CONFERENCE “LOW-DIMENSIONAL TOPOLOGY 2025”

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N-VALUED QUANDLES

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QUANDLES (DISTRIBUTIVE GROUPOIDS)

Definition 1. A quandle is a non-empty set Q with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

(Q1) Idempotency: $x * x = x$ for all $x \in Q$,

(Q2) Invertibility: for any $x, y \in Q$ there exists a unique $z \in Q$ such that $x = z * y$,

(Q3) Self-distributivity: $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$.

An algebraic system satisfying only (Q2) and (Q3) is called a rack.

S. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.), 119 (161), no. 1 (9) (1982), 78–88 (in Russian).

D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23, no. 1 (1982), 37–65.

QUANDLES

EXAMPLE 1. If G is a group, m is an integer, then the binary operation $a *_m b = b^{-m}\, a\, b^m$ turns G into the quandle $\mathrm{Conj}_m(G)$ called the *m-conjugation quandle* on G.

If $m = 1$, this quandle is called *conjugation quandle*.

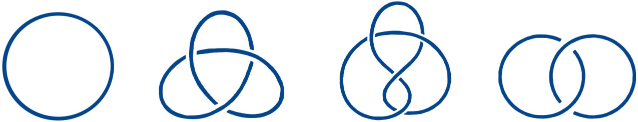
EXAMPLE 2. A group G with the binary operation $a * b = ba^{-1}b$ turns the set G into the quandle $\mathrm{Core}(G)$ called the *core quandle* of G.

If $G = \mathbb{Z}_n$ is the cyclic group of order n, then $\mathrm{Core}(\mathbb{Z})$ is called the *dihedral quandle* and denoted by R_n.

EXAMPLE 3. Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then the set G with binary operation $a *_\varphi b = \varphi(ab^{-1})b$ forms a quandle $\mathrm{Alex}(G,\ \varphi)$ referred as the *generalized Alexander quandle* of G with respect to φ.

DEFINITION OF LINK QUANDLE

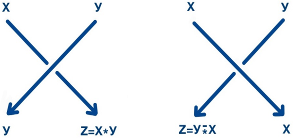
Let L be a link and D_L is its digram



The fundamental quandle of link or simply link quandle $Q(L)$ is generated by elements

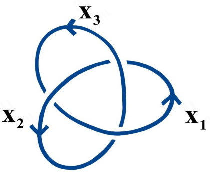
$$x_1, x_2, \ldots, x_n,$$

where x_i corresponds to some connected component of D_L, and any crossing of D_L gives one relation of the form



EXAMPLE: QUANDLE OF THE TREFOIL KNOT

The quandle $Q(K)$ of the trefoil knot K is generated by elements x_1, x_2, x_3:



and is defined by relations

$$x_3 = x_2 * x_1, \quad x_2 = x_1 * x_3, \quad x_1 = x_3 * x_2.$$

RESULT OF MATVEEV AND JOYCE

S. Matveev and D. Joyce proved that $Q(L)$ does not depend on a diagram D_L , i.e. is an invariant of L. In some sense it is a complete invariant of link. It means that

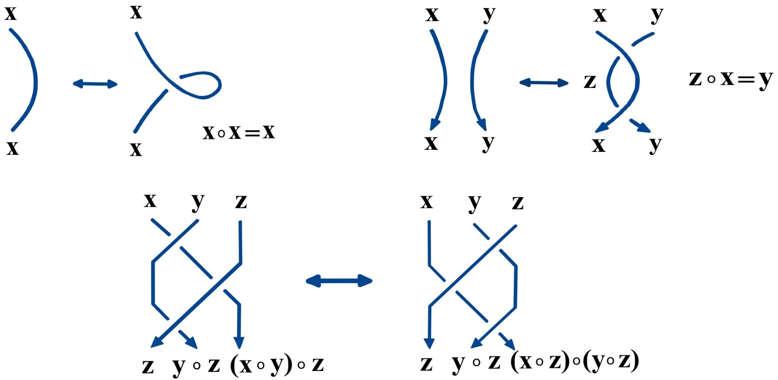
$$Q(L) \cong Q(L') \Rightarrow L \sim L' \text{ or } L \sim -\bar{L}',$$

where $\bar{L}$ is the mirror image of L and $-L$ is the inverse of L.

The main problem is comparison of quandles: for a pair of quandles Q and Q' define are they isomorphic or not?

LINKS AND QUANDLES

A quandle is a groupoid that satisfies three axioms. These axioms motivated by the three Reidemeister moves of link diagrams.

