Euclidean volume of a cone manifold over any hyperbolic knot is an algebraic number

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Euclidean tetrahedron

The calculation of volumes in 3-dimensional space $\mathbb{E}^3, \mathbb{H}^3$, or $\mathbb{S}^3$ is a very old and difficult problem. The first known result belongs to **Tartaglia (1499-1557)** who had described an algorithm for calculating the height of a tetrahedron in $\mathbb{E}^3$ with some concrete lengths of its edges. The formula which expresses the volume of an Euclidean tetrahedron in terms of its edge lengths was given by Euler. More precisely, let T be an Euclidean tetrahedron with edge lengths $d_{ij},\ 1 \leq i < j \leq 4$. Then $V = Vol(T)$ is given by

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Here V is a root of quadratic equation whose coefficients are integer polynomials in $d_{ij},\ 1 \leq i < j \leq 4$.

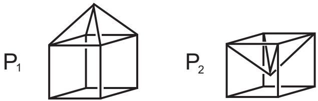
Euclidean polyhedron

Surprisedly, but the result can be generalized for any Euclidean polyhedron in the following way.

Theorem (I. Sabitov, 1996)

Let P be an Euclidean polyhedron with triangular faces. Then $V = Vol(P)$ is a root of an even degree algebraic equation whose coefficients are integer polynomials in squares of edge lengths of P depending on combinatorial type of P only.

Example



(All edge lengths are taken to be 1)

Polyhedra P_1 and P_2 are of the same combinatorial type. Hence, $V_1 = Vol(P_1)$ and $V_2 = Vol(P_2)$ are roots of the same algebraic equation

$$V^{2n} + a_1 V^{2n-2} + \ldots + a_n V^0 = 0.$$

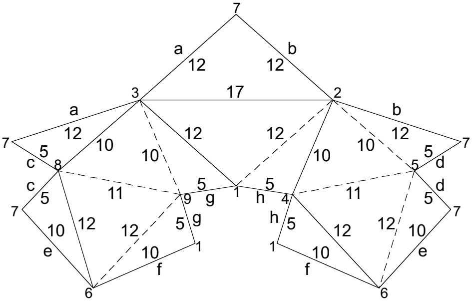
Rigid and flexible Euclidean polyhedra

Cauchy theorem (1813) states that a convex polyhedron with rigid faces is rigid itself. In spite of this, there are non-convex polyhedra with rigid faces which are flexible.

Bricard, 1897 (self-intersecting flexible octahedron)

Connelly, 1977 (the first example of true flexible polyhedron)

The smallest example is given by Steffen (9 vertices, 14 triangular faces).



A net for Steffen'spolyhedron

Rigid and flexible Euclidean polyhedra

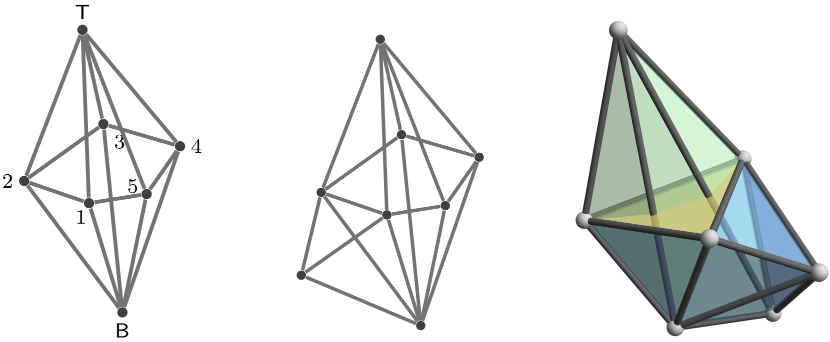


Fig.: Pentagonal bipyramid with triangulated face

[1] M. Gallet, G. Grasegger, J. Legerský, J. Schicho, *Pentagonal bipyramids lead to the smallest flexible embedded polyhedron.* arXiv:2410.13811

Bellows conjecture

An important consequence of Sabitov's theorem is a positive solution of the Bellows Conjecture proposed by R. Connelly and D. Sullivan.

Bellows conjecture. The generalized volume of a flexible polyhedron does not change when it is bending.

Theorem (I. Sabitov, 1998)

All flexible polyhedra keep a constant volume as they are flexed.

A higher-dimensional version of Sabitov's theorem for $\mathbb{E}^n (n \geq 3)$ and other generalisations were obtained by Gaifullin [3].

[2] I. Kh. Sabitov, *A generalized Heron–Tartaglia formula and some of its consequences.* Sbornik: Mathematics **189** (10), 1533–1561 (1998)

[3] A. A. Gaifullin, *Generalization of Sabitov's theorem to polyhedra of arbitrary dimensions.* Discrete Comput. Geom. **52** (2), 195–220 (2014)

Upper half-space model of hyperbolic 3-space

Denote by $\mathbb{H}^3$ a 3-dim *hyperbolic space*.

