Main results

$a, b : S^1 \to Y \qquad [a], [b] \in \pi_1(Y)$

$a \overset{r}{\sim} b \Leftrightarrow a \overset{r}{\approx} b \Leftrightarrow [a]^{-1}[b] \in \gamma^{r+1}\pi_1(Y)$

r-similarity $\overset{r}{\sim}$ on $[X, Y]$ homotopy

analogue of Gusarov's n-eqce of knots

ensemble $D \in \langle Y^X \rangle$

$D = \sum_k v_k \langle a_k \rangle \qquad a_k : X \to Y \qquad v_k \in \mathbb{Z}$

def $D \overset{r}{\doteq} 0$ (D r-coherent) $\Leftrightarrow$

$\forall (T \subseteq X, |T| \leq r) \quad D|_T = 0$ in $\langle Y^{(T)} \rangle$

def $a \overset{r}{\sim} b$ (r-similar) $\quad a, b : X \to Y$

$\Leftrightarrow \exists \; A = \sum_k v_k \langle a_k \rangle \in \langle Y^X \rangle$

such that $a_k \sim a$, $\quad A \overset{r}{\doteq} \langle b \rangle$.

$\overset{r}{\sim}$ - equivalence on $[X, Y]$

Thm 1 $\quad a, b : S^1 \to Y$

$a \overset{r}{\sim} b \; (\Leftarrow) \!\!=\!\! (\Rightarrow) \; [a]^{-1}[b] \in \gamma^{r+1}\pi_1(Y)$

easy difficult

sketch proof of $\Leftarrow$

$[b] = [\dots[g_0, g_1], \dots, g_r] \overset{?}{\Rightarrow} \bar{o} \overset{r}{\sim} b$

universal exple $\quad Y = V_I := \bigvee_{i \in I} S^1$

$I = \{0, \dots, r\}$

$g_i = [e_i] \qquad e_i : S^1 \to V_I$ can. embs

only important: b Brunnian : $\forall i \;\; p_i \circ b \sim \bar{o}$

$p_i : V_I \to V_I \qquad p_i \circ e_k = \begin{cases} \bar{o}, & k = i, \\ e_k & \text{othws} \end{cases}$

[b Brunnian $\Leftrightarrow [b] \in \bigcap_{i \in I} \mathrm{NormClos}([e_i])$

$D := \sum_{J \subseteq I} (-1)^{|J|} \langle p_J \circ b \rangle \in \langle \Omega V_I \rangle \qquad \Omega V_I = V_I^{S^1}$

$p_J : V_I \to V_I \qquad p_J \circ e_k = \bar{o}, \;\; k \in J$

$$D = \sum_{J \subseteq I} (-1)^{|J|} \langle p_J \circ b \rangle \in \langle \Omega V_I \rangle$$

claim $D \overset{r}{=} 0$

proof $T \subseteq S^1$, $|T| \leq r$. $D|_T \overset{?}{=} 0$ in $\langle V_I^{(T)} \rangle$

$b(T) \subseteq V_I$ $\quad \exists i: b(T) \subseteq V_{I \setminus \{i\}}$

$p_i \circ b|_T = b|_T$

$p_{J \cup \{i\}} \circ b|_T = p_J \circ b|_T$

summands of $D|_T$ annihilate $\square$

$$0 \overset{r}{=} D = \langle b \rangle - \sum_{J \neq \emptyset} (-1)^{|J|-1} \langle \underbrace{p_J \circ b}_{\sim \bar{o}} \rangle$$

since b Brunn

$\bar{o} \overset{r}{\sim} b$ $\square$

$\boxed{\Rightarrow}$ in Thm 1 relies on: Fact

$g_1^{x_1} \cdots g_n^{x_n} = 1 \qquad g_1, \dots, g_n \in G$

$x = (x_1, \dots, x_n) \in \mathbb{Z}^n$

Suppose: G nilpotent, $\gamma^{r+1} G = 1$

Then $\exists P_i, Q_j$ of degree at most r:

$$g_1^{x_1} \cdots g_n^{x_n} = 1 \iff \begin{cases} P_i(x) = 0 \\ Q_j(x) \equiv 0 \pmod{m_j} \end{cases}$$

Motivation of introducing $\overset{r}{\approx}$

Conj $X \xrightarrow{a} Y \xrightarrow{b} Z$

$a \overset{p-1}{\sim} \bar{o}$, $b \overset{q-1}{\sim} \bar{o} \Rightarrow b \circ a \overset{pq-1}{\sim} \bar{o}$.

