Satellites and invariants of links

Sergey Melikhov

Steklov Math Institute

50th Birthday of A. V. Malutin, November 2025

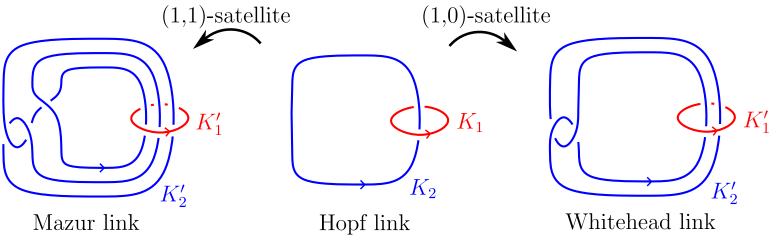
Given a link $L = (K_1, \dots, K_m)$ in S^3, a link $L' = (K'_1, \dots, K'_m)$ is called:

Given a link $L = (K_1, \dots, K_m)$ in S^3, a link $L' = (K'_1, \dots, K'_m)$ is called:

- a *$(p_1, \dots, p_m)$-satellite* of L, if it lies in a tubular neighborhood $T = (T_1, \dots, T_m)$ of L so that each $[K'_i] = p_i[K_i] \in H_1(T_i)$;

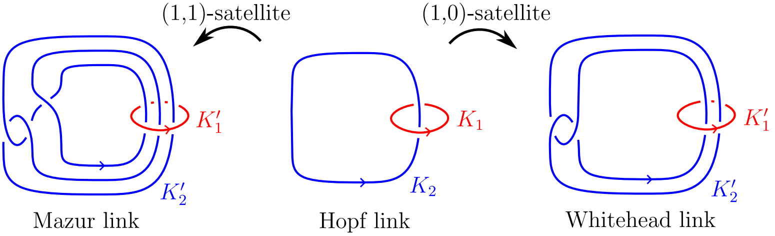
Given a link $L = (K_1, \dots, K_m)$ in S^3, a link $L' = (K'_1, \dots, K'_m)$ is called:

- a *$(p_1, \dots, p_m)$-satellite* of L, if it lies in a tubular neighborhood $T = (T_1, \dots, T_m)$ of L so that each $[K'_i] = p_i[K_i] \in H_1(T_i)$;



Given a link $L = (K_1, \ldots, K_m)$ in S^3, a link $L' = (K'_1, \ldots, K'_m)$ is called:

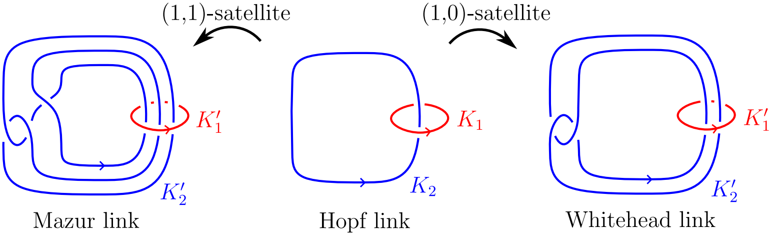
- a *$(p_1, \ldots, p_m)$-satellite* of L, if it lies in a tubular neighborhood $T = (T_1, \ldots, T_m)$ of L so that each $[K'_i] = p_i[K_i] \in H_1(T_i)$;



- a *$(p_1, \ldots, p_m)$-braiding* of L, if moreover each projection $T_i \cong K_i \times D^2 \to K_i$ restricts to a covering map $K'_i \to K_i$;

Given a link $L = (K_1, \dots, K_m)$ in S^3, a link $L' = (K'_1, \dots, K'_m)$ is called:

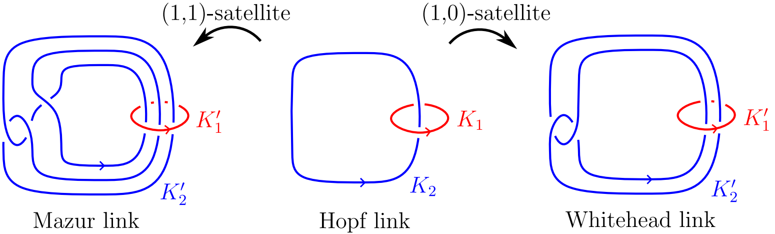
- a *$(p_1, \dots, p_m)$-satellite* of L, if it lies in a tubular neighborhood $T = (T_1, \dots, T_m)$ of L so that each $[K'_i] = p_i[K_i] \in H_1(T_i)$;



- a *$(p_1, \dots, p_m)$-braiding* of L, if moreover each projection $T_i \cong K_i \times D^2 \to K_i$ restricts to a covering map $K'_i \to K_i$;
- a *$(p_1, \dots, p_m)$-cabling* of L, if in addition each $K'_i \subset \partial T_i$.

Given a link $L = (K_1, \dots, K_m)$ in S^3, a link $L' = (K'_1, \dots, K'_m)$ is called:

- a *$(p_1, \dots, p_m)$-satellite* of L, if it lies in a tubular neighborhood $T = (T_1, \dots, T_m)$ of L so that each $[K'_i] = p_i[K_i] \in H_1(T_i)$;



- a *$(p_1, \dots, p_m)$-braiding* of L, if moreover each projection $T_i \cong K_i \times D^2 \to K_i$ restricts to a covering map $K'_i \to K_i$;
- a *$(p_1, \dots, p_m)$-cabling* of L, if in addition each $K'_i \subset \partial T_i$.

In the latter case each K'_i must be the *(p_i, q_i)-cable* of K_i for some q_i coprime to p_i, where $q_i = \mathrm{lk}(K'_i, K_i)$.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

For $k \neq 0$, *strongly k-satellitable*: same but with $p_1, \dots, p_m \in \mathbb{Z}$.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

For $k \neq 0$, *strongly k-satellitable*: same but with $p_1, \dots, p_m \in \mathbb{Z}$.

Example. The linking number is strongly 1-satellitable.

That is, $\operatorname{lk}(L') = p_1 p_2 \operatorname{lk}(L)$ for all $p_1, p_2 \in \mathbb{Z}$, for every 2-component link L and for every (p_1, p_2)-satellite L' of L.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

For $k \neq 0$, *strongly k-satellitable*: same but with $p_1, \dots, p_m \in \mathbb{Z}$.

Example. The linking number is strongly 1-satellitable.

That is, $\mathrm{lk}(L') = p_1 p_2 \,\mathrm{lk}(L)$ for all $p_1, p_2 \in \mathbb{Z}$, for every 2-component link L and for every (p_1, p_2)-satellite L' of L.

Example. Given a link $L = (K_1, \dots, K_m)$, let $l_{ij} = \mathrm{lk}(K_i, K_j)$.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

For $k \neq 0$, *strongly k-satellitable*: same but with $p_1, \dots, p_m \in \mathbb{Z}$.

Example. The linking number is strongly 1-satellitable.

That is, $\mathrm{lk}(L') = p_1 p_2 \, \mathrm{lk}(L)$ for all $p_1, p_2 \in \mathbb{Z}$, for every 2-component link L and for every (p_1, p_2)-satellite L' of L.

Example. Given a link $L = (K_1, \dots, K_m)$, let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

(a) $\lambda(L) := \prod_{i<j} l_{ij}$ is strongly $(m-1)$-satellitable

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

For $k \neq 0$, *strongly k-satellitable*: same but with $p_1, \dots, p_m \in \mathbb{Z}$.

Example. The linking number is strongly 1-satellitable.

That is, $\mathrm{lk}(L') = p_1 p_2 \,\mathrm{lk}(L)$ for all $p_1, p_2 \in \mathbb{Z}$, for every 2-component link L and for every (p_1, p_2)-satellite L' of L.

Example. Given a link $L = (K_1, \dots, K_m)$, let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

(a) $\lambda(L) := \prod_{i<j} l_{ij}$ is strongly $(m-1)$-satellitable

(b) $\lambda^\circ(L) := l_{12} l_{23} \cdots l_{m-1,m} l_{m1}$ is strongly 2-satellitable

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

For $k \neq 0$, *strongly k-satellitable*: same but with $p_1, \dots, p_m \in \mathbb{Z}$.