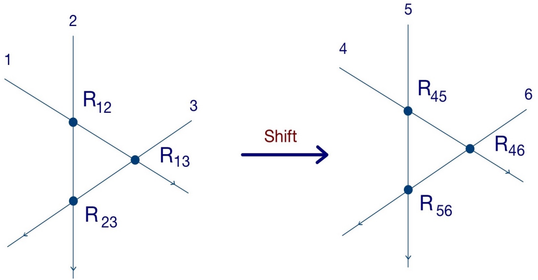


In fact, S. Matveev and D. Joyce have found some set-theoretical solutions for the Yang-Baxter equation. They studied racks and quandles which gives some solutions.

SET-THEORETICAL SOLUTIONS

V. G. Drinfeld (1992) suggested to study set-theoretical solutions, i.e. maps $R : X \times X \to X \times X$ is a bijection for some set X that satisfies the following equation in $Sym(X \times X \times X)$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$



For arbitrary set X the map $P((x, y)) = (y, x)$ that is called flip, gives set-theoretical solution for the Yang-Baxter equation.

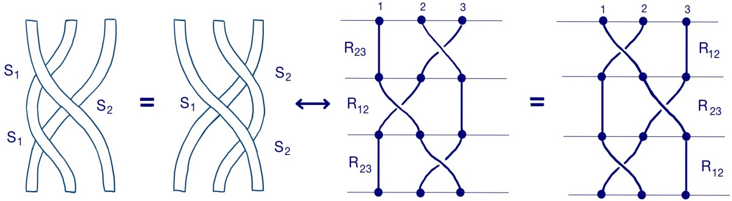
BRAID EQUATION

Let $P : X \times X \to X \times X$ be the flip: $P((x, y)) = (y, x)$ for all $x, y \in X$.

If an operator $R : X \times X \to X \times X$ gives a set-theoretical solution, then the operator $S = PR : X \times X$ gives a solution for the Braid equation

$$S_1 S_2 S_1 = S_2 S_1 S_2,$$

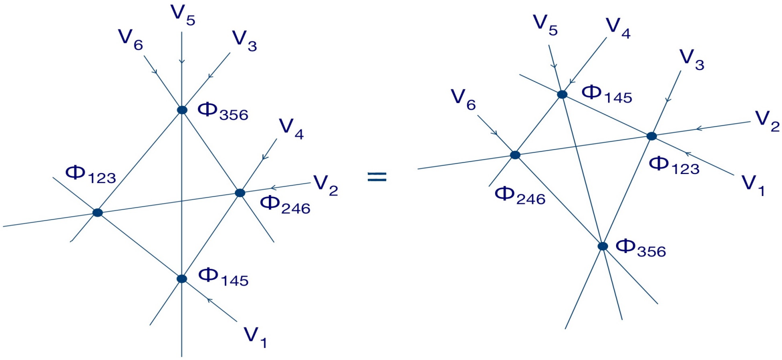
where $S_1 = S \times id$ and $S_2 = id \times S$ are operators on $X \times X \times X$.



TETRAHEDRON EQUATION

Let $\Phi : X^3 \to X^3$ be a bijection for some set X. Then the pair (X, Φ) is said to be a solution of the tetrahedron equation, if the following equality holds in $Sym(X^6)$:

$$\Phi_{123}\Phi_{145}\Phi_{246}\Phi_{356} = \Phi_{356}\Phi_{246}\Phi_{145}\Phi_{123}.$$



HOMOGENEOUS ALGEBRAIC SYSTEMS

Definition 2. Let $\mathcal{A} = (A, f_i, i \in I)$ be an algebraic system with a set of algebraic operations f_i of arity n_i. It is said to be m-homogeneous I-system if all arities n_i are equal to m.

In particular, if $|I| = n$, we will say on m-homogeneous n-system.

If $m = 2$ we will say on groupoid n-system.

EXAMPLE 1. A ring $(K, +, \cdot)$ is groupoid 2-system.

EXAMPLE 2. Semigroup (monoid, group) system $\mathcal{G} = (G, *_i, i \in I)$, where $(G, *_i)$ is a semigroup (monoid, group) for any $i \in I$ are examples of semigroup (monoid, group) I-system.

2-HOMOGENEOUS 2-SYSTEMS

EXAMPLE 3. A semigroup system with two operations is a duplex. That is an algebraic system with two associative binary operations (without added connections between these operations).

Duplexes were introduced by T. Pirashvili.

