathfrak{X}}$ and $\mathfrak{T}^*[1]\mathcal{E}_X$ $\quad\Rightarrow$
to recognize $p_{i\,\alpha}|_{\mathcal{E}^*_{\mathfrak{X}}}$ as $\pi_{\mathcal{E}_X}$-vertical derivations on $\mathcal{E}_X$.

If $\mathfrak{X}$ is a single symmetry (not an appropriate algebra), then $n = 2$.

An isomorphism of bundles $\mathcal{E}^*_{\mathfrak{X}}$ and $\mathfrak{T}^*[1]\mathcal{E}_X$ over $\mathcal{E}_X$ that relates the forms $\mathcal{R}_{\mathfrak{X}}(\rho) \in E_1^{1,n-2}(\mathcal{E}^*_{\mathfrak{X}})$ and $\rho_c \in E_1^{1,0}(\mathfrak{T}^*[1]\mathcal{E}_X)$ is unique (if it exists).

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Reduction of technical formulas (if $\mathcal{R}_{\mathcal{X}}(\rho) = \rho_c$)

First of all, for $\nu \in E_1^{1,n-1}(\mathcal{E}) \subset E_1^{1,n-1}(\mathcal{E}^*)$ we have the two equivalent

$$d_1(s_\nabla \lrcorner \nu) = \mathcal{X}_{\nabla(\nu)} \lrcorner \Omega \quad \Leftrightarrow \quad s_\nabla \lrcorner \nu = -\mathcal{X}_{\nabla(\nu)} \lrcorner \rho \tag{70}$$

If everything is X- ($\mathcal{X}$-) invariant, and ∇ is Hamiltonian, then:

- $[s_\nabla|_{\mathcal{E}^*_{\mathcal{X}}}, s_\nabla|_{\mathcal{E}^*_{\mathcal{X}}}] = 0$ and hence $s_\nabla|_{\mathcal{E}^*_{\mathcal{X}}}$ identifies with a (vertical) Poisson bivector b on the system $\mathcal{E}_X$.
- For the Poisson bivector b and $\beta \in \mathcal{C}\Lambda^1(\mathcal{E}_X) \subset \mathcal{C}\Lambda^1(\mathfrak{T}^*[1]\mathcal{E}_X)$,

$$s_\nabla|_{\mathcal{E}^*_{\mathcal{X}}} \lrcorner \beta \simeq -b(\beta, \cdot) \tag{71}$$

- From $s_\nabla \lrcorner\, d_1\xi = -\mathcal{X}_{\nabla(d_1\xi)} \lrcorner\, \rho$ and $\mathcal{R}_{\mathcal{X}}(\rho) \simeq \mathbb{I} \in \mathrm{End}(D^v(\mathcal{E}_X))$,

$$s_\nabla|_{\mathcal{E}^*_{\mathcal{X}}} \lrcorner\, d\,\mathcal{R}_X(\xi) = \mathcal{X}_{\nabla(d_1\xi)}|_{\mathcal{E}^*_{\mathcal{X}}} \lrcorner\, \mathcal{R}_{\mathcal{X}}(\rho) \simeq X_{\nabla(d_1\xi)}|_{\mathcal{E}_X} \tag{72}$$

i.e., the relation between conservation laws and symmetries of $\mathcal{E}$ given by ∇ is inherited by the reduction, up to sign $\Rightarrow$ the Poisson bracket is.

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Comments on $\mathcal{R}_{\mathcal{X}}(\mathcal{H}_\nabla)$ (provided $\mathcal{R}_{\mathcal{X}}(\rho) = \rho_c$)

The reduced bivector maps constants of X-invariant motion $E_1^{0,0}(\mathcal{E}_X)$ to symmetries of $\mathcal{E}_X$ since

$$\mathcal{R}_{\mathcal{X}}(\mathcal{H}_\nabla) \in E_1^{0,0}(\mathfrak{T}^*[1]\mathcal{E}_X) \tag{73}$$

$\Rightarrow$ Poisson bracket on $E_1^{0,0}(\mathcal{E}_X)$.

Since we have to compute $\mathcal{R}_{\mathcal{X}}(\rho)$ at first, direct computation of $\mathcal{R}_{\mathcal{X}}(\mathcal{H}_\nabla)$ is unnecessary due to

$$\mathcal{H}_\nabla = -\frac{1}{2} s_\nabla \lrcorner\, \rho \quad \Rightarrow \quad \mathcal{R}_{\mathcal{X}}(\mathcal{H}_\nabla) = \frac{1}{2} s_\nabla|_{\mathcal{E}^*_{\mathcal{X}}} \lrcorner\, \mathcal{R}_{\mathcal{X}}(\rho) \tag{74}$$

The RHS and the identification $\mathcal{R}_{\mathcal{X}}(\rho) \simeq \mathbb{I}$ lead to an explicit formula.

