Almost tetrahedral manifolds

Evgeny Fominykh and Yaroslav Nagibin

St. Petersburg State University

Conference "Low-dimensional topology 2025"
November 5, 2025
St. Petersburg

Ideal triangulations of 3-manifolds with boundary

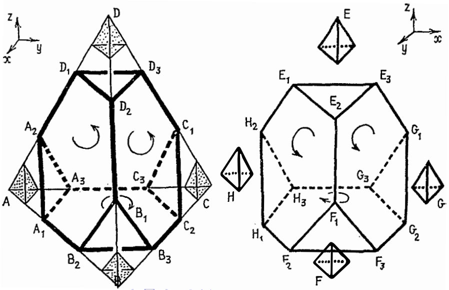
Let M be a compact connected 3-manifold with non-empty boundary
An ideal tetrahedron is a tetrahedron with its vertices removed.

An ideal triangulation of M is a realization of the interior of M as a gluing of some ideal tetrahedra, induced by a simplicial pairing of the faces.

B.G. Casler, 1965

Every M has an ideal triangulation.

ABD – FEH, BCD – EFG,
ADC – EGH, ACB – FHG



A triangulation of M is minimal if there is no triangulation of M with fewer tetrahedra.

The number of tetrahedra in a minimal triangulation of M is denoted $c_\Delta(M)$ and termed the triangulation complexity of M.

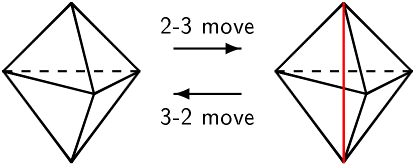
Problem.

How to find the triangulation complexity of a given 3-manifold.

- SnapPy is a modern user interface to the Jeff Weeks's SnapPea kernel. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. Can be used under the Sage. Project by Marc Culler and Nathan Dunfield.
- 3-Manifold Recognizer accepts many different presentations of 3-manifolds, calculates their different invariants and in many cases completely recognizes them. Project by Sergei Matveev, Vladimir Tarkaev and Chelyabinsk topology group.

V. Turaev – A. Vesnin – E. F., 2016

If $\mathcal{T}$ has exactly two edges, and $\mathcal{T}$ does not admit a 3-2 Pachner move, then $\mathcal{T}$ is minimal.



D. Nigomedyanov – E. F., 2023

Let M be a connected compact 3-manifold with boundary. Then

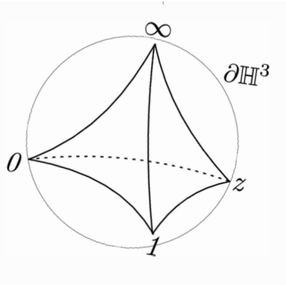
$$c_\Delta(M) \geq \beta_1(M, \mathbb{Z}_2).$$

Two-sided bounds for $c_\Delta(M)$

- Low bounds can be obtained from the information about the hyperbolic volume.

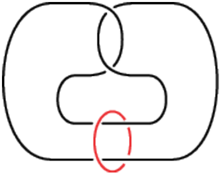
$$c_\Delta(M) \geqslant \frac{vol(M)}{v_3},$$

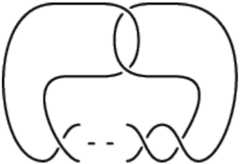
- $v_3 = 1.01494\ldots$
- v_3 is the volume of the regular ideal hyperbolic tetrahedron;
- every geodesic tetrahedron in $\mathbb{H}^3$ has volume at most v_3.



When the volume does not help.

Every twist knot K_n (depicted on the right) can be obtained by Dehn filling the red component of the Whitehead link L (depicted on the left).





$vol(S^3 \setminus L) = v_{oct} =$ volume of a regular ideal octahedron in $\mathbb{H}^3 = 3.6638\dots$

Theorem (M. Gromov and W. Thurston)

Let M be a finite volume hyperbolic manifold with cusps. Let N be a Dehn filling of some cusps of M. Then $vol(N) < vol(M)$.

$$vol(S^3 \setminus K_n) < v_{oct}$$

We call a cusped finite volume hyperbolic 3-manifold M tetrahedral if it can be decomposed into regular ideal tetrahedra.

Theorem.

If the number of tetrahedra is k, then $c_\Delta(M) = k$.

Proof:

- Since M is obtained by gluing k ideal tetrahedra, we have $c_\Delta(M) \leqslant k$.
- $c_\Delta(M) \geqslant \frac{vol(M)}{v_3}$, where $v_3 = 1.01494\ldots$ is the volume of the regular ideal tetrahedron.
- $c_\Delta(M) \geqslant k$, since $vol(M) = k \cdot v_3$.