$\mathbb{H}^3$ can be modelled in $\mathbb{R}^3_+ = \{(x, y, t) \ : \ x, y, t \in \mathbb{R}, t > 0\}$ with metric s given by expression $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$.
The boundary $\partial\mathbb{H}^3 = \{(x, y, 0) \ : \ x, y \in \mathbb{R}\}$ caled *absolute* and consist of points at infinity.

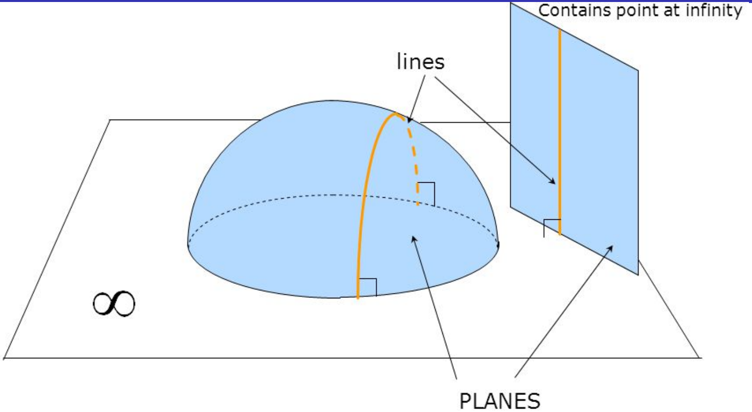
Isometry group $\mathrm{Isom}(\mathbb{H}^3)$ is a group of all actions on $\mathbb{H}^3$ preserving the metric s. Denote by $\mathrm{Isom}^+(\mathbb{H}^3)$ the group of orientation preserving isometries.

$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathbf{PSL}(2, \mathbb{C})$ (Pozitive Special Lorentz group). An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ by the rule

$$g \ : \ (z, t) \mapsto \left(\frac{(az + b)\overline{(cz + d)} + a\overline{c}t^2}{|cz + d|^2 + |c|^2 \, t^2}, \frac{t}{|cz + d|^2 + |c|^2 \, t^2} \right),$$

where $z = x + i\, y$.

Upper half-space model of hyperbolic 3-space



$\mathrm{Isom}^+(\mathbb{H}^3)$ is generated by compositions of even number of reflections with respect to geodesic planes.

$\mathrm{Isom}^+(\mathbb{H}^3) \cong \{\dfrac{a\,z+b}{c\,z+d}\ :\ ad-bc=1\}$ a group of fractional linear mappings in the complex plane $\partial\mathbb{H}^3 = \{(x,y,0)\ :\ z = x+i\,y \in \mathbb{C}\}$.

We will refer a **knot** $\mathcal{K}$ as a smooth simple closed curve in $\mathbb{S}^3$ or $\mathcal{M}$. The **knot exterior** is the compact 3-manifold $X = \mathcal{M} \setminus \eta(\mathcal{K})$ where $\eta(\mathcal{K})$ is an open tubular neighborhood. The group $\pi_1(X)$ is called the **knot group** associated to the knot. We recall that the boundary of X is a torus T and there are two simple closed curves on T called a **longitude** and **meridian** which intersect transversely in a single point. These two curves generate $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$ which is usually referred to as the **peripheral subgroup**.

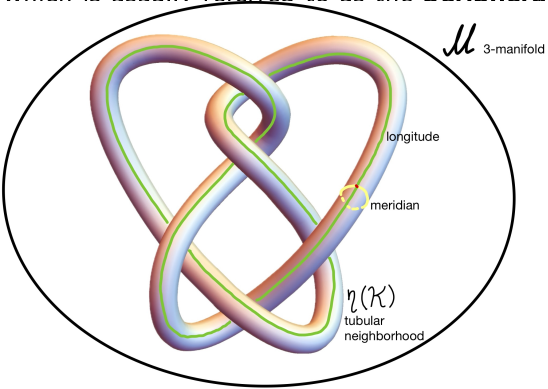


Fig.: *Knot exterior* $X = \mathcal{M} \setminus \eta(\mathcal{K})$

In the 1960s Waldhausen showed that the data consisting of the **knot group plus the peripheral subgroup is a complete knot invariant**. The difficulty is that although a powerful invariant it is extremely difficult to work with this data directly. Usually we admit some loss of information to construct a usable (but non-complete) invariant.

A classical way to study groups is to look at their linear representations and this suggests that one studies representations of knot groups into linear groups. Waldhausen's results suggest that one should also try to keep track of some of the peripheral data.

In begining of 1970's R. Riley in Southampton worked on representations of knots groups in $\mathrm{PSL}(2,\mathbb{C})$, which is the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^3$. After some time he got a faithful representation for the figure-eight knot 4_1, that the image was a discrete group and that the quotient of $\mathbb{H}^3$ by this group was the complement of knot 4_1. Thus, he discovered a **hyperbolic structure on the figure-eight knot complement**. This result by Riley was published much later, in 2013 (Riley died in 2000).

