Fermat Hypersurfaces, Projective Duality, and n-Valued Groups*

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Fermat's Hypersurfaces

For each $d, n \geqslant 1$ define a *Fermat hypersurface*:

$$M_d^n = \{F_d^n = 0\} = \{z_0^d + \dots + z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1} =: \mathbb{P}^{n+1}.$$

The complete linear system $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ gives the Veronese embedding of degree d (with $N = \binom{n+d+1}{d} - 1$ below):

$$\nu_d : \mathbb{P}^{n+1} \hookrightarrow \mathbb{P}((H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)))^*) \cong \mathbb{P}^N$$

$$(z_0 : \dots : z_{n+1}) \mapsto (Y_{d,\dots,0} : \dots : Y_{0,\dots,d}), \ \ Y_{d_0,\dots,d_{n+1}} = z_0^{d_0} \dots z_{n+1}^{d_{n+1}},$$
$$d_0 + d_1 + \dots + d_{n+1} = d.$$

We have

$$M_d^n = \nu_d(\mathbb{P}^{n+1}) \cap H_{F_d^n},$$

where $H_{F_d^n}$ denotes the hyperplane

$$H_{F_d^n} = \Big\{[Y] \in \mathbb{P}^N \ \Big| \ \sum_{j=0}^{n+1} Y_{0,\dots,0,\underbrace{\mathrm{d}}_{j},0,\dots,0} = 0\Big\}.$$

First Examples of Varieties M_d^n

We have the 2-parameter family of *smooth, irreducible* projective varieties:

$$M_d^n = \{z_0^d + \ldots + z_{n+1}^d = 0\} \subset \mathbb{P}^{n+1}$$

$$M_1^n \cong \mathbb{P}^n, \;\; n \geq 1$$

M_d^1 is a curve of genus $(d-1)(d-2)/2, \;\; d \geq 1$

$$M_2^n \cong \mathrm{SO}(n+2)/(\mathrm{SO}(2) \times \mathrm{SO}(n)), \;\; n \geq 1$$

$$M_2^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$$

$$M_2^4 \cong \mathrm{Gr}_{\mathbb{C}}(2, 4)$$

Euler Characteristic of M_d^n

Using the exact sequence for the normal bundle $\mathcal{N}_{X \hookrightarrow \mathbb{P}^{n+1}}$

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^{n+1}}|_X \longrightarrow \mathcal{N}_{X \hookrightarrow \mathbb{P}^{n+1}} \longrightarrow 0$$

and the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus n+2} \longrightarrow \mathcal{T}_{\mathbb{P}^{n+1}} \longrightarrow 0$$

(with $\mathcal{N}_{X \hookrightarrow \mathbb{P}^{n+1}} \cong \mathcal{O}_X(d)$), we have:

Corollary [Dim92, Ch. 5, §3, Exercise 3.7(i)]

For any smooth complex hypersurface X of degree d in $\mathbb{P}^{n+1}$,

$$\chi(X) = n + 2 + \frac{(1-d)^{n+2} - 1}{d}.$$

H^k, H_k, and π_k for Hyperplane Sections

Recall the following classical result:

Lefschetz Hyperplane Theorem [Voi03, Theorem 1.23]

Let $X \subset \mathbb{P}^N$ be an n-dimensional complex projective algebraic variety, let Y be a hyperplane section of X such that $U = X \setminus Y$ is smooth, and let $i : Y \hookrightarrow X$ be the embedding.
Then the induced maps are iso's for $k < n-1$ and for $k = n-1$:

$$H_k(Y;\mathbb{Z}) \xrightarrow{i_*} H_k(X;\mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^k(X;\mathbb{Z}) \xrightarrow{i^*} H^k(Y;\mathbb{Z})$$

$$\pi_k(Y,\mathbb{Z}) \xrightarrow{i_*} \pi_k(X,\mathbb{Z}) \longrightarrow 0$$

(Co-)Homology of M_d^n

Corollary

$$H_k(M_d^n;\mathbb{Z}) \cong H^k(M_d^n;\mathbb{Z}) \cong$$

$$\cong \begin{cases} \mathbb{Z}, & k \text{ even, } k \neq n, \text{ and } 0 \leq k \leq 2n, \\ 0, & k \text{ odd,} \\ \mathbb{Z}^{b_n(d)}, & k = n. \end{cases}$$

$$b_n(d) = \begin{cases} \chi(M_d^n) - n = 2 + \dfrac{(1-d)^{n+2}-1}{d}, & n \text{ even,} \\ \\ n+1-\chi(M_d^n) = \dfrac{1-(1-d)^{n+2}}{d} - 1, & n \text{ odd.} \end{cases}$$