Example. The linking number is strongly 1-satellitable.

That is, $\mathrm{lk}(L') = p_1 p_2 \,\mathrm{lk}(L)$ for all $p_1, p_2 \in \mathbb{Z}$, for every 2-component link L and for every (p_1, p_2)-satellite L' of L.

Example. Given a link $L = (K_1, \dots, K_m)$, let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

(a) $\lambda(L) := \prod_{i<j} l_{ij}$ is strongly $(m-1)$-satellitable

(b) $\lambda^\circ(L) := l_{12} l_{23} \cdots l_{m-1,m} l_{m1}$ is strongly 2-satellitable

(c) if m is even, then $l_{12} l_{34} \cdots l_{m-1,m}$ is strongly 1-satellitable

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \ldots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \ldots, p_m)$-cabling/$(p_1, \ldots, p_m)$-braiding/$(p_1, \ldots, p_m)$-satellite L' of L.

---

v is k-satellitable $\Rightarrow$ $v(L') = v(L)$ whenever L' is a $(1, \ldots, 1)$-satellite of L

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

---

v is k-satellitable $\Rightarrow$ $v(L') = v(L)$ whenever L' is a $(1, \dots, 1)$-satellite of L
$\Rightarrow$ v is an invariant of F-isotopy

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

---

v is k-satellitable $\Rightarrow$ $v(L') = v(L)$ whenever L' is a $(1, \dots, 1)$-satellite of L
$\Rightarrow$ v is an invariant of F-isotopy

F-isotopy: equivalence relation on links generated by ambient isotopy and the operation of replacing a given link with any its $(1, \dots, 1)$-satellite.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

---

v is k-satellitable $\Rightarrow$ $v(L') = v(L)$ whenever L' is a $(1, \dots, 1)$-satellite of L
$\Rightarrow$ v is an invariant of F-isotopy

F-isotopy: equivalence relation on links generated by ambient isotopy and the operation of replacing a given link with any its $(1, \dots, 1)$-satellite. (Named after R. Fox who first considered it.)

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

---

v is k-satellitable $\Rightarrow$ $v(L') = v(L)$ whenever L' is a $(1, \dots, 1)$-satellite of L
$\Rightarrow$ v is an invariant of F-isotopy

F-isotopy: equivalence relation on links generated by ambient isotopy and the operation of replacing a given link with any its $(1, \dots, 1)$-satellite. (Named after R. Fox who first considered it.)

Example. Since every knot is F-isotopic to an unknot, there are no non-constant k-satellitable invariants of knots.

An abelian group-valued invariant v of m-component links is called *k-cableable*/*k-braidable*/*k-satellitable* if

$$v(L') = (p_1 \cdots p_m)^k v(L)$$

for all $p_1, \dots, p_m \in \mathbb{Z} \setminus \{0\}$, for every m-component link L and for every $(p_1, \dots, p_m)$-cabling/$(p_1, \dots, p_m)$-braiding/$(p_1, \dots, p_m)$-satellite L' of L.

---

v is k-satellitable $\Rightarrow$ $v(L') = v(L)$ whenever L' is a $(1, \dots, 1)$-satellite of L $\Rightarrow$ v is an invariant of F-isotopy

F-isotopy: equivalence relation on links generated by ambient isotopy and the operation of replacing a given link with any its $(1, \dots, 1)$-satellite. (Named after R. Fox who first considered it.)

Example. Since every knot is F-isotopic to an unknot, there are no non-constant k-satellitable invariants of knots.

Example. The following *almost* satisfy the definition of a 1-braidable invariant — namely, they do so for links L with no unknotted components:

- bridge number of knots (Schubert 1954) and links (Williams 1992)
- braid index of links (Williams 1992)

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n} \in \mathbb{Z}/\delta_{i_1\dots i_n}$.

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n} \in \mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L) = \begin{cases} \bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n}\in\mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L)=\begin{cases}\bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L)=0,\\ 0 & \text{otherwise.}\end{cases}$$

Then each $\bar{\bar\mu}_{i_1\dots i_n}(L)\in\mathbb{Z}$ and

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n}\in\mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L)=\begin{cases}\bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L)=0,\\ 0 & \text{otherwise.}\end{cases}$$

Then each $\bar{\bar\mu}_{i_1\dots i_n}(L)\in\mathbb{Z}$ and

- each $\bar{\bar\mu}_{i_1\dots i_n}$ with precisely k occurrences of each index is k-satellitable

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n}\in\mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L)=\begin{cases}\bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L)=0,\\ 0 & \text{otherwise.}\end{cases}$$

Then each $\bar{\bar\mu}_{i_1\dots i_n}(L)\in\mathbb{Z}$ and

- each $\bar{\bar\mu}_{i_1\dots i_n}$ with precisely k occurrences of each index is k-satellitable
- $\bar{\bar\mu}_{1122}$ is not strongly 2-satellitable: $\bar{\bar\mu}_{1122}(W)\neq(1\cdot 0)^2\cdot\bar{\bar\mu}_{1122}(H)$

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n} \in \mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L) = \begin{cases} \bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then each $\bar{\bar\mu}_{i_1\dots i_n}(L) \in \mathbb{Z}$ and

- each $\bar{\bar\mu}_{i_1\dots i_n}$ with precisely k occurrences of each index is k-satellitable
- $\bar{\bar\mu}_{1122}$ is not strongly 2-satellitable: $\bar{\bar\mu}_{1122}(W) \neq (1 \cdot 0)^2 \cdot \bar{\bar\mu}_{1122}(H)$
- each $\bar{\bar\mu}_{i_1\dots i_n}$ with pairwise distinct indices is strongly 1-satellitable

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n} \in \mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L) = \begin{cases} \bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then each $\bar{\bar\mu}_{i_1\dots i_n}(L) \in \mathbb{Z}$ and

- each $\bar{\bar\mu}_{i_1\dots i_n}$ with precisely k occurrences of each index is k-satellitable
- $\bar{\bar\mu}_{1122}$ is not strongly 2-satellitable: $\bar{\bar\mu}_{1122}(W) \neq (1\cdot 0)^2 \cdot \bar{\bar\mu}_{1122}(H)$
- each $\bar{\bar\mu}_{i_1\dots i_n}$ with pairwise distinct indices is strongly 1-satellitable

Example/Corollary. The following invariant is 0-satellitable:

$$\bar{\bar{\bar\mu}}_{i_1\dots i_n}(L) = \begin{cases} 1 & \text{if } \bar{\bar\mu}_{i_1\dots i_n}(L) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example/Theorem. Milnor's $\bar\mu$-invariants $\bar\mu_{i_1\dots i_n} \in \mathbb{Z}/\delta_{i_1\dots i_n}$. Let

$$\bar{\bar\mu}_{i_1\dots i_n}(L) = \begin{cases} \bar\mu_{i_1\dots i_n}(L) & \text{if } \delta_{i_1\dots i_n}(L) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then each $\bar{\bar\mu}_{i_1\dots i_n}(L) \in \mathbb{Z}$ and

- each $\bar{\bar\mu}_{i_1\dots i_n}$ with precisely k occurrences of each index is k-satellitable
- $\bar{\bar\mu}_{1122}$ is not strongly 2-satellitable: $\bar{\bar\mu}_{1122}(W) \neq (1 \cdot 0)^2 \cdot \bar{\bar\mu}_{1122}(H)$
- each $\bar{\bar\mu}_{i_1\dots i_n}$ with pairwise distinct indices is strongly 1-satellitable

Example/Corollary. The following invariant is 0-satellitable:

$$\bar{\bar{\bar\mu}}_{i_1\dots i_n}(L) = \begin{cases} 1 & \text{if } \bar{\bar\mu}_{i_1\dots i_n}(L) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

v is 0-satellitable: $v(L') = v(L)$ whenever L' is a $(p_1, \dots, p_m)$-satellite of L with $p_i \neq 0$.

Problem (V. I. Arnold–P. M. Akhmetiev). Does there exist a finite type invariant which is k-cableable for some k but is not a function of the pairwise linking numbers?