T. Pirashvili *Sets with two associative operations*, Cent.Eur.J.Math.1(2), (2003), 169–183.

EXAMPLE 4. n-Tuple semigroup is an algebraic system

$$\mathcal{S} = (S, *_i, i \in I), \quad |I| = n,$$

such that $(S, *_i)$ is a semigroup for any $i \in I$ and with the following axiom which connects these operations,

$$(a *_i b) *_j c = a *_i (b *_j c), \quad a, b, c \in S, \quad i, j \in I.$$

n-Tuple semigroup were introduced by N. Koreshkov.

N. Koreshkov, *n-Tuple algebras of associative type*, Izv. Vyssh. Uchebn. Zaved. Mat., No. 12, 34–42 (2008).

2-HOMOGENEOUS 2-SYSTEMS

EXAMPLE 5. A dimonoid is a set X together with two binary operations $\vdash$ and $\dashv$ satisfying the following axioms:

$$\begin{cases} x \dashv (y \dashv z) \overset{1}{=} (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z), \\ (x \vdash y) \dashv z \overset{3}{=} x \vdash (y \dashv z), \\ (x \dashv y) \vdash z \overset{4}{=} x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z. \end{cases}$$

for all x, y and $z \in X$. Observe that relations 1 and 5 are the "associativity" of the products $\vdash$ and $\dashv$ respectively.

Dimonoids were introduced by J.-L.Loday in his construction of a universal enveloping algebra for the Leibniz algebra.

J.-L.Loday, *Algebres ayant deux operations associatives (dialgebres)*. C. R. Acad. Sci. Paris Ser. I Math. 321 (1995), no. 2, 141–146.

DIMONOIDS

a) Let M be a monoid. Put $D = M \times M$ and define the products by

$$(m, n) \dashv (m', n') := (m, nm'n'),$$

$$(m, n) \vdash (m', n') := (mnm', n').$$

$D = (D, \dashv, \vdash)$ is a dimonoid.

b) Let G be a group and X be a G-set. The following formulas define a dimonoid structure on $X \times G$:

$$(x, g) \dashv (y, h) := (x, gh),$$

$$(x, g) \vdash (y, h) := (g \cdot x, gh).$$

We see that a dimonoid is an example of semigroup 2-system.

SKEW BRACES

EXAMPLE 6. A group system $(G, \cdot, \circ)$ with two operation is said to be a skew (left) brace if

$$g_1 \circ (g_2 \cdot g_3) = (g_1 \circ g_2) \cdot g_1^{-1} \cdot (g_1 \circ g_3)$$

for all $g_1, g_2, g_3 \in G$, where g_1^{-1} denotes the inverse of g_1 in $(G, \cdot)$.

We call $(G, \cdot)$ the additive group and $(G, \circ)$ the multiplicative group of the skew left brace $(G, \cdot, \circ)$. A skew left brace $(G, \cdot, \circ)$ is said to be a (left) brace if $(G, \cdot)$ is an abelian group.
Braces were introduced by W. Rump as an algebraic system related to the quantum Yang—Baxter equation. L. Guarnieri and L. Vendramin defined for the same purposes a more general notion of a skew left brace.

W. Rump, *Braces, radical rings and the quantum Yang–Baxter equations*, J. Algebra 307 (2007), 153–170.

L. Guarnieri L. and Vendramin, *Skew braces and the Yang–Baxter equation*, Math. Comp. 86 (2017), 2519–2534.

QUESTION 1. Let $\mathcal{G} = (G, *_i, i \in I)$ be a semigroup system. Define a product of semigroup operations,

$$g(*_i *_j)h = (g *_i h) *_j h, \quad g, h \in G.$$

Find necessary and sufficient conditions under which $(G, *_i *_j)$ is a semigroup.

GROUP SYSTEMS AND MULTI-GROUPS

Let $\mathcal{G} = (G, *_i, i \in I)$ is semigroup (monoid, group) system, where $(G, *_i)$ is a semigroup (monoid, group) for any $i \in I$.

Definition 3. We will call $\mathcal{G}$ by multi-semigroup (multi-monoid, multi-group) if the operations are connected by the following condition

$$(a *_i b) *_j c = a *_i (b *_j c), \quad a, b, c \in G, \quad i, j \in I.$$

GROUP SYSTEMS AND MULTI-GROUPS

Let $M_n(\mathbb{k})$ be a set of $n \times n$ matrices over a field $\mathbb{k}$.
Let us fix a matrix $M \in M_n(k)$ and define a set of multiplications $*_{t_0,t_1} : M_n(k) \times M_n(k) \to M_n(k)$,

$$A *_{t_0,t_1} B = t_0 AB + t_1 AMB, \quad t_0, t_1 \in k.$$

PROPOSITION 1. [BKT-24]

1) $(M_n(k), *_{1,t_1}, t_1 \in k)$ is a multi-semigroup.