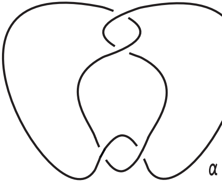


Fig.: Figure-eight knot 4_1

Then he showed that the same idea works for several other knots. In 1975 R. Riley found examples of **hyperbolic structures on some knot and link complements** in the three-dimensional sphere. Seven of them, so called **excellent knots**, were described later in his paper (1982).

Later, in the spring of 1977, W.P. Thurston announced an existence theorem for Riemannian metrics of constant negative curvature on 3-manifolds. In particular, it turned out that the **knot complement of a simple knot (excepting torical and satellite) admits a complete hyperbolic structure**[1]. This fact allowed to consider knot theory from the viewpoint of geometry and Kleinian group theory.

In 1980 W. Thurston constructed a **hyperbolic** 3-**manifold homeomorphic to the complement of knot** 4_1 in $\mathbb{S}^3$ by gluing faces of two regular ideal tetrahedra. This manifold has a complete hyperbolic structure.

[1] Thurston wrote that he was motivated by Riley's beautiful examples

Introduction

The A-**polynomial of a knot** was introduced in [4] and has become a powerful knot invariant. It encodes not only topological but also geometric information about the knot complement, especially in the case of hyperbolic knots. The notion of A-polynomial can be generalised to the case of hyperbolic manifolds with a single cusp [5].

[4] D. Cooper, M. Culler, H. Gillet, D. D. Long, P. B. Shalen, *Plane curves associated to character varieties of* 3*–manifolds.* Invent. Math. **118** (1), 47–84 (1994)

[5] A. Champanerkar, *A–polynomial and Bloch invariants of hyperbolic* 3*–manifolds.* PhD Thesis, Columbia University (2003)

The other concept that we deal with, namely **degeneration and regeneration of hyperbolic cone-manifold structures**, has been studied in many works over the years. In this regard, we refer to the results of Boileau, Leeb, and Porti [6, 7, 8].

[6] M. Boileau, B. Leeb, J. Porti, *Geometrization of 3–dimensional orbifolds.* Ann. Math. **162** (1), 195–250 (2005)

[7] M. Boileau, J. Porti, *Geometrization of 3–orbifolds of cyclic type.* Astérisque **272** (2001), 214 pp. (with an appendix by M. Heusener and J. Porti)

[8] J. Porti, *Regenerating hyperbolic and spherical cone structures from Euclidean ones.* Topology **37** (2), 365–392 (1998)

Introduction

A 3-dimensional **cone manifold** is a manifold $(\mathcal{M}, \mathcal{K})$ which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to the standard sphere and $(\mathcal{M}, \mathcal{K})$ is equipped with a complete path metric such that the restriction of the metric to each tetrahedron is isometric to a geodesic tetrahedron of constant curvature. The cone manifold is **hyperbolic**, **Euclidean** or **spherical** if the curvature is $-1, 0$, or $+1$ respectively.

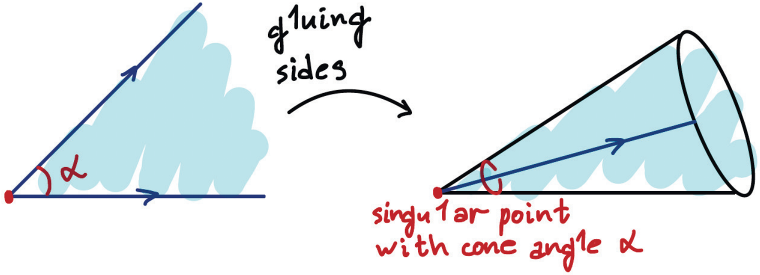


Fig.: 2-*dimensional cone manifold*

Introduction

Let $\mathcal{K}$ be a knot (in $\mathbb{S}^3$ or any other closed orientable 3-manifold $\mathcal{M}$), and let $\mathcal{C}_\alpha = \mathcal{C}_\alpha(\mathcal{M}, \mathcal{K})$ be the corresponding cone manifold with underlying topological space $\mathcal{M}$ and cone angle α along a singular geodesic in $\mathcal{M}$ isotopic to $\mathcal{K}$

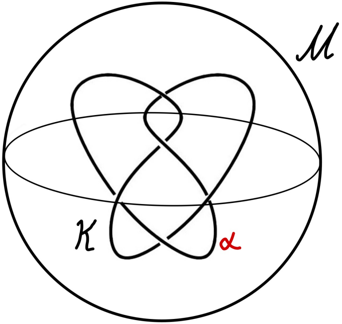
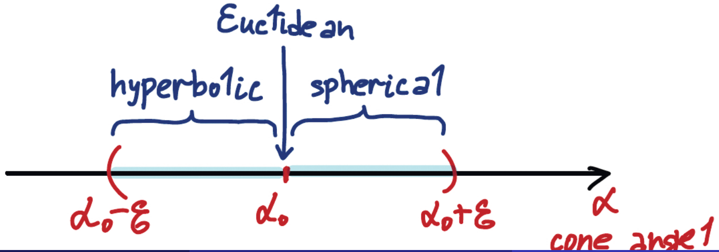