Tabulated cusped manifolds

Burton – Callahan – Hildebrand – Thistlethwaite – Weeks census

	Orientable manifolds	
Tetrahedra	Total	Tetrahedral
1	0	0
2	2	2
3	9	0
4	56	4
5	234	2
6	962	7
7	3 552	1
8	12 846	13
9	44 250	1
Total	61 911	30

[S. Garoufalidis – M. Goerner – V. Tarkaev – A. Vesnin – E.F., 2016

There exists 11,580 orientable tetrahedral manifolds up to 25 tetrahedra.
There exists 25,194 non-orientable tetrahedral manifolds up to 21 tetrahedra.

Exact values of $c_\Delta(M)$: infinite families of tetrahedral manifolds

Remark.

If N is a k-fold covering of a tetrahedral manifold M, then N is also tetrahedral and

$$c_\Delta(N) = k \cdot c_\Delta(M).$$

This gives infinite families of manifolds with known complexity.

Example: Let N_k be the total space of the punctured torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^k$.

[S. Anisov, 2005]

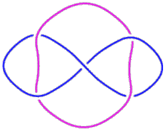
$c_\Delta(N_k) = 2k.$

Proof: N_k is the k-fold covering of the figure-eight knot complement N_1.

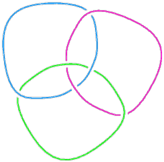
We say that a triangulation of a cusped finite volume hyperbolic 3-manifold M with k tetrahedra is almost tetrahedral if

$$(k-1)\cdot v_3 < vol(M) < k\cdot v_3.$$

If such a triangulation exists, we say M is almost tetrahedral.



Whitehead link (L5a1) complement ($m129$)
4 tetrahedra, $vol = 3.66386237671\ldots$



Borromean rings (L6a4) complement ($t12067$)
8 tetrahedra, $vol = 7.32772475342\ldots$

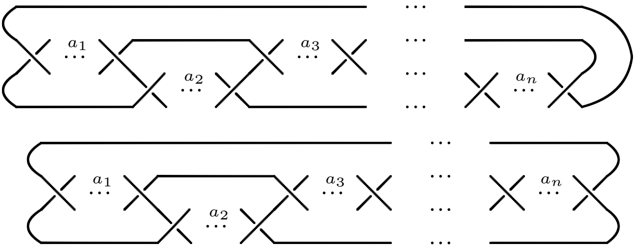
Tabulated cusped manifolds

Burton – Callahan – Hildebrand – Thistlethwaite – Weeks census

	Orientable manifolds		
Tetrahedra	Total	Tetrahedral	Almost Tetrahedral
1	0	0	0
2	2	2	0
3	9	0	9
4	56	4	50
5	234	2	144
6	962	7	358
7	3 552	1	675
8	12 846	13	1 467
9	44 250	1	3 239
Total	61 911	30	5 942

Theorem.

If T is an almost tetrahedral decomposition of M with k tetrahedra, then $c_{\triangle}(M) = k$.



We can represent a two-bridge link $K(p/q)$ by using Conway's notation as

$$p/q = [a_1, a_2, \ldots, a_{n-1}, a_n] = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}.$$

Here a_j denotes a number of half-twists.

Exact values of $c_\Delta(M)$: an infinite family of almost tetrahedral manifolds

[M. Ishikawa – K. Nemoto, 2016]

If $p/q = [2, 1, 1, \ldots, 1, 2]$, then $c_{tet}(S^3 \setminus K(p,q)) = 2n - 2$.

Proof:

- [M. Sakuma – J. Weeks, 1995] and [M. Ishikawa – K. Nemoto, 2016]: constructed ideal triangulations of $S^3 \setminus K(p,q)$ with $2n - 2$ tetrahedra.
- [C. Petronio – A. Vesnin, 2009]
  based on [D. Futer – E. Kalfagianni – J. Purcell:, 2008]:

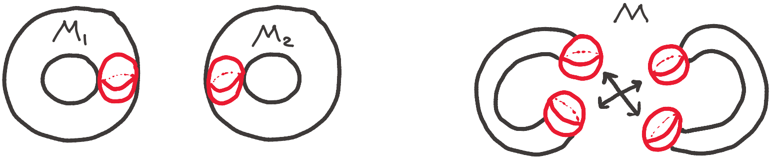
$$vol(S^3 \setminus K(p,q)) > (2n - 2.66) \cdot v_3.$$

Let

- M_1 and M_2 be finite volume hyperbolic 3-manifolds with ideal triangulations consisting of t_1 and t_2 tetrahedra, respectively.
- $S_1 \subset M_1$ and $S_2 \subset M_2$ are incompressible 3-punctured spheres $\implies$ S_1 and S_2 are isotopic to totally geodesic spheres.
- cut M_1 and M_2 open along S_1 and S_2, respectively, and then glue copies of the 3-punctured spheres together to yield a 3-manifold M.