2) If $detM \neq 0$ then $(GL_n(k), *_{1,0}, *_{0,1})$ is a multi-group.

GROUP SYSTEMS AND MULTI-GROUPS

Let $M_n(\Bbbk)$ be a set of $n \times n$ matrices over a field $\Bbbk$.

$$A *_{s,t,M_1,M_2} B = sAM_1B + tAM_2B, \quad s, t \in \Bbbk, \quad M_1, M_2 \in M_n(\Bbbk).$$

QUESTION 2. What can we say on this product? What algebraic systems one can construct using these operations? Is there connection of these operations with non-standard matrix multiplications which were studied in

V. G. Bardakov, A. A. Simonov, *Rings and groups of matrices with a nonstandard product*, (Russian) Sibirsk. Mat. Zh. 53 (2013), no. 4, 741–750.

GROUP SYSTEMS AND MULTI-GROUPS

THEOREM 1. [TK-24]

The groupoid system $(M_n(\Bbbk), *_{s,t,M_1,M_2}, s, t \in \Bbbk, M_1, M_2 \in M_n(\Bbbk))$ is a multi-semigroup.

THEOREM 2. [TK-24]

1) Let $M \in M_n(\Bbbk)$, $detM \neq 0$. Then $(GL_n(\Bbbk), *_M)$ is a group with the product $A *_M B = AMB$, with the unit element $E^{(*_M)} = M^{-1}$ and the inverse $\bar{A}^{(*_M)} = M^{-1}A^{-1}M^{-1}$.

2) The algebraic system $(GL_n(\Bbbk), *_M, M \in GL_n(\Bbbk))$ is a multi-group.

Tatyana A. Kozlovskaya. *Multi-groups.* Matematika i mekhanika - Tomsk State University Journal of Mathematics and Mechanics. 87. 2024. pp. 34-43.

QUANDLE SYSTEMS

V. G. Bardakov, D. A. Fedoseev, *Multiplication of quandle structures*, Algebra and Logic, 63:2 (2024), 75–97.

They considered quandle system $\mathcal{Q} = (Q, *_i, i \in I)$, where $(Q, *_i)$ is a quandle for any $i \in I$ and defined a multiplication $*_i *_j$ by the rule

$$p(*_i *_j)q = (p *_i q) *_j q, \quad p, q \in Q.$$

In the general case the algebraic system $(Q, *_i *_j)$ is not a quandle, but if the operations are satisfied the axioms

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z), \ \ (x *_j y) *_i z = (x *_i z) *_j (y *_i z), \quad x, y, z \in Q,$$

then $(Q, *_i *_j)$ and $(Q, *_j *_i)$ are quandles.

QUANDLE SYSTEMS

V. Turaev, Multi-quandles of topological pairs, arXiv:2205.00951

Definition 4. Quandle systems which satisfy axioms:

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z),$$
$$(x *_j y) *_i z = (x *_i z) *_j (y *_i z), \quad x, y, z \in Q,$$

for all $i, j \in I$ are called multi-quandles.

V. Turaev gave a topological interpretation of multi-quandles.

RACK SYSTEMS

THEOREM 3. [TK-24] Let $Q = (V, G)$, where V is a vector space of dimension n over a field $\mathbb{k}$, G is a subgroup of $\mathrm{GL}_n(\mathbb{k})$. Then the algebraic system $(Q, \circ_n, n \in \mathbb{Z})$, where

$$(a, A) \circ_n (b, B) = (A^n b, A^n B A^{-n}), \quad a, b \in V, \quad A, B \in G$$

satisfies the axioms:

1) Left self-distributivity,

$$(a, A) \circ_n ((b, B) \circ_n (c, C)) = ((a, A) \circ_n (b, B)) \circ_n ((a, A) \circ_n (c, C)),$$

$$a, b, c \in V, \quad A, B, C \in G.$$

2) Left divisibility,
for any (a, A), $(b, B) \in Q$ there is unique $(u, X) \in Q$ such that

$$(a, A) \circ_n (u, X) = (b, B).$$

RACK SYSTEMS

COROLLARY 1. [TK-24] The algebraic system $(Q, \circ_n^{op}, n \in \mathbb{Z})$, where the opposite operations are defined by the rules

$$(a, A) \circ_n^{op} (b, B) = (b, B) \circ_n (a, A)$$

is a rack system.