Fig.: *Cone manifold* $\mathcal{C}_\alpha = \mathcal{C}_\alpha(\mathcal{M}, \mathcal{K})$

Introduction

We shall assume that $\mathcal{K}$ is a hyperbolic knot (i.e. $\mathcal{C}_0 = \mathcal{C}_0(\mathcal{M}, \mathcal{K}) = \mathcal{M} \setminus \mathcal{K}$ has a complete hyperbolic metric of finite volume) and that a hyperbolic structure on the cone-manifold $\mathcal{C}_\alpha$ exists for any $\alpha \in (\alpha_0 - \varepsilon, \alpha_0)$, where $\alpha_0 > \varepsilon > 0$, and **degenerates (up to rescaling) into a Euclidean structure** as $\alpha \to \alpha_0$.

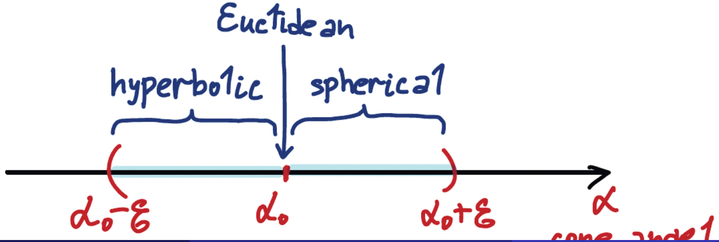
On the other hand, given a Euclidean cone manifold $\mathcal{C}_{\alpha_0}$, a hyperbolic or spherical structure can often be "**regenerated**": namely, it will be hyperbolic for $\alpha \in (\alpha_0 - \varepsilon, \alpha_0)$ and spherical for $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$. Such cone manifolds exist under some weak cohomological assumptions (Porti-1998)



Since $\mathcal{C}_\alpha$ converges in the Gromov–Hausdorff metric (after an appropriate rescaling) to a Euclidean cone-manifold $\mathcal{C}_{\alpha_0}$, one can define the associated *normalised Euclidean volume* as

$$\mathrm{vol}\,\mathcal{C}_{\alpha_0} = \lim_{\alpha\to\alpha_0^-} \frac{\mathrm{Vol}\,\mathcal{C}_\alpha}{\ell_\alpha^3},$$

where $\ell_\alpha = \ell_\alpha(\mathcal{M},\mathcal{K})$ is the length of the singular geodesic of $\mathcal{C}_\alpha = \mathcal{C}_\alpha(\mathcal{M},\mathcal{K})$. From here on "vol" denotes the normalised Euclidean volume in contrast to "Vol" that refers to the standard hyperbolic one. Such a construction for the normalised Euclidean volume appears in (Porti-1998)



Introduction

The hyperbolic volume of $\mathcal{C}_\alpha$ is an important quantity: due to the Mostow–Prasad–Kojima rigidity, the volume of $\mathcal{C}_0$ is a topological invariant whenever it admits a complete hyperbolic metric (of finite volume) [9]. There is also a large number of results concerning rigidity of cone–manifolds in the hyperbolic and other geometries [10].

[9] S. Kojima, *Deformations of hyperbolic 3-cone-manioflds.* J. Diff. Geom. **49** (3), 469–516 (1998)

[10] J. Porti, H. Weiss, *Deforming Euclidean cone 3–manifolds.* Geometry & Topology **11** (3), 1507–1538 (2007)

Introduction

Concerning the normalised Euclidean volume the following results are known. Local rigidity of hyperbolic cone manifolds was proven in [12] for the case of knot and link cone manifolds and in [11, 13] for the general case of 3–dimensional cone manifolds with cone angles less or equal to π. Global rigidity also takes place under some additional conditions. Namely, it was shown in [10] that if $(\mathcal{M}, \mathcal{K})$ is not a Seifert pair then both hyperbolic and spherical structures can be "regenerated" from the Euclidean one and global rigidity follows from Gromov–Prasad–Kojima theorem. In this case, the normalised Euclidean volume can be considered as a topological invariant of the knot type of $\mathcal{K}$ in $\mathcal{M}$.

[11] C. D. Hodgson, S. P. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery.* J. Diff. Geom. **48** (1), 1–59 (1998)

[12] H. Weiss, *Local rigidity of 3–dimensional cone–manifolds.* J. Differential Geom. **71** (3), 437–506 (2005)

[13] H. Weiss, *Global rigidity of 3–dimensional cone–manifolds.* J. Differential Geom. **76** (3), 495–523 (2007)

Introduction

The number-theoretic nature of hyperbolic volume is usually highly intricate (see, e.g., [14]). The main result of our work with A. Kolpakov and A. Mednykh [15] is that the normalised Euclidean volume is an algebraic number. In many cases we give a method to compute its minimal polynomial.