True if $b \overset{q-1}{\approx} \bar{o}$

Conj $\overset{r}{\sim} \iff \overset{r}{\approx}$

Fissile ensembles

$H \in \langle Z^{(\Delta^n)} \rangle \qquad H = \sum_k w_k \langle h_k \rangle$

$\varepsilon(H) := \sum_k w_k$

def H fissile if

$H \in \langle Z^{(\Delta^n)} \rangle \qquad H = \sum_k w_k \langle h_k \rangle$

<u>def</u> H <u>fissile</u> : (1) $\varepsilon(H) = 1$

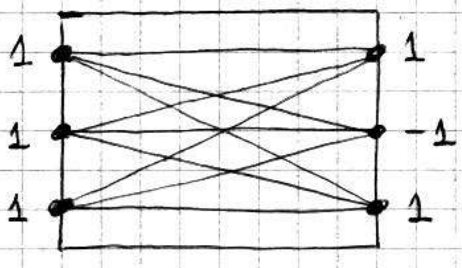
(2) $\forall$ (faces $A, B \subseteq \Delta^n$, $A \cap B = \emptyset$)

$$H\big|_{A \cup B} = F \# G := \sum_{ij} u_i v_j \langle (f_i, g_j) \rangle$$

$F \in \langle Z^{(A)} \rangle$, $\quad F = \sum_i u_i \langle f_i \rangle \quad \varepsilon(F) = 1$

$G \in \langle Z^{(B)} \rangle$, $\quad G = \sum_j v_j \langle g_j \rangle \quad \varepsilon(G) = 1$

$\left(\Rightarrow \quad F = H\big|_A, \quad G = H\big|_B \right)$

Z 

independence

Δ^1

Strong r-similarity $\overset{r}{\approx}$ on $[X, Y]$

<u>def</u> $a, b : X \to Y \qquad a \overset{r}{\approx} b \iff$

$\forall n \geq 0$ $\exists$ fissile $H \in \langle (Y^X)^{\cdot(\Delta^n)} \rangle$,

$H = \sum_k w_k \langle h_k \rangle$, such that

$h_k \sim \mathrm{const}(a) : \Delta^n \longrightarrow Y^X \ni a$

and $H \overset{r}{=} \langle \mathrm{const}(b) \rangle$

in $\langle Y^{(\Delta^n \times X)} \rangle$.

conj : $\overset{r}{\sim} \iff \overset{r}{\approx}$ evidence?

<u>Thm 2</u> $\quad a, b : S^1 \longrightarrow Y$

$a \overset{r}{\approx} b \iff [a]^{-1}[b] \in \gamma^{r+1}\pi_1(Y)$

difficult by Thm 1

<u>about proof of $\Leftarrow$</u> We may assume:

$Y = V_I$, $I = \{0, \dots, r\}$, $a = \bar{0}$, b Brunn.

$\| \quad \bigvee_{i \in I} S^1$

We seek fissile $H \in \langle (\Omega V_I)^{(\Delta^n)} \rangle$,

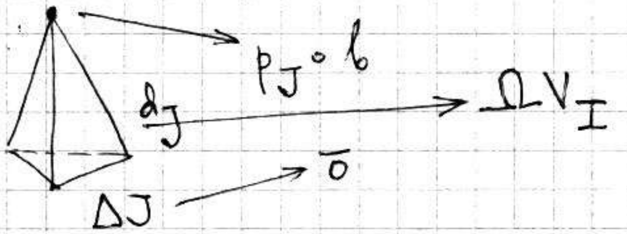
$H = \sum_k w_k \langle h_k \rangle$, $\quad h_k \sim \bar{0} : \Delta^n \to \Omega V_I$

$H \overset{r}{=} \langle \mathrm{const}(b) \rangle$ in $\langle Y^{(\Delta^n \times V_I)} \rangle$.

$H = \sum_k w_k \langle h_k \rangle, \qquad h_k : \Delta^n \to \Omega V_I$

h_k assembled from "parts" d_J :

$I \supseteq J \mapsto \left(d_J : C\Delta J \to \Omega V_I \right)$



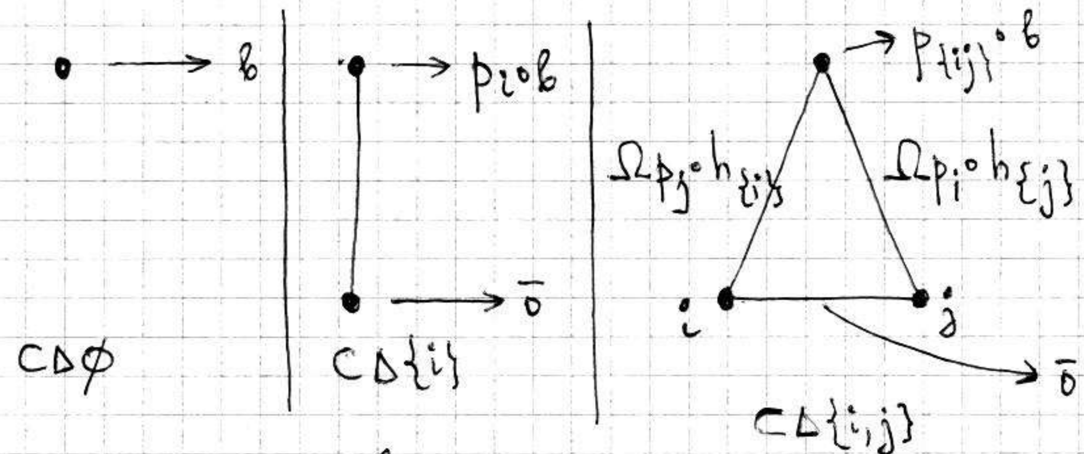
(1) $d_\emptyset : C\Delta\emptyset \to \Omega V_I, \quad \bullet \mapsto b$