Problem (V. I. Arnold–P. M. Akhmetiev). Does there exist a finite type invariant which is k-cableable for some k but is not a function of the pairwise linking numbers?

- implicit in the abstract of Akhmetiev's talk (February 2021, Moscow Geometric Topology Seminar)
- rather explicit in Akhmetiev's 2016 book "Finite-Type Invariants of Magnetic Lines", but with the k-cableable condition replaced by its precursor version

Problem (V. I. Arnold–P. M. Akhmetiev). Does there exist a finite type invariant which is k-cableable for some k but is not a function of the pairwise linking numbers?

- implicit in the abstract of Akhmetiev's talk (February 2021, Moscow Geometric Topology Seminar)
- rather explicit in Akhmetiev's 2016 book "Finite-Type Invariants of Magnetic Lines", but with the k-cableable condition replaced by its precursor version

Motivation: topological lower bounds for the energy of a magnetic field

Problem (V. I. Arnold–P. M. Akhmetiev). Does there exist a finite type invariant which is k-cableable for some k but is not a function of the pairwise linking numbers?

- implicit in the abstract of Akhmetiev's talk (February 2021, Moscow Geometric Topology Seminar)
- rather explicit in Akhmetiev's 2016 book "Finite-Type Invariants of Magnetic Lines", but with the k-cableable condition replaced by its precursor version

Motivation: topological lower bounds for the energy of a magnetic field

> "What invariants of knots can be extended to invariants of divergence-free vector fields?" [Arnold's Problems, Problem 1990-16] (discussed in detail in [Arnold–Khesin 1998; §8.A], not only for knots but also for links)

Problem (V. I. Arnold–P. M. Akhmetiev). Does there exist a finite type invariant which is k-cableable for some k but is not a function of the pairwise linking numbers?

- implicit in the abstract of Akhmetiev's talk (February 2021, Moscow Geometric Topology Seminar)
- rather explicit in Akhmetiev's 2016 book "Finite-Type Invariants of Magnetic Lines", but with the k-cableable condition replaced by its precursor version

Motivation: topological lower bounds for the energy of a magnetic field

> "What invariants of knots can be extended to invariants of divergence-free vector fields?" [Arnold's Problems, Problem 1990-16] (discussed in detail in [Arnold–Khesin 1998; §8.A], not only for knots but also for links)

> "The dream is to define such a hierarchy of invariants for generic vector fields such that, [when] all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy." [Arnold–Khesin 1998; §7.C "Higher-order linking integrals"]

Problem (V. I. Arnold–P. M. Akhmetiev). Does there exist a finite type invariant which is k-cableable for some k but is not a function of the pairwise linking numbers?

- implicit in the abstract of Akhmetiev's talk (February 2021, Moscow Geometric Topology Seminar)
- rather explicit in Akhmetiev's 2016 book "Finite-Type Invariants of Magnetic Lines", but with the k-cableable condition replaced by its precursor version

Motivation: topological lower bounds for the energy of a magnetic field

> "What invariants of knots can be extended to invariants of divergence-free vector fields?" [Arnold's Problems, Problem 1990-16] (discussed in detail in [Arnold–Khesin 1998; §8.A], not only for knots but also for links)

> "The dream is to define such a hierarchy of invariants for generic vector fields such that, [when] all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy." [Arnold–Khesin 1998; §7.C "Higher-order linking integrals"]

No-go theorems: S. S. Podkorytov (2004), S. Baader–J. Marché (2012), E. A. Kudryavtseva (2016)

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)
- May 2025: "not proved" [video: MGT Seminar, May 16, `mathnet.ru`]

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)
- May 2025: "not proved" [video: MGT Seminar, May 16, `mathnet.ru`]

Previous versions of the claimed result (2011–2016):

- Theorem 8 in `arXiv:1105.5876v2`
- Theorem 4.2 in *J. Phys.: Conf. Ser.* **544** (2014), Paper No. 012015
- Theorem 9 in the book (Lambert Acad. Publ., Saarbrücken, 2016)

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)
- May 2025: "not proved" [video: MGT Seminar, May 16, `mathnet.ru`]

Previous versions of the claimed result (2011–2016):

- Theorem 8 in `arXiv:1105.5876v2`
- Theorem 4.2 in *J. Phys.: Conf. Ser.* **544** (2014), Paper No. 012015
- Theorem 9 in the book (Lambert Acad. Publ., Saarbrücken, 2016)

Initial version of the conjecture in *J. Geom. Phys.* **53** (2005), 180–196

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)
- May 2025: "not proved" [video: MGT Seminar, May 16, `mathnet.ru`]

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)
- May 2025: “not proved” [video: MGT Seminar, May 16, `mathnet.ru`]

Remark. There is a formula for the behavior under cabling for:

- The Alexander polynomial of a knot (Seifert, 1950)
- The Alexander polynomial of a link (Sumners–Woods, 1977)
- The Conway potential function of a link (Cimasoni, 2005)

The Conway polynomial of an m-component link

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Conjecture (P. M. Akhmetiev). There exists a 4-cableable invariant of 3-component links which is a polynomial in the coefficients c_0, c_1 of the Conway polynomial of the link and of its proper sublinks, but is not a function of the pairwise linking numbers.

Claimed to be proved by Akhmetiev a number of times in 2019–2025:

- Assertion B.1 in *Proc. Steklov Inst. Math.* **308** (2020), 42–55
- Theorems 14, 17 in *J. Geom. Phys.* **170** (2021), Paper No. 104379
- 5 talks at Moscow Geometric Topology Seminar (2019: abstract only; 2021, 2024, two talks in 2025: abstract and video at `mathnet.ru`)
- May 2025: "not proved" [video: MGT Seminar, May 16, `mathnet.ru`]

Remark. There is a formula for the behavior under cabling for:

- The Alexander polynomial of a knot (Seifert, 1950)
- The Alexander polynomial of a link (Sumners–Woods, 1977)
- The Conway potential function of a link (Cimasoni, 2005)

Communicated in 2021 (and again at/after each subsequent talk)

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers.

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. ($\Rightarrow$ Arnold–Akhmetiev Problem)

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. (⇒ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks;

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. (⇒ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; (⇒ Akhmetiev Conjecture)

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. (⇒ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; (⇒ Akhmetiev Conjecture)

(b) $\bar{\bar{\omega}}$ is strongly $(m+1)$-satellitable;

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. (⇒ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; (⇒ Akhmetiev Conjecture)

(b) $\bar{\bar{\omega}}$ is strongly $(m+1)$-satellitable;

(c) $\bar{\bar{\omega}}$ is of type $2 + \frac{m(m+1)}{2}$ and of colored type 1;

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. ($\Rightarrow$ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; ($\Rightarrow$ Akhmetiev Conjecture)

(b) $\bar{\bar{\omega}}$ is strongly $(m+1)$-satellitable;

(c) $\bar{\bar{\omega}}$ is of type $2 + \frac{m(m+1)}{2}$ and of colored type 1;
(v is of colored type n: the standard extension $\bar{v}$ to singular links vanishes on singular links with $\geq n+1$ self-intersections of components)

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. ($\Rightarrow$ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; ($\Rightarrow$ Akhmetiev Conjecture)

(b) $\bar{\bar{\omega}}$ is strongly $(m+1)$-satellitable;

(c) $\bar{\bar{\omega}}$ is of type $2 + \frac{m(m+1)}{2}$ and of colored type 1;

(v is of colored type n: the standard extension $\bar{v}$ to singular links vanishes on singular links with $\geq n+1$ self-intersections of components)

(d) $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L, if $m = 3, 4$.

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. (⇒ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; (⇒ Akhmetiev Conjecture)

(b) $\bar{\bar{\omega}}$ is strongly $(m+1)$-satellitable;

(c) $\bar{\bar{\omega}}$ is of type $2 + \frac{m(m+1)}{2}$ and of colored type 1;

(v is of colored type n: the standard extension $\bar{v}$ to singular links vanishes on singular links with $\geq n+1$ self-intersections of components)

(d) $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L, if $m = 3, 4$.