[14] D. Zagier, *The Dilogarithm Function*, in: P. Cartier, P. Moussa, B. Julia, P. Vanhove (eds.), Frontiers in Number Theory, Physics, and Geometry II. Springer–Verlag, Berlin (2007)

[15] N. Abrosimov, A. Kolpakov, A. Mednykh, *Euclidean volumes of hyperbolic knots.* Proc. Amer. Math. Soc., 152 (2024), 869–881.
DOI: 10.1090/proc/16353

Introduction

In the proof of the main result we use a modified version of the *A-polynomial*, while the standard one was introduced by Cooper, Culler, Gillet, Long, and Shalen in [4]. Our version contributes the real length of the singular geodesic instead of the complex one, and will be called the *Riley polynomial*. It appears very suitable for computational purposes. We also provide a pseudocode that computes the minimal polynomial of the normalised volume $\mathrm{vol}\,\mathcal{C}_{\alpha_0}$. This code can be used in any computer algebra system capable of computing resultants and factorising multivariable polynomials, such as SageMath or Mathematica.

Riley polynomial as a real version of A–polynomial

We assume that the fundamental group $\pi_1(\mathcal{M} \setminus \mathcal{K})$ has a holonomy representation in $SL_2(\mathbb{C})$. This condition is surely satisfied if the underlying manifold $\mathcal{M}$ is $\mathbb{S}^3$ (see [16]). Otherwise, one has to require additional properties: for example, that $\mathcal{M}$ be a $\mathbb{Z}_2$–homological 3–sphere, and the image of each peripheral subgroup of $\mathcal{K}$ is not $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ under the holonomy map into $PSL_2(\mathbb{C})$ (see [16] for more details).

Let $A(M, L)$ be the A–polynomial of $(\mathcal{M}, \mathcal{K})$ as defined in [1] that corresponds to the $SL_2(\mathbb{C})$ representations of the fundamental group of $\pi_1(\mathcal{M} \setminus \mathcal{K})$.

[16] F. Gonzalez-Acuña, J. M. Montesinos–Amilibia, *On the character variety of group representations in $SL(2, C)$ and $PSL(2, C)$*. Math. Z. **214**, 627–652 (1993)

Riley polynomial as a real version of A–polynomial

Following [17], choose the canonical longitude–meridian pair (λ, μ) in the fundamental group $\pi_1(\mathcal{M} \setminus \mathcal{K})$ in such a way that μ is the oriented boundary of a meridian disc of $\mathcal{K}$ and the longitude curve λ is null–homologous outside of $\mathcal{K}$. Let $h : \pi_1(\mathcal{M} \setminus \mathcal{K}) \to SL_2(\mathbb{C})$ be the holonomy map of the cone manifold $\mathcal{C}_\alpha(\mathcal{M}, \mathcal{K})$.
Then, up to conjugation in $SL_2(\mathbb{C})$,

$$h(\mu) = \pm \begin{bmatrix} \exp(i\,\alpha/2) & 0 \\ 0 & \exp(-i\,\alpha/2) \end{bmatrix}, \quad h(\lambda) = \begin{bmatrix} \exp(\gamma/2) & 0 \\ 0 & \exp(-\gamma/2) \end{bmatrix},$$

where $\gamma = \ell + i\varphi$, ℓ is the length of $\mathcal{K}$, and φ, $-2\pi \le \varphi < 2\pi$, is the angle of the lifted holonomy of $\mathcal{K}$. For the sake of simplicity, we will refer to $\gamma = \ell + i\varphi$ as the complex length of the singular geodesic $\mathcal{K}$.

[17] H. M. Hilden, M. T. Lozano, J. M. Montesinos–Amilibia, *On volumes and Chern–Simons invariants of geometric 3–manifolds.* J. Math. Sci. Univ. Tokyo **3**, 723–744 (1996)

Riley polynomial as a real version of A–polynomial

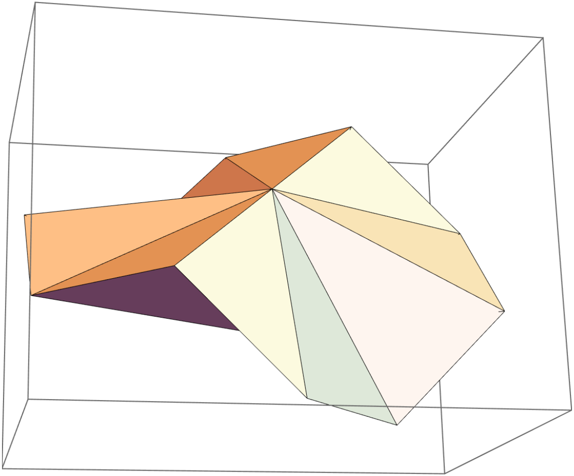


Fig.: *Fundamental polyhedron for the figure-eight knot cone manifold*

Riley polynomial as a real version of A–polynomial

An important property of the A–polynomial is that the cone angle α and complex length γ of $\mathcal{K}$ are related by the equation $A(L, M) = 0$, where $L = \exp(\gamma/2)$ and $M = \exp(i\,\alpha/2)$.