Tatyana A. Kozlovskaya. *Multi-groups.* Matematika i mekhanika - Tomsk State University Journal of Mathematics and Mechanics. 87. 2024. pp. 34-43.

n-VALUED GROUPS

BRIEF HISTORY

In 1971, V. M. Buchstaber and S. P. Novikov proposed a construction motivated by the theory of characteristic classes.

This construction describes a multiplication in which the product of any pair of elements is a multiset of n points.

An axiomatic definition of n-valued groups and the first results of their algebraic theory were obtained in a subsequent series of works by V. M. Buchstaber.

Currently, the theory of n-valued groups and their applications in various areas of mathematics and mathematical physics are being developed by a number of authors.

n-VALUED GROUPS

Let X be a non-empty set. An n-valued multiplication on X is a map

$$\mu : X \times X \to (X)^n = Sym^n X,$$

$$\mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k,$$

where $(X)^n = Sym^n X$ is the n-th symmetric power of X.

V. M. Buchstaber, S. P. Novikov, Formal groups, power systems and Adams operators, Russian Mat. Sb. (N.S.), 84 (126), 1971, 81–118.

n-VALUED GROUPS

Definition 1. The map μ defines n-valued group $\mathcal{X} = (X, \mu, e, inv)$ on X if the next axioms hold:
Associativity. The n^2-sets:

$$[x*(y*z)_1, x*(y*z)_2, \dots, x*(y*z)_n], \; [(x*y)_1*z, (x*y)_2*z, \dots, (x*y)_n*z]$$

coincide for all $x, y, z \in X$.
Unit. There exists an element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for all $x \in X$.
Inverse. It is defined a map $inv : X \to X$ such that

$$e \in inv(x) * x \qquad \text{and} \qquad e \in x * inv(x)$$

for all $x \in X$.

2-VALUED GROUP STRUCTURE ON $\mathbb{Z}_+$.

EXAMPLE 1. (Buchstaber-Novikov group)

$$\mu : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2 : \quad x * y = [x + y, |x - y|], \quad x, y \in \mathbb{Z}_+$$

Unit:

$$e = 0.$$

Inverse:

$$inv(x) = x.$$

Associativity:

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

coincide for all $x, y, z \in \mathbb{Z}_+$.

COSET GROUPS

THEOREM 1. [Buchstaber, 2006]

Let G be a group with multiplication m, $A \leq \mathrm{Aut}(G)$ and $|A| = n$. The set of orbits $X = G/A$ can be equipped with an n-multiplication

$$\mu\colon X \times X \to (X)^n, \ \ \mu(x, y) = \pi(m(\pi^{-1}(x), \pi^{-1}(y))),$$

where $\pi\colon G \to X$ is the canonical projection. Then X with multiplication μ is a n-valued group with the unit $e_X = \pi(e_G)$ and the inverse $inv(x)$ of $x \in X$ is $\pi((\pi^{-1}(x))^{-1})$. This group is called the coset group of (G, A).

V. M. Buchstaber, n-valued groups: theory and applications, Moscow Math. J., 6, no. 1 (2006), 57–84.

COSET GROUPS

EXAMPLE 2. (Buchstaber-Novikov group)
Let $G = \mathbb{Z}$ be the infinite cyclic group, $A = \{\mathrm{id}, -\mathrm{id}\}$ a subgroup of its automorphisms of order 2. The set of orbits

$$X = \{\{0\}, \{\pm a\} \mid a \in \mathbb{N}\}$$

can be identified with the set of non-negative integers $\mathbb{Z}_+$ with the 2–valued multiplication

$$\mu\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2, \quad x * y = [x + y, |x - y|], \quad x, y \in \mathbb{Z}_+.$$

Hence, Buchstaber-Novikov group is a coset group.

DOUBLE COSET GROUPS

THEOREM 2. [Buchstaber, 2006]

Let G is a group and H - its subgroup of cardinality n. Denote by X the space of double coset classes $H\backslash G/H$. Define the n-valued multiplication $\mu\colon X \times X \to (X)^n$ by the formula

$$\mu(x, y) = \{Hg_1H\} * \{Hg_2H\} = [\{Hg_1hg_2H\}, \quad h \in H].$$

and the inverse element $inv_X(x) = \{Hg^{-1}H\}$, where $x = \{HgH\}$. This construction for n-valued groups called a double coset group.