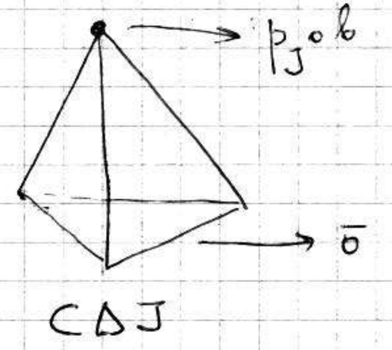
(2) $i \in J, \qquad K = J \setminus \{i\}$

$$\begin{tikzcd} C\Delta K \arrow[r, "d_K"] \arrow[d, "in"'] & \Omega V_I \arrow[d, "\Omega p_i"] \\ C\Delta J \arrow[r, "d_J"] & \Omega V_I \end{tikzcd}$$

(3) $d_J\big|_{C\Delta J} = \bar{o}$

Induction on $|J|$

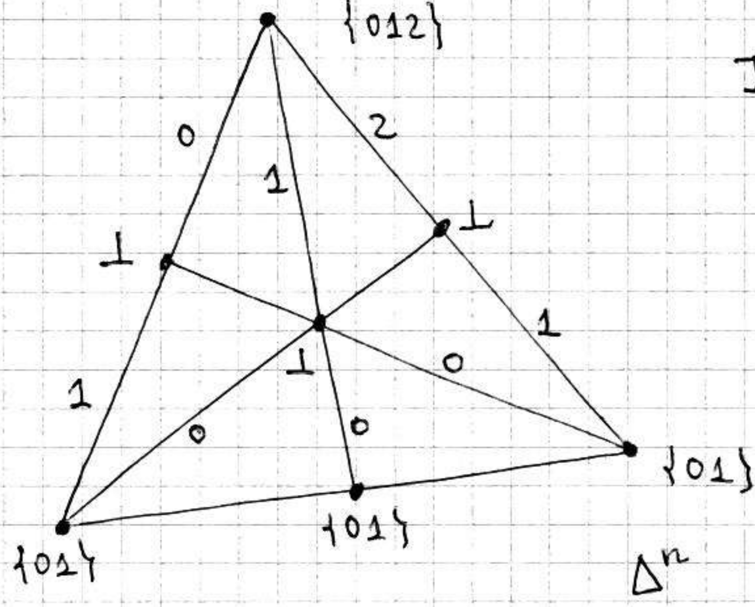




$\Omega_0 V_I$ contractible

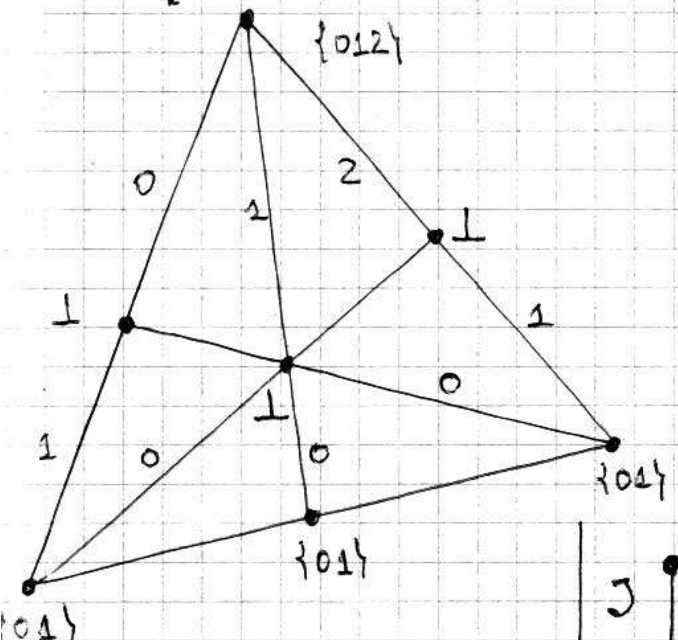
(somewhat non-constructive)