Let $A(\overline{M}, \overline{L}) = \overline{A(M, L)}$ be the complex conjugate of the A–polynomial. The coefficients of both polynomials are always integers. Consider

$$\widehat{A}(M, \overline{L}) = M^{\deg_M A(M,L)} A(M^{-1}, \overline{L}).$$

Once $M = \exp(i\,\alpha/2)$, we have $\overline{M} = M^{-1}$ and $A(\overline{M}, \overline{L}) = A(M^{-1}, \overline{L})$. Also, if $L = \exp(\frac{\ell + i\,\varphi}{2})$ then $\overline{L} = \exp(\frac{\ell - i\,\varphi}{2})$ and the quantity $W = L\overline{L} = \exp(\ell)$ is associated with the real length ℓ of the knot $\mathcal{K}$.

We need to obtain the Riley polynomial that relates the variables $M = \exp(i\,\alpha/2)$ and $W = L\overline{L} = \exp(\ell)$. In order to do this, we consider L, $\overline{L}$ and M as independent variables.

Riley polynomial as a real version of A–polynomial

$$\mathrm{res}(P, Q) = \prod_{(x,y):P(x)=0,Q(y)=0} (x - y)$$

$\mathrm{res}(P, Q)$ is a polynomial in coefficients of P and Q. $\mathrm{res}(P, Q) = 0$ if and only if the polynomials P and Q have a common root.

Let us compute the consecutive resultants $R_1 = Res_L(A(M, L), \widehat{A}(M, \overline{L}))$ and $R_2 = Res_{\overline{L}}(R_1, W - L\overline{L})$, see [18] for basic theory of resultants as applied to Laurent polynomials. We define a **Riley polynomial** as a factor of R_2 that corresponds to the hyperbolic structure on $\mathcal{C}_0$. This factor corresponds to the so-called "excellent component" of the character variety of $(\mathcal{M}, \mathcal{K})$, that is the component containing the character of the complete structure. Note that by construction $R(M, W)$ is a two–variable polynomial with integer coefficients.

[18] A. G. Khovanskii, L. Monin, *The resultant of developed systems of Laurent polynomials.* Mosc. Math. J. **17** (4), 717–740 (2017)

Key property of defined Riley polynomial

Proposition

The identity $R(M, W) = 0$ *holds, whenever* $M = \exp(i\,\alpha/2)$ *and* $W = L\overline{L} = \exp(\ell)$.

Euclidean volumes and algebraic numbers

Theorem (Abrosimov, Kolpakov, Mednykh – 2024)

Let $\mathcal{C}_\alpha = \mathcal{C}_\alpha(\mathcal{M}, \mathcal{K})$ be a cone–manifold with underlying 3–manifold $\mathcal{M}$ and singular set a knot $\mathcal{K}$ with cone angle α. Assume that fundamental group $\pi_1(\mathcal{M} \setminus \mathcal{K})$ has a holonomy representation in $SL_2(\mathbb{C})$, and $\mathcal{C}_\alpha$ admits a hyperbolic structure for $\alpha \in (\alpha_0 - \varepsilon, \alpha_0)$ that degenerates into a Euclidean structure as $\alpha \to \alpha_0$. Then the normalised Euclidean volume of $\mathcal{C}_{\alpha_0}$ is an algebraic number.

[19] N. Abrosimov, A. Kolpakov, A. Mednykh, *Euclidean volumes of hyperbolic knots.* Proc. Amer. Math. Soc., 152 (2024), 869–881.
DOI: 10.1090/proc/16353

Outline of the proof

The limit $\operatorname{vol}\mathcal{C}_{\alpha_0} = \lim\limits_{\alpha\to\alpha_0^-} \dfrac{\operatorname{Vol}\mathcal{C}_\alpha}{\ell_\alpha^3}$ exists by [Corollary C, Porti-1998]. It remains to show that its value is among the roots of a polynomial with integer coefficients. For this purpose, we express $\operatorname{vol}\mathcal{C}_{\alpha_0}$ as a root of the associated Riley polynomial of $(\mathcal{M},\mathcal{K})$ which we consider as a "real" version of the A–polynomial, as opposed to the original "complex" one.

Let us recall the Schläfli formula that in our case takes the following simple form:

$$\mathrm{d}\operatorname{Vol}\mathcal{C}_\alpha(\mathcal{M},\mathcal{K}) = -\frac{1}{2}\,\ell_\alpha \mathrm{d}\alpha. \tag{1}$$

As $\alpha\to\alpha_0^-$, we have that $\ell_\alpha\to 0$ and $\operatorname{Vol}\mathcal{C}_\alpha\to 0$ [Proposition C, Porti-1998]. Thus $M_0 = \exp(i\alpha_0/2)$ is among the roots of $R(M_0,\exp(0)) = R(M_0,1)$. In particular, M_0 is an algebraic number.