V. M. Buchstaber, n-valued groups: theory and applications, Moscow Math. J., 6, no. 1 (2006), 57–84.

Let X be a non-empty set with n-valued multiplications $*$ and $\bar{*}$:

$$x * y = [(x * y)_1, (x * y)_2, \dots, (x * y)_n],$$
$$x \bar{*} y = [(x \bar{*} y)_1, (x \bar{*} y)_2, \dots, (x \bar{*} y)_n], \quad x, y \in X,$$

i.e. $x * y$ and $x \bar{*} y$ are multisets in $Sym^n(X)$.

n-VALUED RACKS AND QUANDLES

Definition 1. [BKT-24] n-Valued rack is a triple $\mathcal{X} = (X, *, \bar{*})$ such that the following axioms hold:

(M1) Invertibility: for any $x, y \in X$ the element x lies in the n^2-multiset $(x * y)\bar{*}y$ and in the n^2-multiset $(x\bar{*}y) * y$;

(M2) Self-distributivity: for any $x, y, z \in X$ the n^2-multiset $(x * y) * z$ is a subset of the n^3-multiset $(x * z) * (y * z)$.

An (n-valued) rack $\mathcal{X} = (X, *, \bar{*})$ is said to be an n-valued quandle if the next axiom holds

(M3) Idempotency: for any $x \in X$ the multiset $x * x$ contains x.

EXAMPLE.

Let G be a group and I is a subset of integer numbers. For any $i \in I$ define i-conjugation quandle $Conj_i(G)$ on G. If I is finite and contains n elements, we can define n-valued multiplication

$$g * h = [g *_i h, \quad i \in I], \quad g, h \in G.$$

In this case $g \bar{*} h = [g \bar{*}_i h, \quad i \in I]$ and it is easy to check that it is an n-valued quandle.

This algebraic system $(G, \{*_i\}_{i \in I})$ is called n-multi-quandle.

V. Turaev, Multi-quandles of topological pairs, arXiv:2205.00951

COSET CONSTRUCTION

Let Q be a quandle with multiplication m and its inverse $\bar{m}$ that means that for any $q, h \in Q$ the following holds

$$\bar{m}(m(q,h),h) = m(\bar{m}(q,h),h) = q.$$

Further, let $A \subset \mathrm{Aut}(Q)$ be a subgroup of order n. Then the set of orbits $X = Q/A$ can be equipped with an n-multiplication

$$\mu \colon X \times X \to (X)^n$$

by the rule

$$\mu(x,y) = x * y = \pi(m(\pi^{-1}(x), \pi^{-1}(y))),$$

where $\pi \colon Q \to X$ is the canonical projection.

COSET CONSTRUCTION

THEOREM 1. [BKT-24]

The multiplication μ defines an n-valued quandle structure on the orbit space $X = Q/A$, called the *coset n-valued quandle* of (G, A), with the inverse operation

$$x \bar{*} y = \pi(\bar{m}(\pi^{-1}(x), \pi^{-1}(y))).$$

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, *n-valued quandles and associated bialgebras.* TMF, Theoret. and Math. Phys., 220:1 (2024), 1080–1096 DOI: https://doi.org/10.4213/tmf10656

EXAMPLE 1.

Let us consider the conjugation quandle $Q = Conj(S_3)$ for the group S_3 and subgroup of inner automorphisms $A \subset Inn(Q)$ that is generated by multiplication on s_1. It is evident that $A \cong S_2$. The orbits of Q under the action of A are

$$1, s_1, \{s_2, s_1 s_2 s_1\}, \{s_1 s_2, s_2 s_1\}.$$

Let us denote these classes by $(x_0, \ldots, x_3)$. The multiplication table for this 2-quandle is given by

Q/A	x_0	x_1	x_2	x_3
x_0	$[x_0, x_0]$	$[x_0, x_0]$	$[x_0, x_0]$	$[x_0, x_0]$
x_1	$[x_1, x_1]$	$[x_1, x_1]$	$[x_2, x_2]$	$[x_2, x_2]$
x_2	$[x_2, x_2]$	$[x_2, x_2]$	$[x_1, x_2]$	$[x_1, x_2]$
x_3	$[x_3, x_3]$	$[x_3, x_3]$	$[x_3, x_3]$	$[x_3, x_3]$

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COSET CONSTRUCTION

EXAMPLE 2. Let us consider the finite cyclic group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ of order 4 and the automorphism $\varphi\colon \mathbb{Z}_4 \to \mathbb{Z}_4$,

$$\varphi(0) = 0, \quad \varphi(1) = 3, \quad \varphi(2) = 2, \quad \varphi(3) = 1.$$

This automorphism has order 2 and the 2-valued coset group $X = (\mathbb{Z}_4, \langle\varphi\rangle)$ has three elements,

$$a = \{0\}, \quad b = \{1, 3\}, \quad c = \{2\},$$

with the multiplication table

*	a	b	c
a	[a, a]	[b, b]	[c, c]
b	[b, b]	[a, c]	[b, b]
c	[c, c]	[b, b]	[a, a]

.