As the variational Schouten bracket $[s_\nabla, \cdot]$ shows,

s_∇ is $\mathcal{X}$-invariant, e.g., if X is a Hamiltonian symmetry of $\mathcal{E}$ (w.r.t. ∇).

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Theorem. Let $X = E_\varphi|_\mathcal{E}$ be a symmetry of a $(1+1)$-evolution system

$$F = 0\,, \qquad F^i = u^i_t - f^i \tag{75}$$

with m dependent variables $u^1, \ldots, u^m$ s.t., for some $k_1, \ldots, k_m \geqslant -1$,

$$\det\left(\frac{\partial \varphi^j}{\partial u^i_{k_i+1}}\right) \neq 0 \qquad \text{on} \quad \{\varphi = 0\} \quad \text{(no prolongation)} \tag{76}$$

and none of φ^j depend on u^i_t, $u^i_{k_i+2}$, or their total derivatives. If

$$\nabla(\bar\psi)^i = \nabla^{ij\,k}\bar D_{kx}(\bar\psi_j)\,, \qquad \bar D_x = D_x|_\mathcal{E}\,, \qquad \bar\psi = \psi|_\mathcal{E} \tag{77}$$

is an X-invariant Hamiltonian operator of $\mathcal{E}$, then on the algebra $E^{0,0}_1(\mathcal{E}_X)$, there exists a Poisson bracket inherited from $\{\cdot,\cdot\}_\nabla$. In coordinates of the form t, x, $u^i_0, \ldots, u^i_{k_i}$ on $\mathcal{E}_X$, the bivector defining this bracket is given by

$$-\frac{1}{2}\Big(\hat\nabla^{ij\,k}w_{j\,k}\wedge\partial_{u^i_0} + \mathcal{L}_{\widetilde{D}_x}(\hat\nabla^{ij\,k}w_{j\,k})\wedge\partial_{u^i_1} + \ldots + \mathcal{L}^{k_i}_{\widetilde{D}_x}(\hat\nabla^{ij\,k}w_{j\,k})\wedge\partial_{u^i_{k_i}}\Big) \tag{78}$$

where $\hat\nabla^{ij\,k} = \nabla^{ij\,k}|_{\mathcal{E}_X}$, $\widetilde{D}_x = D_x|_{\mathcal{E}_X}$, $w_{j\,k} = \mathcal{L}^{k}_{\widetilde{D}_x}(w_{j\,0})$, and $w_{j\,0}$ are defined by the relations $\partial_{u^i_{k_i}} = -w_{j\,0}\;\partial\varphi^j/\partial u^i_{k_i+1}\big|_{\mathcal{E}_X}$.

Example (the KdV)

Let us consider

$$u_t - (6uu_x + u_{xxx}) = 0\,, \quad \nabla = \bar{D}_x^3 + 4u\bar{D}_x + 2u_x\,, \quad \bar{D}_x = D_x|_{\mathcal{E}} \qquad (79)$$

We take t, x, u, u_x, u_{xx}, ... as coordinates on $\mathcal{E}$. The conservation law corresponding to $\overline{\psi}$ with the component $u_{xx} + 3u^2$ gives rise to $X = E_\varphi|_{\mathcal{E}}$,

$$\varphi = u_{5x} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x \qquad (80)$$

X is Hamiltonian $\Rightarrow \nabla$ is X-invariant. The system $\mathcal{E}_X$:

$$u_t = 6uu_x + u_{xxx}\,, \qquad u_{5x} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = 0 \qquad (81)$$

The variables t, x, u, u_x, u_{xx}, u_{xxx}, $u_{4x} = u_{xxxx}$ are coordinates on $\mathcal{E}_X$.
Let us apply the theorem. Here $\varphi^1 = \varphi$, $k_1 = 4$, $\partial\varphi^1/\partial_{u^1_{k_1+1}} = 1$,

$$\widetilde{D}_x = \partial_x + u_x\partial_u + \ldots + u_{4x}\partial_{u_{xxx}} - (10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x)\partial_{u_{4x}} \qquad (82)$$