Proof: Based on Conway potential function and Fibonacci and Lucas polynomials

Main Theorem. For each $m \geq 3$ there exists an $(m+1)$-cableable finite type invariant $\bar{\bar{\omega}}$ of m-component links which is not a function of the pairwise linking numbers. ($\Rightarrow$ Arnold–Akhmetiev Problem)

Moreover,

(a) $\bar{\bar{\omega}}$ is a polynomial in the coefficients c_0, c_1 of the Conway polynomials of the link and of its proper sublinks; ($\Rightarrow$ Akhmetiev Conjecture)

(b) $\bar{\bar{\omega}}$ is strongly $(m+1)$-satellitable;

(c) $\bar{\bar{\omega}}$ is of type $2 + \frac{m(m+1)}{2}$ and of colored type 1;

(v is of colored type n: the standard extension $\bar{v}$ to singular links vanishes on singular links with $\geq n+1$ self-intersections of components)

(d) $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L, if $m = 3, 4$.

Proof: Based on Conway potential function and Fibonacci and Lucas polynomials

Corollary. The following invariant is 0-solenoidal:

$$\bar{\bar{\bar{\omega}}}(L) = \begin{cases} \dfrac{\bar{\bar{\omega}}(L)}{\lambda(L)\lambda^{\circ}(L)} & \text{if } \lambda(L) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \cdots + c_r z^{2r}\right)$$

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L) z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L) z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For $m = 2$:

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L)z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For $m = 2$:

- $\alpha(L) = \mathrm{lk}(L)$

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$
$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L) z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For $m = 2$:

- $\alpha(L) = \mathrm{lk}(L)$
- $\beta(L)$ is an integer lift of $\bar{\mu}_{1122}$; “generalized Sato–Levine invariant”

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L) z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For $m = 2$:

- $\alpha(L) = \mathrm{lk}(L)$
- $\beta(L)$ is an integer lift of $\bar{\mu}_{1122}$; "generalized Sato–Levine invariant"
- $\beta(L)$ generates the group of colored type 1 invariants modulo colored type 0 invariants (Kirk–Livingston, 1998)

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$
$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L)z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For $m = 2$:

- $\alpha(L) = \mathrm{lk}(L)$
- $\beta(L)$ is an integer lift of $\bar{\mu}_{1122}$; "generalized Sato–Levine invariant"
- $\beta(L)$ generates the group of colored type 1 invariants modulo colored type 0 invariants (Kirk–Livingston, 1998)
- $\alpha(L)$ and $\beta(L)$ form a complete set of invariants of self C_2-equivalence (aka Δ-link homotopy) (Nakanishi–Ohyama, 2003)

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar\nabla_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar\nabla_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L)z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For arbitrary m:

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L)z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For arbitrary m:

- $\alpha(L) = \sum_T \prod_{\{i,j\}\in E(T)} l_{ij}$, where T runs over all spanning trees of K_m (Hosokawa–Hartley–Hoste)

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$

$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L)z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For arbitrary m:

- $\alpha(L) = \sum_T \prod_{\{i,j\}\in E(T)} l_{ij}$, where T runs over all spanning trees of K_m (Hosokawa–Hartley–Hoste)
- $\beta(L)$ is a colored type 1 invariant

Given a PL link $L = (K_1, \dots, K_m)$, the formal power series

$$\bar{\nabla}_L(z) = \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

is invariant under PL isotopy (= under addition and deletion of local knots)

Remark. Every k-satellitable invariant is an invariant of PL isotopy (since PL isotopy implies F-isotopy)

$$\nabla_L(z) = z^{m-1}\left(c_0 + c_1 z^2 + \dots + c_r z^{2r}\right)$$
$$\bar{\nabla}_L(z) = z^{m-1}\Big(\alpha(L) + \beta(L) z^2 + \dots\Big),$$

where $\alpha(L) = c_0(L)$ and $\beta(L) = c_1(L) - c_0(L)\big(c_1(K_1) + \dots + c_1(K_m)\big)$.

For arbitrary m:

- $\alpha(L) = \sum_T \prod_{\{i,j\} \in E(T)} l_{ij}$, where T runs over all spanning trees of K_m (Hosokawa–Hartley–Hoste)
- $\beta(L)$ is a colored type 1 invariant
- $\beta(L)$ is an invariant of self C_2-equivalence

Addendum 1 to Main Theorem. $\bar{\bar{\omega}}(L)$ is a polynomial in $\beta(L)$, in the $\beta(\Lambda)$ for proper sublinks $\Lambda \subset L$ and in the pairwise linking numbers.

Addendum 1 to Main Theorem. $\bar{\bar{\omega}}(L)$ is a polynomial in $\beta(L)$, in the $\beta(\Lambda)$ for proper sublinks $\Lambda \subset L$ and in the pairwise linking numbers.

Addendum 2 to Main Theorem. For a link $L = (K_1, K_2, K_3)$ let $l_{ij} = \operatorname{lk}(K_i, K_j)$, $\lambda = l_{12} l_{23} l_{32}$ and $\beta_{ij} = \beta(K_i, K_j)$. Then

$$\bar{\bar{\omega}}(L) = \beta(L)\lambda - \alpha(L) \sum_{(i,j,k)\in\langle 3\rangle!} l_{ij} l_{jk} \beta_{ik} - \lambda \sum_{(i,j,k)\in\langle 3\rangle!} l_{ij} l_{jk} \frac{2l_{ik}^2 + l_{ij} l_{jk} + 1}{12},$$

where $\langle 3\rangle!$ denotes the set $\{(1,2,3), (2,3,1), (3,1,2)\}$ of all circular shifts of $(1,2,3)$.

Addendum 1 to Main Theorem. $\bar{\bar{\omega}}(L)$ is a polynomial in $\beta(L)$, in the $\beta(\Lambda)$ for proper sublinks $\Lambda \subset L$ and in the pairwise linking numbers.

Addendum 2 to Main Theorem. For a link $L = (K_1, K_2, K_3)$ let $l_{ij} = \mathrm{lk}(K_i, K_j)$, $\lambda = l_{12}l_{23}l_{32}$ and $\beta_{ij} = \beta(K_i, K_j)$. Then

$$\bar{\bar{\omega}}(L) = \beta(L)\lambda - \alpha(L) \sum_{(i,j,k)\in\langle 3\rangle!} l_{ij}l_{jk}\beta_{ik} - \lambda \sum_{(i,j,k)\in\langle 3\rangle!} l_{ij}l_{jk}\frac{2l_{ik}^2 + l_{ij}l_{jk} + 1}{12},$$

where $\langle 3\rangle!$ denotes the set $\{(1,2,3), (2,3,1), (3,1,2)\}$ of all circular shifts of $(1,2,3)$.

Terms correctly predicted by P. M. Akhmetiev (2005[±]/2014; 2021):

$$\beta(L)\lambda - \alpha(L) \sum_{(i,j,k)\in\langle 3\rangle!} l_{ij}l_{jk}\beta_{ik} - \lambda \sum_{(i,j,k)\in\langle 3\rangle!} l_{ij}l_{jk}\frac{2l_{ik}^2 + l_{ij}l_{jk}}{12}$$

Remainder: 0 (2021) / a degree 9 polynomial in the l_{ij} (April 2025)

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1,\dots,x_n)=\Omega_L(x_{c(1)},\dots,x_{c(m)})$$

$\Lambda=(L,c)$ colored link: $L=(K_1,\dots,K_m)$, $\quad c\colon\{1,\dots,m\}\to\{1,\dots,n\}$

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1, \dots, x_n) = \Omega_L(x_{c(1)}, \dots, x_{c(m)})$$

$\Lambda = (L, c)$ colored link: $L = (K_1, \dots, K_m)$, $\quad c\colon \{1, \dots, m\} \to \{1, \dots, n\}$

One-variable case: $\Omega_\Lambda(x) = \dfrac{\nabla_\Lambda(x - x^{-1})}{x - x^{-1}}$

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1, \dots, x_n) = \Omega_L(x_{c(1)}, \dots, x_{c(m)})$$

$\Lambda = (L, c)$ colored link: $L = (K_1, \dots, K_m)$, $\quad c\colon \{1, \dots, m\} \to \{1, \dots, n\}$

One-variable case: $\Omega_\Lambda(x) = \dfrac{\nabla_\Lambda(x - x^{-1})}{x - x^{-1}}$

Multi-variable case: Treat $z = x - x^{-1}$ as a quadratic equation in x.