Outline of the proof

Let us observe that the expression for $\mathrm{vol}(\mathcal{M}, \mathcal{K})$ can be rewritten by using the L'Hôpital rule as follows

$$\mathrm{vol}\,\mathcal{C}_{\alpha_0} = \lim_{\alpha \to \alpha_0^-} \frac{\mathrm{Vol}\,\mathcal{C}_\alpha}{\ell_\alpha^3} = \lim_{\alpha \to \alpha_0^-} \frac{(\mathrm{Vol}\,\mathcal{C}_\alpha)'_\alpha}{(\ell_\alpha^3)'_\alpha} = \lim_{\alpha \to \alpha_0^-} \frac{-\frac{1}{2}\ell_\alpha}{3\,\ell_\alpha^2 (\ell_\alpha)'_\alpha} = \\ = -\frac{1}{3} \lim_{\alpha \to \alpha_0^-} \frac{1}{(\ell_\alpha^2)'_\alpha}, \quad (2)$$

where we use the Schläfli formula (1) in order to differentiate $\mathrm{Vol}\,\mathcal{C}_\alpha$. Here and below, we shall use f'_x as a shortcut for $\frac{\mathrm{d}f}{\mathrm{d}x}$, for any expression f that depends on a variable x explicilty or implicitly.

Outline of the proof

Moreover, as $\alpha \to \alpha_0$ we have that $\ell = \ell_\alpha \to 0$ and $W = \exp(\ell) \to 1$. Thus we can introduce a new variable X and write $W = 1 + X$, where $X \to 0$ as $\alpha \to \alpha_0$. Then $\ell = \ln(1 + X)$, $\ell^2 = \ln^2(1 + X) = X^2 - X^3 + O(X^4)$ as $X \to 0$. Hence $(\ell^2)'_\alpha = (X^2)'_\alpha + O(X^2)$. By using [Corollary C, Porti-1998], we can replace $O(X^2)$ with $O(|\alpha - \alpha_0|)$, as $\alpha \to \alpha_0$. The resulting asymptotic expansion $(\ell^2)'_\alpha = (X^2)'_\alpha + O(|\alpha - \alpha_0|)$, as $\alpha \to \alpha_0$, allows us to use only polynomial expressions in the rest of the proof.

By computing the resultant of $R(M, W)$ with $(W - 1 - X)$ in W first, and then computing one more resultant of the obtained expression with $(Y - X^2)$ in X, we find the minimal polynomial $P(M, Y)$ for Y over the ring $\mathbb{Q}[M]$.

Outline of the proof

Now let us consider Y as an implicit function $Y = Y(M)$ defined by the equation $P(M, Y(M)) = 0$ together with the condition $Y(M_0) = 0$, for $M_0 = \exp(i\alpha_0/2)$. This allows us to compute the derivative $Y'(M)$ in terms of $P(M, Y)$ and $Y(M)$ itself. Since $P'_M(M, Y(M)) + P'_Y(M, Y(M))\, Y'(M) = 0$, let us put $Q(M, Y, Z) = P'_M(M, Y) + P'_Y(M, Y)\, Z$, where $Z = Y'(M)$ is a new variable.

By taking the resultant of $P(M, Y)$ and $Q(M, Y, Z)$ in Y, we finally obtain the minimal polynomial $S(M, Z)$ for $Z = (X^2)'_\alpha$ over the ring $\mathbb{Q}[M]$, after choosing the appropriate irreducible factor.

Outline of the proof

Converting all the $\frac{d}{d\alpha}$ derivatives to the $\frac{d}{dM}$ ones via the chain rule allows us to rewrite (2) as

$$\operatorname{vol} \mathcal{C}_{\alpha_0} = \frac{2i}{3M_0 Z_0}, \tag{3}$$

where $Z_0 = \lim_{\alpha \to \alpha_0^-} Y'(M)$, for $M = \exp(i\alpha/2)$, is among the roots of the polynomial $S(M_0, Z)$, for $M_0 = \exp(i\alpha_0/2)$.

As M_0 is an algebraic number and $S(M, Z)$ is a polynomial with integer coefficients, we conclude that Z_0 is algebraic. Hence, $\operatorname{vol} \mathcal{C}_{\alpha_0}$ is also algebraic. $\square$

Computing the minimal polynomial for normalised volume

Here we provide a pseudocode that computes the minimal polynomial of $\mathrm{vol}(\mathcal{M},\mathcal{K})$ starting from the $SL_2(\mathbb{C})$ A–polynomial of $(\mathcal{M},\mathcal{K})$ as input. This algorithm can be used in any computer algebra system that has enough functionality in commutative algebra, such as SageMath or Mathematica.

Data: $A(M,L) =$ the A–polynomial of $(\mathcal{M},\mathcal{K})$.
Result: The minimal polynomial of $\mathrm{vol}\,\mathcal{C}_{\alpha_0}$.