Assemble $h_k : \Delta^k \to \Omega V_I$



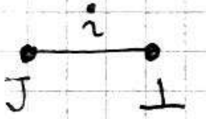
$I = \{0, \dots, r\}$

$H = \sum_k w_k \langle h_k \rangle$, $\quad h_k : \Delta^n \longrightarrow \Omega V_I$

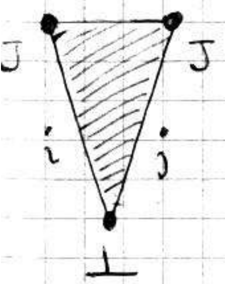


$\{0,\dots,r\}$
$\parallel$
$J \subseteq I$

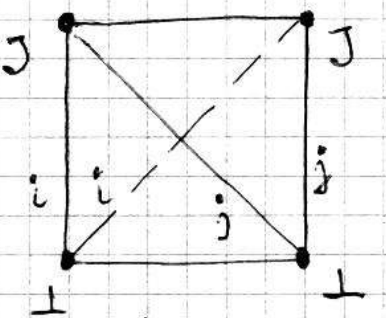
$\Rightarrow \quad J = K$



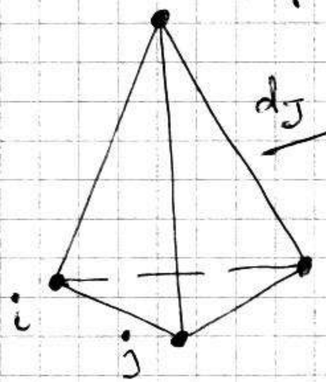
$i \in J$



$\Rightarrow \quad i = j$



$\big\downarrow$ affine map



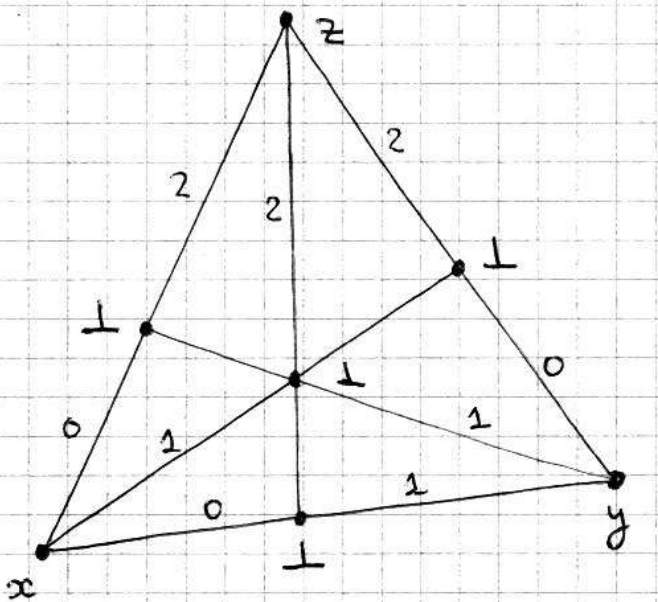
$\xrightarrow{d_J} \Omega V_I$

$C\Delta J$

$w_k = ?$ $\qquad H \overset{r}{=} \langle \mathrm{const}(b) \rangle$ follows from:

$\langle \mathrm{const}(b) \rangle - H = \sum u_i$ (special ensmble)

(special ensmble) $\overset{r}{=} 0$

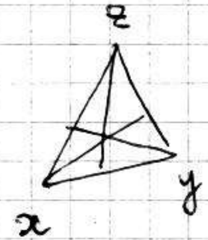


$x \geq \{0,1\}, \quad y \geq \{0,1\}, \quad z \geq \{2\}$

$:=$

intersection $= \emptyset$

$$:= \sum_{x \geq \{01\},\, y \geq \{01\},\, z \geq \{2\}} (-1)^{|x|-2} (-1)^{|y|-2} (-1)^{|z|-1}$$



For Δ^1: $\langle \mathrm{const}(\ell) \rangle - H =$

$$= \boxed{\begin{array}{l} z \qquad z \qquad z \\ \bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet \\ z \supseteq \emptyset \end{array}} - \sum_{\substack{J,K \neq \emptyset \\ J \cap K = \emptyset}} \boxed{\begin{array}{l} x \qquad 1 \qquad y \\ \bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet \\ \quad i(J) \quad i(K) \\ x \supseteq J, \; y \supseteq K \end{array}}$$

$$\emptyset \neq J \mapsto i(J) \in J$$

fissility:

$$H\Big|_{\partial\Delta^1} = \sum_{J,K \neq \emptyset} (-1)^{|J|-1} (-1)^{|K|-1} \quad \overset{J}{\bullet} \cdots\cdots \overset{K}{\bullet}$$