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1,\dots,x_n)=\Omega_L(x_{c(1)},\dots,x_{c(m)})$$

$\Lambda=(L,c)$ colored link: $L=(K_1,\dots,K_m)$, $\quad c\colon\{1,\dots,m\}\to\{1,\dots,n\}$

One-variable case: $\Omega_\Lambda(x)=\dfrac{\nabla_\Lambda(x-x^{-1})}{x-x^{-1}}$

Multi-variable case: Treat $z=x-x^{-1}$ as a quadratic equation in x.

Select a root $x(z)=\frac{z}{2}\pm\sqrt{1+\frac{z^2}{4}}$ and expand the radical as a formal power series in z according to Newton' formula $(1+t)^r=1+rt+\frac{r(r-1)}{2}t^2+\dots$

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1, \dots, x_n) = \Omega_L(x_{c(1)}, \dots, x_{c(m)})$$

$\Lambda = (L, c)$ colored link: $L = (K_1, \dots, K_m)$, $\quad c\colon \{1, \dots, m\} \to \{1, \dots, n\}$

One-variable case: $\Omega_\Lambda(x) = \dfrac{\nabla_\Lambda(x - x^{-1})}{x - x^{-1}}$

Multi-variable case: Treat $z = x - x^{-1}$ as a quadratic equation in x.

Select a root $x(z) = \frac{z}{2} \pm \sqrt{1 + \frac{z^2}{4}}$ and expand the radical as a formal power series in z according to Newton' formula $(1+t)^r = 1 + rt + \frac{r(r-1)}{2}t^2 + \dots$

Set $\mho_\Lambda(z_1, \dots, z_n) = \Omega_\Lambda\big(x(z_1), \dots, x(z_n)\big) \in \mathbb{Q}[[z_1, \dots, z_n]]$.

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1,\dots,x_n)=\Omega_L(x_{c(1)},\dots,x_{c(m)})$$

$\Lambda=(L,c)$ colored link: $L=(K_1,\dots,K_m)$, $\quad c\colon\{1,\dots,m\}\to\{1,\dots,n\}$

One-variable case: $\Omega_\Lambda(x)=\dfrac{\nabla_\Lambda(x-x^{-1})}{x-x^{-1}}$

Multi-variable case: Treat $z=x-x^{-1}$ as a quadratic equation in x.

Select a root $x(z)=\frac{z}{2}\pm\sqrt{1+\frac{z^2}{4}}$ and expand the radical as a formal power series in z according to Newton' formula $(1+t)^r=1+rt+\frac{r(r-1)}{2}t^2+\dots$

Set $\mho_\Lambda(z_1,\dots,z_n)=\Omega_\Lambda\big(x(z_1),\dots,x(z_n)\big)\in\mathbb{Q}[[z_1,\dots,z_n]]$.

Theorem (M., 2003). (a) Both choices of the root lead to the same $\mho_\Lambda$.

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1,\dots,x_n)=\Omega_L(x_{c(1)},\dots,x_{c(m)})$$

$\Lambda=(L,c)$ colored link: $L=(K_1,\dots,K_m)$, $\quad c\colon\{1,\dots,m\}\to\{1,\dots,n\}$

One-variable case: $\Omega_\Lambda(x)=\dfrac{\nabla_\Lambda(x-x^{-1})}{x-x^{-1}}$

Multi-variable case: Treat $z=x-x^{-1}$ as a quadratic equation in x.

Select a root $x(z)=\frac{z}{2}\pm\sqrt{1+\frac{z^2}{4}}$ and expand the radical as a formal power series in z according to Newton' formula $(1+t)^r=1+rt+\frac{r(r-1)}{2}t^2+\dots$

Set $\mho_\Lambda(z_1,\dots,z_n)=\Omega_\Lambda\big(x(z_1),\dots,x(z_n)\big)\in\mathbb{Q}[[z_1,\dots,z_n]]$.

Theorem (M., 2003). (a) Both choices of the root lead to the same $\mho_\Lambda$.

(b) The coefficient of $\mho_\Lambda$ at a term of total degree k is of type $k+1$.

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1,\dots,x_n)=\Omega_L(x_{c(1)},\dots,x_{c(m)})$$

$\Lambda=(L,c)$ colored link: $L=(K_1,\dots,K_m)$, $\quad c\colon\{1,\dots,m\}\to\{1,\dots,n\}$

One-variable case: $\Omega_\Lambda(x)=\dfrac{\nabla_\Lambda(x-x^{-1})}{x-x^{-1}}$

Multi-variable case: Treat $z=x-x^{-1}$ as a quadratic equation in x.

Select a root $x(z)=\frac{z}{2}\pm\sqrt{1+\frac{z^2}{4}}$ and expand the radical as a formal power series in z according to Newton' formula $(1+t)^r=1+rt+\frac{r(r-1)}{2}t^2+\dots$

Set $\mho_\Lambda(z_1,\dots,z_n)=\Omega_\Lambda\big(x(z_1),\dots,x(z_n)\big)\in\mathbb{Q}[[z_1,\dots,z_n]]$.

Theorem (M., 2003). (a) Both choices of the root lead to the same $\mho_\Lambda$.

(b) The coefficient of $\mho_\Lambda$ at a term of total degree k is of type $k+1$.

(c) The total degree of every nonzero term of $\mho_\Lambda$ has the same parity as m.

Conway potential function (sign-refined multi-variable Alexander polynomial)

$$\Omega_\Lambda(x_1,\dots,x_n)=\Omega_L(x_{c(1)},\dots,x_{c(m)})$$

$\Lambda=(L,c)$ colored link: $L=(K_1,\dots,K_m)$, $\quad c\colon\{1,\dots,m\}\to\{1,\dots,n\}$

One-variable case: $\Omega_\Lambda(x)=\dfrac{\nabla_\Lambda(x-x^{-1})}{x-x^{-1}}$

Multi-variable case: Treat $z=x-x^{-1}$ as a quadratic equation in x.

Select a root $x(z)=\frac{z}{2}\pm\sqrt{1+\frac{z^2}{4}}$ and expand the radical as a formal power series in z according to Newton' formula $(1+t)^r=1+rt+\frac{r(r-1)}{2}t^2+\dots$

Set $\mho_\Lambda(z_1,\dots,z_n)=\Omega_\Lambda\big(x(z_1),\dots,x(z_n)\big)\in\mathbb{Q}[[z_1,\dots,z_n]]$.

Theorem (M., 2003). (a) Both choices of the root lead to the same $\mho_\Lambda$.

(b) The coefficient of $\mho_\Lambda$ at a term of total degree k is of type $k+1$.

(c) The total degree of every nonzero term of $\mho_\Lambda$ has the same parity as m.

(d) $\mho_\Lambda(4y_1,\dots,4y_n)\in\mathbb{Z}[[y_1,\dots,y_n]]$.

Theorem. Let $L = (K_1, \dots, K_m)$ be a link and let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

$$\mho_L(z_1, \dots, z_m) = \sum_T \prod_{\{i,j\} \in E(T)} l_{ij} \prod_{v \in V(T)} z_v^{\deg(v)-1} + (\text{terms of total degrees} \geq m),$$

where T runs over all spanning trees of K_m.

Theorem. Let $L = (K_1, \dots, K_m)$ be a link and let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

$$\mho_L(z_1, \dots, z_m) = \sum_T \prod_{\{i,j\} \in E(T)} l_{ij} \prod_{v \in V(T)} z_v^{\deg(v)-1} + (\text{terms of total degrees} \geq m),$$

where T runs over all spanning trees of K_m.

Remark. $\prod_{v \in V(T)} z_v^{\deg(v)-1}$ has total degree $m-2$ for each T.