Hence, a is the unit element and $inv = id$ is the identity map.

EXAMPLE 3. Let us consider the dihedral quandle $\mathrm{R}_4 = \{a_0, a_1, a_2, a_3\}$. This quandle has inner automorphisms $S_i = S_{a_i}$, $i = 0, 1, 2, 3$, which act by the rules

$$a_0 S_0 = a_0, \quad a_1 S_0 = a_3, \quad a_2 S_0 = a_2, \quad a_3 S_0 = a_1,$$

$$a_0 S_1 = a_2, \quad a_1 S_1 = a_1, \quad a_2 S_1 = a_0, \quad a_3 S_1 = a_3.$$

Further, $S_2 = S_0$, $S_3 = S_1$, $S_0 S_1 = S_1 S_0$ and $\mathrm{Inn}(\mathrm{R}_4) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider the coset 2-valued quandle $Q = (\mathrm{R}_4, \langle S_0 \rangle)$. This quandle has elements $A = \{a_0\}$, $B = \{a_1, a_3\}$, $C = \{a_2\}$, and the multiplication table

*	A	B	C
A	[A, A]	[C, C]	[A, A]
B	[B, B]	[B, B]	[B, B]
C	[C, C]	[A, A]	[C, C]

.

EXAMPLE 4. We construct coset 2-valued quandle on $Q = Core(\mathbb{Z})$. Let $\varphi \in \mathrm{Aut}(Q)$, $\varphi(a) = -a$, $a \in \mathbb{Z}$. The set of orbits: $X = \{\{a, -a\} \mid a \in \mathbb{Z}\}$ with a 2-valued multiplication

$$\{a, -a\} * \{b, -b\} = [\{2b - a, -2b + a\}, \{2b + a, -2b - a\}], \quad a, b, c \in \mathbb{Z},$$

gives a coset 2-valued quandle. We can identified the set of orbits with the set of non-negative integers $\mathbb{Z}_+$ with the 2-valued multiplication

$$*\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2$$

given by the formula

$$a * b = [2b + a, |2b - a|], \quad a, b \in \mathbb{Z}_+.$$

ORDERED n-VALUED GROUPS

Definition 2. [BKT-24] An ordered n-valued group is a non-empty set G with n-tuple multiplication

$$x * y = (x *_1 y, x *_2 y, \ldots, x *_n y),$$

such that the following axioms hold:
1) associativity:

$$(x *_i y) *_j z = x *_i (y *_j z), \quad i, j \in \{1, 2, \ldots, n\},$$

for all $x, y, z \in X$.
2) unit element:
for any operation $*_i$ there exists an element $e_i \in G$ such that

$$x *_i e_i = e_i *_i x = x.$$

ORDERED n-VALUED GROUPS

If $e_1 = e_2 = \ldots = e_n = e$, then we say that an ordered n-valued group has a common unit element $e \in G$. In this case for any $x \in G$ we have

$$x * e = (x *_1 e, x *_2 e, \ldots, x *_n e) = (x, x, \ldots, x),$$

$$e * x = (e *_1 x, e *_2 x, \ldots, e *_n x) = (x, x, \ldots, x);$$

3) invertibility: for any operation $*_i$ for any $x \in G$ the exists an inverse element $\bar{x}^{(i)}$ such that

$$x *_i \bar{x}^{(i)} = \bar{x}^{(i)} *_i x = e_i.$$

It is easy to see that $(G, *_i)$ is a group for any $i \in \{1, 2, \ldots, n\}$ and $(G, *_1, ..., *_n)$ is n-multi-group.

PROPOSITION 1. [BKT-24]

Let $(G, *_1, ..., *_n)$ be an n-multi-group. If $e_i = e_j$ for some $i, j \in \{1, 2, \ldots, n\}$, then $*_i = *_j$.

n-VALUED RACKS AND QUANDLES

THEOREM 2. [BKT-24]

Let X be a non empty set with n quandle (rack) operations $*_i$ such that for any pair $1 \leq i, j \leq n$ of indices, the following identities hold

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z), \quad x, y, z \in X.$$

Then an n-valued multiplication

$$x * y = [x *_1 y, x *_2 y, \ldots, x *_n y]$$

defines an n-valued quandle (respectively, rack) structure on X.