Then

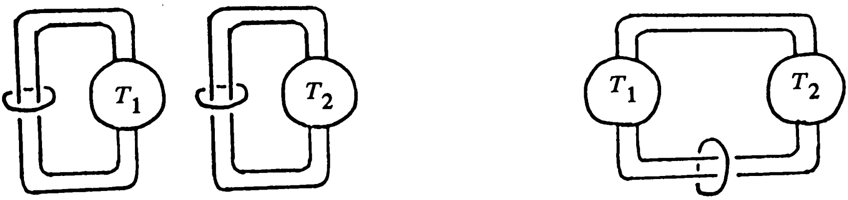
1) M is a hyperbolic 3-manifold; 2) $vol(M) = vol(M_1) + vol(M_2)$; 3) M has a triangulation consisting of $t_1 + t_2$ tetrahedra.



How to construct a new almost tetrahedral manifold

tetrahedral mfd + almost tetrahedral mfd = new almost tetrahedral mfd

Tetrahedra	Tetrahedral	Almost Tetrahedral
4	0	2
5	2	0
6	0	9
7	0	23
8	0	16
9	0	63
10	29	?



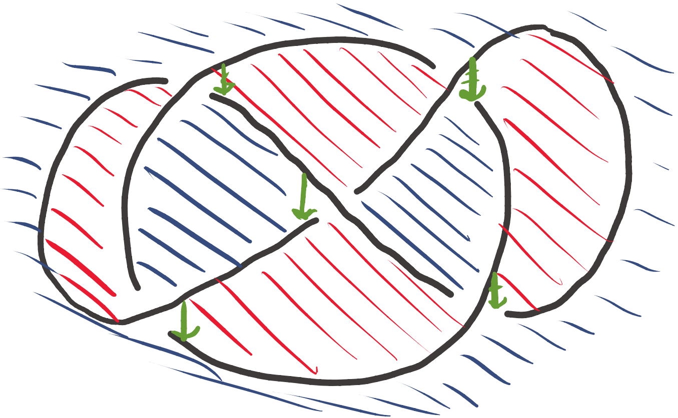
(a) (b)

[C. Adams, 1986]

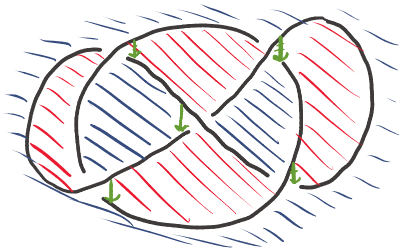
Let L_1 and L_2 be links in S^3 such that $S^3 \setminus L_1$ and $S^3 \setminus L_2$ are hyperbolic and L_1 and L_2 have projections as in Figure (a). Let L be the link with projection as in Figure (b). Then $S^3 \setminus L$ is hyperbolic and

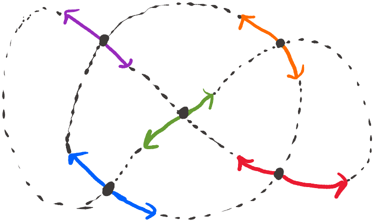
$$vol(S^3 \setminus L) = vol(S^3 \setminus L_1) + vol(S^3 \setminus L_2).$$

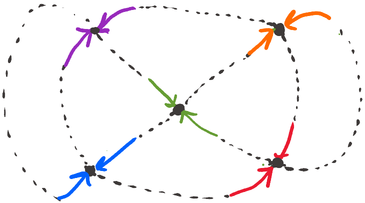
Moto-o TAKAHASHI, ON THE CONCRETE CONSTRUCTION OF HYPERBOLIC STRUCTURES OF 3-MANIFOLDS, TSUKUBA J. MATH. Vol. 9 No. 1 (1985). 41–83



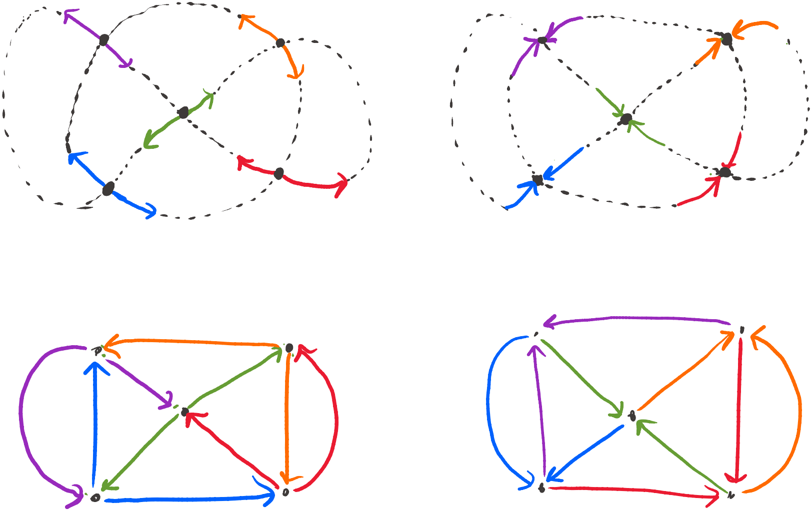
How to triangulate an alternating link complement? Step 2