Theorem. Let $L = (K_1, \dots, K_m)$ be a link and let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

$$\mho_L(z_1, \dots, z_m) = \sum_T \prod_{\{i,j\} \in E(T)} l_{ij} \prod_{v \in V(T)} z_v^{\deg(v)-1}$$
$$+(\text{terms of total degrees} \geq m),$$

where T runs over all spanning trees of K_m.

Remark. $\prod_{v \in V(T)} z_v^{\deg(v)-1}$ has total degree $m-2$ for each T.

Corollary: the Hosokawa–Hartley–Hoste formula for $\alpha(L)$

Theorem. Let $L = (K_1, \dots, K_m)$ be a link and let $l_{ij} = \mathrm{lk}(K_i, K_j)$. Then

$$\mho_L(z_1, \dots, z_m) = \sum_T \prod_{\{i,j\} \in E(T)} l_{ij} \prod_{v \in V(T)} z_v^{\deg(v)-1} + (\text{terms of total degrees} \geq m),$$

where T runs over all spanning trees of K_m.

Remark. $\prod_{v \in V(T)} z_v^{\deg(v)-1}$ has total degree $m-2$ for each T.

Corollary: the Hosokawa–Hartley–Hoste formula for $\alpha(L)$

Similar formulas:

- L. Traldi (1988), A. Yu. Buryak (2011) for $\Omega_L(1+v_1, \dots, 1+v_m)$
- J. Levine (1999) for $\Delta_L(1+u_1, \dots, 1+u_m)$

Theorem. Let $L = (K_1, \dots, K_m)$ be an m-component link, for an $S \subset [m]$ let $L_S = (K_{s_1}, \dots, K_{s_n})$, where $S = \{s_1, \dots, s_n\}$, and let $l_{ij} = \mathrm{lk}(L_{\{i,j\}})$. Then

$$\omega(L) = \beta(L) - \left(\sum_{S \subsetneq [m]} \omega(L_S) \sum_F \prod_{\{i,j\} \in E(F)} l_{ij} \right) + \text{(a polynomial in } l_{ij}),$$

where F runs over all rooted forests with all roots in S and with the non-roots being precisely all the elements of $[m] \setminus S$.

A *rooted forest* is a graph whose every component is a rooted tree (that is, a tree with a distinguished vertex) containing at least one edge.

Theorem. Let $L=(K_1,\dots,K_m)$ be an m-component link, for an $S\subset[m]$ let $L_S=(K_{s_1},\dots,K_{s_n})$, where $S=\{s_1,\dots,s_n\}$, and let $l_{ij}=\operatorname{lk}(L_{\{i,j\}})$. Then

$$\omega(L)=\beta(L)-\left(\sum_{S\subsetneq[m]}\omega(L_S)\sum_F\prod_{\{i,j\}\in E(F)}l_{ij}\right)+\text{(a polynomial in } l_{ij}),$$

where F runs over all rooted forests with all roots in S and with the non-roots being precisely all the elements of $[m]\setminus S$.

A *rooted forest* is a graph whose every component is a rooted tree (that is, a tree with a distinguished vertex) containing at least one edge.

Corollary. $\omega(L)=\sum_\Lambda P_\Lambda\beta(\Lambda)+Q$, where Λ runs over all sublinks of L and each P_Λ as well as Q are polynomials in the pairwise linking numbers of L.

$L = (K_1, \dots, K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar{\mho}_L(z_1, \dots, z_m) := \frac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$$

$L = (K_1, \dots, K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar{\mho}_L(z_1, \dots, z_m) := \frac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$$

Step 2. Let $\ell_{ij} = \sqrt{\mathrm{lk}(K_i, K_j)} \in [0, \infty) \cup i[0, \infty)$ and $\ell = \prod_{i<j} \ell_{ij}$.

$L = (K_1, \dots, K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar{\mho}_L(z_1, \dots, z_m) := \frac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$$

Step 2. Let $\ell_{ij} = \sqrt{\operatorname{lk}(K_i, K_j)} \in [0, \infty) \cup i[0, \infty)$ and $\ell = \prod_{i<j} \ell_{ij}$.

$$\bar{\bar{\mho}}_L(z_1, \dots, z_m) := \frac{\ell^{4-m}\, \bar{\mho}_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \bar{\mho}_{(K_i, K_j)}(\ell z_i, \ell z_j)}$$

$L = (K_1, \dots, K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar{\mho}_L(z_1, \dots, z_m) := \frac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$$

Step 2. Let $\ell_{ij} = \sqrt{\operatorname{lk}(K_i, K_j)} \in [0, \infty) \cup i[0, \infty)$ and $\ell = \prod_{i<j} \ell_{ij}$.

$$\bar{\bar{\mho}}_L(z_1, \dots, z_m) := \frac{\ell^{4-m}\, \bar{\mho}_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \bar{\mho}_{(K_i,K_j)}(\ell z_i, \ell z_j)} = \frac{\ell^{2-m}\, \bar{\mho}_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \frac{\bar{\mho}_{(K_i,K_j)}(\ell_{ij} \frac{\ell}{\ell_{ij}} z_i,\ \ell_{ij} \frac{\ell}{\ell_{ij}} z_j)}{\ell_{ij}^2}}$$

$L = (K_1, \dots, K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar{\mho}_L(z_1, \dots, z_m) := \frac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$$

Step 2. Let $\ell_{ij} = \sqrt{\operatorname{lk}(K_i, K_j)} \in [0, \infty) \cup i[0, \infty)$ and $\ell = \prod_{i<j} \ell_{ij}$.

$$\bar{\bar{\mho}}_L(z_1, \dots, z_m) := \frac{\ell^{4-m}\, \bar{\mho}_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \bar{\mho}_{(K_i,K_j)}(\ell z_i, \ell z_j)} = \frac{\ell^{2-m}\, \bar{\mho}_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \frac{\bar{\mho}_{(K_i,K_j)}(\ell_{ij} \frac{\ell}{\ell_{ij}} z_i,\ \ell_{ij} \frac{\ell}{\ell_{ij}} z_j)}{\ell_{ij}^2}}$$

where $\dfrac{\bar{\mho}_{(K_i,K_j)}(\ell_{ij} u, \ell_{ij} v)}{\ell_{ij}^2} = 1 + au^2 + buv + cv^2 + \dots$

$L = (K_1, \dots, K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar\mho_L(z_1, \dots, z_m) := \frac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$$

Step 2. Let $\ell_{ij} = \sqrt{\mathrm{lk}(K_i, K_j)} \in [0, \infty) \cup i[0, \infty)$ and $\ell = \prod_{i<j} \ell_{ij}$.

$$\bar{\bar\mho}_L(z_1, \dots, z_m) := \frac{\ell^{4-m}\,\bar\mho_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \bar\mho_{(K_i,K_j)}(\ell z_i, \ell z_j)} = \frac{\ell^{2-m}\,\bar\mho_L(\ell z_1, \dots, \ell z_m)}{\prod_{i<j} \frac{\bar\mho_{(K_i,K_j)}(\ell_{ij}\frac{\ell}{\ell_{ij}} z_i,\ \ell_{ij}\frac{\ell}{\ell_{ij}} z_j)}{\ell_{ij}^2}}$$

where $\dfrac{\bar\mho_{(K_i,K_j)}(\ell_{ij} u, \ell_{ij} v)}{\ell_{ij}^2} = 1 + au^2 + buv + cv^2 + \dots$

due to $\bar\mho_{(K_i,K_j)}(u, v) = \ell_{ij}^2 + au^2 + buv + cv^2 + \dots$

$L=(K_1,\dots,K_m)$. **Step 1.** Make $\mho_L$ invariant under PL isotopy:

$$\bar\mho_L(z_1,\dots,z_m):=\frac{\mho_L(z_1,\dots,z_m)}{\nabla_{K_1}(z_1)\cdots\nabla_{K_m}(z_m)}$$

Step 2. Let $\ell_{ij}=\sqrt{\operatorname{lk}(K_i,K_j)}\in[0,\infty)\cup i[0,\infty)$ and $\ell=\prod_{i<j}\ell_{ij}$.