V.G. Bardakov, T.A. Kozlovskaya, D.V. Talalaev, *n-valued quandles and associated bialgebras.* TMF, Theoret. and Math. Phys., 220:1 (2024), 1080–1096.

In particular, if we take the powers on quandle operations, then we get

COROLLARY 1. [BKT-24] Let $(Q, \circ)$ be an n-quandle (it means that for any $q, h \in Q$ we have $q *^n h = q$). We define an n-valued multiplication on Q by the formula

$$x * y = [x \circ y, x \circ^2 y, \ldots, x \circ^n y], \quad x, y \in Q.$$

Then $(Q, *)$ is an n-valued quandle.

2-MULTI-RACKS ON $\mathbb{Z}$

We construct some rack operations on the set of integers $\mathbb{Z}$. The operation $x * y = 2y - x$ gives a quandle structure on $\mathbb{Z}$ and $(\mathbb{Z}, *)$ is the core quandle $\mathrm{Core}\,\mathbb{Z}$. Let us consider a more general operation

$$x \underset{\varepsilon,a,b}{*} y = \varepsilon x + ay + b, \quad x, y \in \mathbb{Z}$$

for some $\varepsilon \in \{\pm 1\}$, $a, b \in \mathbb{Z}$. We can consider this operation as a deformation of the operation $*$.

2-MULTI-RACKS ON $\mathbb{Z}$

PROPOSITION 2. [BKT-24]

1) The set of integers $\mathbb{Z}$ with an operation

$$x \underset{\varepsilon,a,b}{*} y = \varepsilon x + ay + b, \quad x, y \in \mathbb{Z}, \quad \varepsilon \in \{\pm 1\}, a, b \in \mathbb{Z},$$

is a quandle if and only if it is the trivial quandle: $\varepsilon = 1$, $b = 0$, or it is the core quandle with the operation $x * y = -x + 2y$.
2) On the set of integers the operation

$$x \underset{\varepsilon,b}{*} y = \varepsilon x + b, \quad x, y \in \mathbb{Z}$$

defines a rack for $\varepsilon \in \{\pm 1\}$, $b \in \mathbb{Z}$.

2-MULTI-RACKS ON $\mathbb{Z}$

We will call the rack operations from this proposition by linear operations. The next proposition gives all 2-multi racks on $\mathbb{Z}$ with linear operations.

PROPOSITION 3. [BKT-24] Let b, b_1, and b_2 are integer numbers. The next pairs of multiplications define 2-multi-rack on $\mathbb{Z}$:

1) $x *_1 y = x, \quad x *_2 y = x;$

2) $x *_1 y = 2y - x, \quad x *_2 y = x;$

3.1) $x *_1 y = x + b_1, \quad x *_2 y = x + b_2;$

3.2) $x *_1 y = x, \quad x *_2 y = -x + b;$

3.3) $x *_1 y = -x + b, \quad x *_2 y = x;$

3.4) $x *_1 y = -x + b, \quad x *_2 y = -x + b.$

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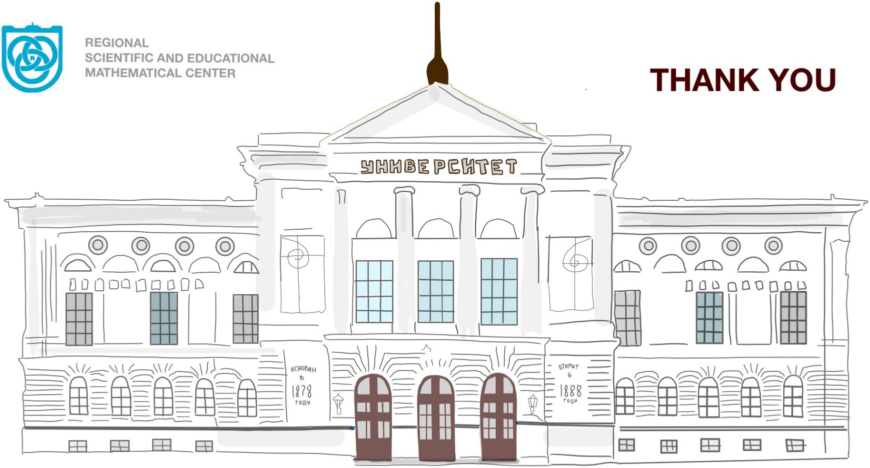

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