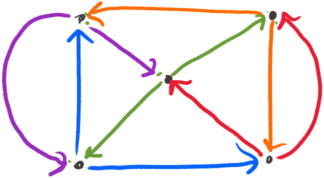


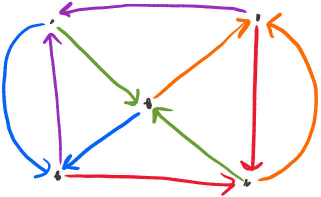


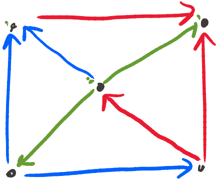
How to triangulate an alternating link complement? Step 3

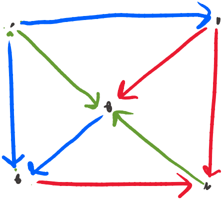


How to triangulate an alternating link complement? Step 4

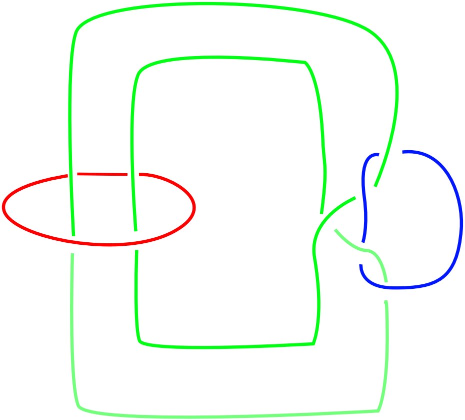






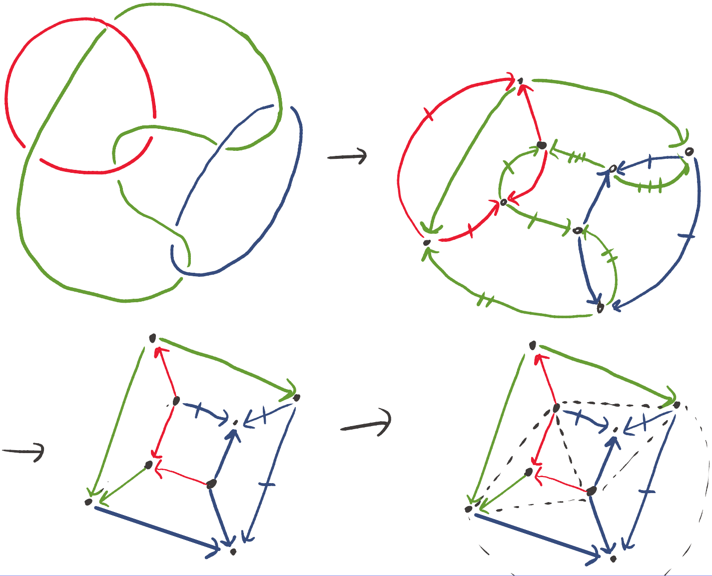


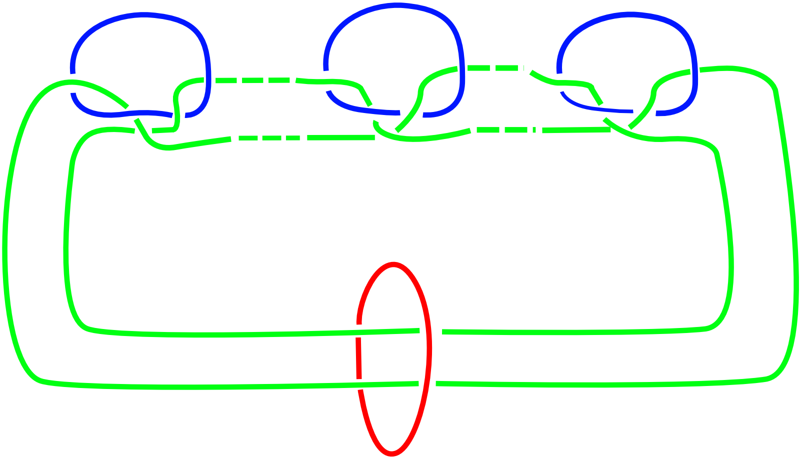
Tetrahedral manifold $S^3 \setminus L8a20$.



Tetrahedral manifold $S^3 \setminus L8a20$.

Tetrahedral manifold $S^3 \setminus L8a20$.





Tetrahedral manifold = 3-fold covering of $S^3 \setminus L8a20$.