$$\bar{\bar\mho}_L(z_1,\dots,z_m):=\frac{\ell^{4-m}\,\bar\mho_L(\ell z_1,\dots,\ell z_m)}{\prod_{i<j}\bar\mho_{(K_i,K_j)}(\ell z_i,\ell z_j)}=\frac{\ell^{2-m}\,\bar\mho_L(\ell z_1,\dots,\ell z_m)}{\prod_{i<j}\frac{\bar\mho_{(K_i,K_j)}(\ell_{ij}\frac{\ell}{\ell_{ij}}z_i,\ \ell_{ij}\frac{\ell}{\ell_{ij}}z_j)}{\ell_{ij}^2}}$$

where $\dfrac{\bar\mho_{(K_i,K_j)}(\ell_{ij}u,\ell_{ij}v)}{\ell_{ij}^2}=1+au^2+buv+cv^2+\dots$

due to $\bar\mho_{(K_i,K_j)}(u,v)=\ell_{ij}^2+au^2+buv+cv^2+\dots$

Step 3. $\bar{\bar{\bar\mho}}_L(z_1,\dots,z_m)=\dfrac{\bar{\bar\mho}_L(z_1,\dots,z_m)}{\prod\limits_{(i,j,k)\in\langle m\rangle^{\underline{\langle 3\rangle}}}\left(1+\frac{1}{12}\ell_{ij}^2\ell_{ik}^2\ell^2z_jz_k\right)}$, where $\langle m\rangle^{\underline{\langle 3\rangle}}$

denotes the set of all injections $\langle 3\rangle\to\langle m\rangle$ that respect the cyclic order.

Addendum 3 to Main Theorem. For a link L of $m \geq 3$ components $\bar{\bar{\omega}}(L)$ is the coefficient of $\bar{\bar{\bar{\mho}}}_L(z_1, \dots, z_m)$ at $z_1 \cdots z_m$.

Addendum 3 to Main Theorem. For a link L of $m \geq 3$ components $\bar{\bar{\omega}}(L)$ is the coefficient of $\bar{\bar{\bar{\mho}}}_L(z_1,\dots,z_m)$ at $z_1\cdots z_m$.

In terms of the coefficient $\bar{\omega}(L)$ of $\bar{\bar{\mho}}_L(z_1,\dots,z_m)$ at $z_1\cdots z_m$:

$$\bar{\bar{\omega}}(L) = \bar{\omega}(L) - \tfrac{1}{12}\lambda \sum_{(i,j,k)\in\langle m\rangle^{\underline{\langle 3\rangle}}} l_{ij}l_{ik} \sum_{(i_1,\dots,i_{m-2})\in([m]\setminus\{j,k\})!} l_{ji_1}l_{i_1i_2}\cdots l_{i_{m-3}i_{m-2}}l_{i_{m-2}k}$$

where $l_{ij} = \operatorname{lk}(K_i,K_j)$ and $\lambda = \prod_{i<j} l_{ij}$,

Addendum 3 to Main Theorem. For a link L of $m \geq 3$ components $\bar{\bar{\omega}}(L)$ is the coefficient of $\bar{\bar{\bar{\mho}}}_L(z_1, \dots, z_m)$ at $z_1 \cdots z_m$.

In terms of the coefficient $\bar{\omega}(L)$ of $\bar{\bar{\mho}}_L(z_1, \dots, z_m)$ at $z_1 \cdots z_m$:

$$\bar{\bar{\omega}}(L) = \bar{\omega}(L) - \tfrac{1}{12}\lambda \sum_{(i,j,k)\in\langle m\rangle^{\underline{\langle 3\rangle}}} l_{ij} l_{ik} \sum_{(i_1,\dots,i_{m-2})\in([m]\setminus\{j,k\})!} l_{ji_1} l_{i_1 i_2} \cdots l_{i_{m-3} i_{m-2}} l_{i_{m-2} k}$$

where $l_{ij} = \mathrm{lk}(K_i, K_j)$ and $\lambda = \prod_{i<j} l_{ij}$,

In terms of the coefficient $\omega(L)$ of $\mho_L(z_1, \dots, z_m)$ at $z_1 \cdots z_m$:

$$\bar{\omega}(L) = \lambda\omega(L) - \sum_{\{j,k\}\subset[m]} \frac{\lambda}{l_{jk}}\omega_{jk} \sum_{(i_1,\dots,i_{m-2})\in([m]\setminus\{j,k\})!} l_{ji_1} l_{i_1 i_2} \cdots l_{i_{m-3} i_{m-2}} l_{i_{m-2} k}$$

where $\omega_{ij} = \omega(K_i, K_j)$.

$\omega(L)$ is also the coefficient of $\bar{\mho}_L(z_1, \dots, z_m) = \dfrac{\mho_L(z_1, \dots, z_m)}{\nabla_{K_1}(z_1) \cdots \nabla_{K_m}(z_m)}$ at $z_1 \cdots z_m$.

Conjecture. $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L.

Conjecture. $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L.

$\Leftrightarrow$ $\lambda\omega(L)$ is not a function of invariants of proper sublinks of L.

Conjecture. $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L.

$\Leftrightarrow$ $\lambda\omega(L)$ is not a function of invariants of proper sublinks of L.

Proposition 1. $\omega(L)$ is not a function of invariants of proper sublinks of L.

Conjecture. $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L.

$\Leftrightarrow$ $\lambda\omega(L)$ is not a function of invariants of proper sublinks of L.

Proposition 1. $\omega(L)$ is not a function of invariants of proper sublinks of L.

Proof. $\omega(L)$ has a remarkably simple crossing change formula for a positive self-intersection of the ith component of L:

$$\omega(L_+) - \omega(L_-) = \sum_{(j_1,\dots,j_{m-1})\in([m]\setminus\{i\})!} l_{i'j_1} l_{j_1j_2} \cdots l_{j_{m-2}j_{m-1}} l_{j_{m-1}i''}, \qquad (\times)$$

where $L_\pm = (K_1, \dots, K_{i_\pm}, \dots, K_m)$, the singular knot between K_{i_+} and K_{i_-} is smoothed to a two-component link $(K_{i'}, K_{i''})$ and $l_{jk} = \operatorname{lk}(K_j, K_k)$.

Conjecture. $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L.

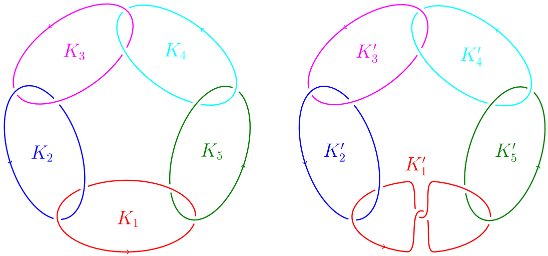
$\Leftrightarrow$ $\lambda\omega(L)$ is not a function of invariants of proper sublinks of L.

Proposition 1. $\omega(L)$ is not a function of invariants of proper sublinks of L.

Proof. $\omega(L)$ has a remarkably simple crossing change formula for a positive self-intersection of the ith component of L:

$$\omega(L_+) - \omega(L_-) = \sum_{(j_1,\dots,j_{m-1})\in([m]\setminus\{i\})!} l_{i'j_1} l_{j_1 j_2} \cdots l_{j_{m-2}j_{m-1}} l_{j_{m-1}i''}, \qquad (\times)$$

where $L_\pm = (K_1, \dots, K_{i_\pm}, \dots, K_m)$, the singular knot between K_{i_+} and K_{i_-} is smoothed to a two-component link $(K_{i'}, K_{i''})$ and $l_{jk} = \mathrm{lk}(K_j, K_k)$.



L_m $(m = 5)$ L'_m

$\omega(L'_m) - \omega(L_m) = 1.$ □

Conjecture. $\bar{\bar{\omega}}(L)$ is not a function of invariants of proper sublinks of L.