1. Let $d = \deg_M A(M,L)$;
2. Let $\overline{L}$ be a new variable. Let R_1, R_2, R_3 be three auxiliary variable;
3. Let W be a new variable;
4. Let $\widehat{A}(M,L) := M^d \cdot A(M^{-1},\overline{L})$;
5. Let R_1 be the resultant of $A(M,L)$ and $\widehat{A}(M,L)$ in L;
6. Let R_2 be the resultant of R_1 and $W - L\cdot\overline{L}$ in $\overline{L}$;

Computing the minimal polynomial for normalised volume

7. Factorise R_2 and isolate its irreducible factor that corresponds to the excellent component of the character variety of $(\mathcal{M}, \mathcal{K})$, that is the component containing the character of the complete hyperbolic structure. We refer to it as *the Riley polynomial* $R(M, W)$;
8. Let R_1 be the resultant of $R(M, W)$ and $W - X - 1$ in W;
9. Let R_2 be the resultant of R_1 and $Y - X^2$ in X;
10. Factorise R_2 and isolate its irreducible factor that corresponds to the minimal polynomial of Y over the field $\mathbb{Q}(M)$ of rational functions in M;
11. Set $Y = Y(M)$ to be a function of M. Let $Y' = Y'(M)$ be the derivative of $Y(M)$ with respect to M;
12. Differentiate $P(M, Y(M))$ with respect to M. Store the output as R_1;
13. Substitute $Y'(M)$ in R_1 by a new variable Z. Store the output as $Q(M, Y, Z)$;
14. Let R_2 be the resultant of $P(M, Y)$ and $Q(M, Y, Z)$ in Y;

Computing the minimal polynomial for normalised volume

15. Factorise R_2 and isolate its irreducible factor $S(M,Z)$ that corresponds to the minimal polynomial of Z over $\mathbb{Q}(M)$;
16. Let V be a new variable. Let I be the complex unit and let R_1 be $3 \cdot M \cdot Z \cdot V - 2 \cdot I$. Let R_2 be $R(M,1)$;
17. Let R_3 be the resultant of R_1 and $S(M,Z)$ in Z. Let R_4 be the resultant of R_2 and R_3 in M;
18. Factorise R_4 and isolate its irreducible factor $F(V)$ that corresponds to the minimal polynomial of V over $\mathbb{Q}$;
19. Output $F(V)$.

Applications of the main theorem

One of the most interesting cases of applications of Theorem 4 is that of two–bridge knots. Any two–bridge knot $\mathcal{K}$ of slope p/q, where $p > 1$ and $2 < q < p - 1$, is hyperbolic [20].

Let $\mathcal{C}_\alpha = \mathcal{C}_\alpha(\mathbb{S}^3, \mathcal{K})$. Then according to [21] there exists an angle $\alpha_0 \in [2\pi/3, \pi)$ such that $\mathcal{C}_\alpha$ is hyperbolic for all $\alpha \in [0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for all $\alpha \in (\alpha_0, 2\pi - \alpha_0)$. Thus we can define the normalised Euclidean volume $\mathrm{vol}(\mathcal{K}) = \mathrm{vol}(\mathbb{S}^3, \mathcal{K})$ of $\mathcal{K}$. Because of Weiss' rigidity theorem [12, 13], $\mathrm{vol}(\mathcal{K})$ will be also a topological invariant of $\mathcal{K}$ together with its hyperbolic volume.

[20] G. Burde, H. Zieschang, M. Heusener, *Knots. 3rd edition.* De Gruyter Studies in Mathematics 5. Berlin (2014).

[21] J. Porti, *Spherical cone structures on 2–bridge knots and links.* Kobe J. Math. **21** (1–2), 61–70 (2004)

Example: knot 5_2 (triple-twist knot)

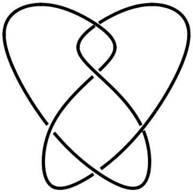


Fig.: Knot 5_2

The A-polynomial for $SL_2(\mathbb{C})$ presentation of knot 5_2 is

$$A(M, L) = 1 + L(-1 + 2M^2 + 2M^4 - M^8 + M^{10}) \\ + L^2(M^4 - M^6 + 2M^{10} + 2M^{12} - M^{14}) + L^3M^{14}.$$

Example: knot 5_2 (triple-twist knot)

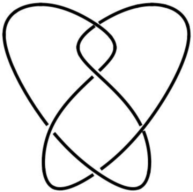


Fig.: Knot 5_2

By performing the algorithm we obtain that the normalised Euclidean volume has numerical value

$$1/\left(6\sqrt{-6+68\sqrt{2}+4\sqrt{983}+946\sqrt{2}}\right) = 0.00990963099945638\ldots$$

This number is algebraic with minimal polynomial

$$785065068490752\,x^8 + 412091172864\,x^6 + 64457856\,x^4 - 864\,x^2 - 1.$$


END