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The corresponding vector field $w_{1\,0}$ from $\partial_{u^i_{k_i}} = -w_{j\,0}\ \partial\varphi^j/\partial u^i_{k_i+1}\big|_{\mathcal{E}_X}$ and $w_{1\,1} = [\widetilde{D}_x, w_{1\,0}]$, $w_{1\,2} = [\widetilde{D}_x, w_{1\,1}]$, ... have the form

$$w_{1\,0} = -\partial_{u_{4x}}, \quad w_{1\,1} = \partial_{u_{xxx}}, \quad w_{1\,2} = -\partial_{u_{xx}} + 10u\partial_{u_{4x}}, \tag{83}$$

$$w_{1\,3} = \partial_{u_x} - 10u\partial_{u_{xxx}} - 10u_x\partial_{u_{4x}}, \tag{84}$$

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$$w_{1\,5} = -10u\partial_{u_x} + 10u_x\partial_{u_{xx}} - (10u_{xx} - 70u^2)\partial_{u_{xxx}}, \qquad \ldots \tag{86}$$

(no additional evaluation of vertical components is required).

The bivector from the theorem reads

$$-\frac{1}{2}\big((w_{1\,3} + 4uw_{1\,1} + 2u_x w_{1\,0})\wedge\partial_u + \ldots + \mathcal{L}^4_{\widetilde{D}_x}(w_{1\,3} + 4uw_{1\,1} + 2u_x w_{1\,0})\wedge\partial_{u_{4x}}\big)$$

It defines the Poisson bracket inherited from $\{\cdot,\cdot\}_\nabla$.

The KdV admits the Hamiltonian operator $D_x|_{\mathcal{E}}$, which is also X-invariant.

The operators $D_x|_{\mathcal{E}}$ and ∇ form a Poisson pencil $\Rightarrow$ their reductions form a Poisson pencil. The invariant reduction of $D_x|_{\mathcal{E}}$ results in the bivector

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One can take

$$\omega_L = p\,\theta \wedge dx + \big(6up\,\theta + p\,\theta_{xx} - p_x\theta_x + p_{xx}\theta\big) \wedge dt\,, \tag{88}$$

where $\theta = du - u_x dx - u_t\, dt$, $\theta_x = du_x - u_{xx} dx - u_{tx} dt$, ...
The lift of X to the cotangent equation is the following degree-0 symmetry

$$\mathcal{X} = \varphi\partial_u - l_\varphi^*(p)\partial_p + \dots\,, \tag{89}$$

$$l_\varphi^*(p) = -p_{5x} - 10up_{xxx} - 10u_x p_{xx} - 10u_{xx}p_x - 30u^2 p_x \tag{90}$$

The system $\mathcal{E}^*_{\mathcal{X}}$ is the infinite prolongation of

$$u_t = 6uu_x + u_{xxx},\;\; u_{5x} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0\,, \tag{91}$$

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The reduction of the canonical variational 1-form ρ is given by $\vartheta|_{\mathcal{E}^*_{\mathcal{X}}}$, where

$$\mathcal{L}_{\mathcal{X}}(\omega_L|_{\mathcal{E}^*}) = d_0\vartheta \tag{93}$$

From the dx-component of this equation, one unambiguously finds

$$\vartheta = -\,p\,\bar\theta_{4x} + p_x\bar\theta_{xxx} - (10up + p_{xx})\bar\theta_{xx} + (-10u_x p + 10up_x + p_{xxx})\bar\theta_x - (10up_{xx} + (10u_{xx} + 30u^2)p + p_{4x})\bar\theta$$

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Final comments

I would like to gratefully acknowledge Alexander Verbovetsky for suggesting the idea of reducing Poisson brackets by interpreting them as conservation laws of the cotangent equations.

If $\mathcal{E}_X$ is of infinite type, an interpretation of a reduced Hamiltonian operator via of the geometry of $\mathcal{E}_X$ requires either a compatibility complex for the linearization of $\mathcal{E}^*_{\mathcal{X}}$, or *an intrinsic definition of a cotangent equation*, or ...

(?) The reduction of ρ is related to the reduced variational principle.

(?) The cotangent equation to $\mathcal{E}_{X,X_1}$ is not $\mathcal{E}^*_{\mathcal{X},\mathcal{X}_1}$ if $[X_1, X] = cX$, $c \neq 0$. Apparently, it is given by $\mathcal{E}^*_{\mathcal{X},\mathcal{X}_1 - c\mathbb{X}_\rho}$, where $\mathbb{X}_\rho$ corresponds to ρ: $\mathbb{X}_\rho \lrcorner \Omega = \rho$

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Literature on Reduction Methods

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Thank you!