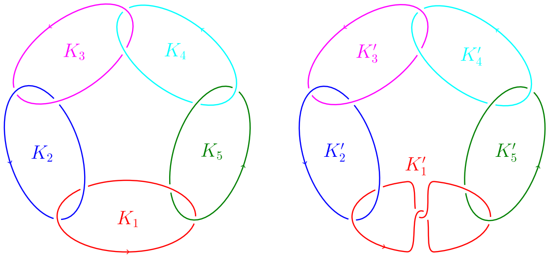
$\Leftrightarrow$ $\lambda\omega(L)$ is not a function of invariants of proper sublinks of L.

Proposition 1. $\omega(L)$ is not a function of invariants of proper sublinks of L.

Proof. $\omega(L)$ has a remarkably simple crossing change formula for a positive self-intersection of the ith component of L:

$$\omega(L_+) - \omega(L_-) = \sum_{(j_1,\dots,j_{m-1})\in([m]\setminus\{i\})!} l_{i'j_1} l_{j_1j_2}\cdots l_{j_{m-2}j_{m-1}} l_{j_{m-1}i''}, \qquad (\times)$$

where $L_\pm = (K_1,\dots,K_{i_\pm},\dots,K_m)$, the singular knot between K_{i_+} and K_{i_-} is smoothed to a two-component link $(K_{i'},K_{i''})$ and $l_{jk} = \mathrm{lk}(K_j,K_k)$.



L_m $(m=5)$ L'_m

$\omega(L'_m) - \omega(L_m) = 1.$ □

Proposition 2. If $m=3$, $\lambda\omega(L)$ (and hence $\bar{\bar{\omega}}(L)$) is not a function of invariants of proper sublinks of L.

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proof. When all the $l_{ij} = \ell$, formula $(\times)$ takes the form

$$\omega(L_+) - \omega(L_-) = \ell^{m-2} \sum_{j,k \in [m] \setminus \{i\},\ j \neq k} l_{i'j} l_{ki''}$$

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proof. When all the $l_{ij} = \ell$, formula $(\times)$ takes the form

$$\omega(L_+) - \omega(L_-) = \ell^{m-2} \sum_{j,k\in[m]\setminus\{i\},\ j\neq k} l_{i'j} l_{ki''}$$

and its 3-component case takes the form

$$\omega(K_{i_+}, K_j, K_k) - \omega(K_{i_-}, K_j, K_k) = \ell\big(l_{i'j} l_{ki''} + l_{i'k} l_{ji''}\big).$$

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proof. When all the $l_{ij} = \ell$, formula $(\times)$ takes the form

$$\omega(L_+) - \omega(L_-) = \ell^{m-2} \sum_{j,k\in[m]\setminus\{i\},\ j\neq k} l_{i'j} l_{ki''}$$

and its 3-component case takes the form

$$\omega(K_{i_+}, K_j, K_k) - \omega(K_{i_-}, K_j, K_k) = \ell\big(l_{i'j} l_{ki''} + l_{i'k} l_{ji''}\big).$$

Thus $\omega(L)$ has exactly the same crossing change formula as

$$\omega'(L) := \ell^{m-3} \textstyle\sum_{i<j<k} \omega(K_i, K_j, K_k).$$

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proof. When all the $l_{ij} = \ell$, formula $(\times)$ takes the form

$$\omega(L_+) - \omega(L_-) = \ell^{m-2} \sum_{j,k\in[m]\setminus\{i\},\ j\neq k} l_{i'j} l_{ki''}$$

and its 3-component case takes the form

$$\omega(K_{i_+}, K_j, K_k) - \omega(K_{i_-}, K_j, K_k) = \ell\big(l_{i'j} l_{ki''} + l_{i'k} l_{ji''}\big).$$

Thus $\omega(L)$ has exactly the same crossing change formula as

$$\omega'(L) := \ell^{m-3} \textstyle\sum_{i<j<k} \omega(K_i, K_j, K_k).$$

Hence $\omega(L) - \omega'(L)$ is a finite type invariant of link homotopy.

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proof. When all the $l_{ij} = \ell$, formula $(\times)$ takes the form

$$\omega(L_+) - \omega(L_-) = \ell^{m-2} \sum_{j,k \in [m] \setminus \{i\},\ j \neq k} l_{i'j} l_{ki''}$$

and its 3-component case takes the form

$$\omega(K_{i_+}, K_j, K_k) - \omega(K_{i_-}, K_j, K_k) = \ell \big(l_{i'j} l_{ki''} + l_{i'k} l_{ji''} \big).$$

Thus $\omega(L)$ has exactly the same crossing change formula as

$$\omega'(L) := \ell^{m-3} \textstyle\sum_{i<j<k} \omega(K_i, K_j, K_k).$$

Hence $\omega(L) - \omega'(L)$ is a finite type invariant of link homotopy.

But all such invariants are known to be polynomials in the pairwise linking numbers for 4- and 5-component links [Mellor–Thurston, 2000]. □

Proposition 3. For $m = 4, 5$ and for links L with all $l_{ij} = \ell$, $\omega(L)$ is a function of ℓ and of the invariants $\omega(\Lambda)$ for 3-components sublinks Λ of L.

Proof. When all the $l_{ij} = \ell$, formula $(\times)$ takes the form

$$\omega(L_+) - \omega(L_-) = \ell^{m-2} \sum_{j,k\in[m]\setminus\{i\},\ j\neq k} l_{i'j} l_{ki''}$$

and its 3-component case takes the form

$$\omega(K_{i_+}, K_j, K_k) - \omega(K_{i_-}, K_j, K_k) = \ell\big(l_{i'j} l_{ki''} + l_{i'k} l_{ji''}\big).$$

Thus $\omega(L)$ has exactly the same crossing change formula as

$$\omega'(L) := \ell^{m-3} \textstyle\sum_{i<j<k} \omega(K_i, K_j, K_k).$$

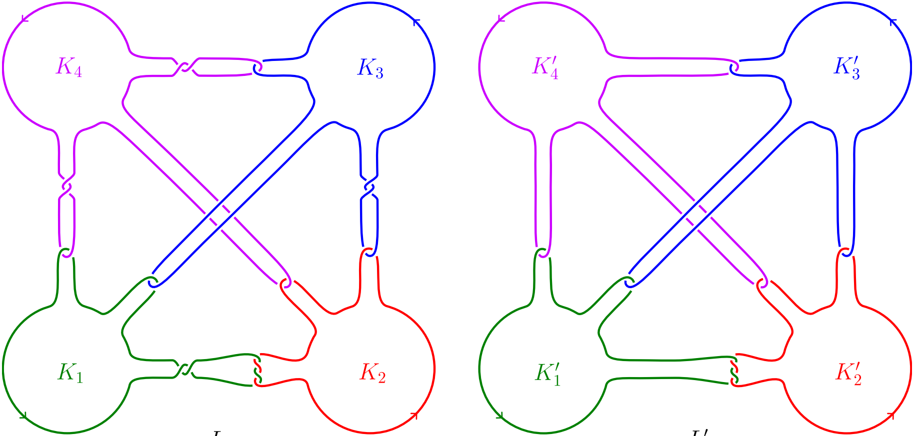
Hence $\omega(L) - \omega'(L)$ is a finite type invariant of link homotopy.

But all such invariants are known to be polynomials in the pairwise linking numbers for 4- and 5-component links [Mellor–Thurston, 2000]. □

(But not for 6-component links [X.-S. Lin, 2000].)

Proposition 4. For $m = 4$, $\lambda\omega(L)$ (and hence $\bar{\bar{\omega}}(L)$) is not a function of invariants of proper sublinks of L.

Proposition 4. For $m = 4$, $\lambda\omega(L)$ (and hence $\bar{\bar{\omega}}(L)$) is not a function of invariants of proper sublinks of L.



L $\qquad$ L'

Links $L = S\begin{pmatrix} 2\ 1\ 1 & 1\ \ 0\ \ -1 \\ \ \ 1\ 1 , & \ -1\ \ \ 0 \\ \ \ \ \ 1 & \ \ \ \ \ \ 1 \end{pmatrix}$ and $L' = S\begin{pmatrix} 2\ 1\ 1 & 0\ 0\ 0 \\ \ \ 1\ 1 , & \ \ 0\ 0 \\ \ \ \ \ 1 & \ \ \ \ 0 \end{pmatrix}$.