Cohomology Ring of M_d^n

Let $i : M_d^n \hookrightarrow \mathbb{P}^{n+1}$, $H^2(M_d^n, \mathbb{Z}) = \mathbb{Z}h$, where $h = i^* c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$, $u = [M_d^n] \in H^{2n}(M_d^n, \mathbb{Z})$, and Q be the intersection form. Define

$$H^n(M_d^n)_{\mathrm{van}} = \mathrm{Ker}(L : H^n(M_d^n, \mathbb{Z}) \xrightarrow{\cup h} H^{n+2}(M_d^n, \mathbb{Z}))\ .$$

For n odd, $H^{n+2}(M_d^n) = 0$, so $H^n(M_d^n) \cong H^n(M_d^n)_{\mathrm{van}}$.

For n even, $H^n(M_d^n) \cong \mathbb{Z}h^{n/2} \oplus H^n(M_d^n)_{\mathrm{van}}$.

$$\begin{cases} h^k \cup h^\ell = h^{k+\ell}, & \text{for each } 0 \le k, \ell, k+\ell \le n. \\ h^n = du. & \\ h^k = 0, & \text{for each } k > n. \\ h \cup \alpha = 0, & \text{for each } \alpha \in H^n(M_d^n)_{\mathrm{van}}. \\ \alpha \cup \beta = Q(\alpha, \beta)u, & \text{for each } \alpha, \beta \in H^n(M_d^n)_{\mathrm{van}}. \end{cases}$$

Homotopy Groups of M_d^n

Corollary

Let $d \geq 2$, $n \geq 3$. The computations of $\pi_k(M_d^n)$ for $k \leq n-1$ are as follows:

$$\pi_k(M_d^n) \cong \begin{cases} 0, & k = 1,\ n \geq 3, \\ \mathbb{Z}, & k = 2,\ n \geq 3, \\ 0, & 3 \leq k \leq n-1,\ n \geq 4. \end{cases}$$

M_d^1 is a curve of genus $(d-1)(d-2)/2$. Also, $M_1^n = \mathbb{C}P^{n-1}$. The cases of the surfaces M_d^2 and the threefolds M_d^3 will be considered soon.

E_8 form

Consider the surface

$$M_4^2 = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{P}^3 .$$

Its intersection form $Q_{M_4^2}$ on $H^2(M_4^2;\mathbb{Z})$ is even, unimodular, of signature -16,

$$Q_{M_4^2} \cong 3H \oplus 2(-E_8),$$

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}$$

In particular, $\dim_+ Q_{M_4^2} = 3$, $\dim_- Q_{M_4^2} = 19$, and $b_2(M_4^2) = 22$.

M_4^2 is a K3 Surface

A K3 surface is a compact complex surface that is simply connected and has trivial first Chern class (Calabi–Yau surface).

There is a *unique diffeomorphism type* of K3 surface, although the moduli space of all complex K3 surfaces has complex dimension 20 [Sco05].

$$
\begin{aligned}
c_1 &= 0 \\
b_0 &= b_4 = 1 \\
b_1 &= b_3 = 0 \\
\chi &= 24 \\
\tau &= -16 \\
b_2 &= \chi - 2 = 22
\end{aligned}
$$

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

The Hodge diamond of any K3 surface

K3 Surfaces

The real part of a K3 surface. Reproduced from [Dol20]

Surfaces M_d^2

From the Freedman's classification of 4-manifolds, we get:

Theorem

If d is odd, the variety M_d^2 is or.-preserving homeomorphic to

$$(\mathbb{C}P^2)^{\# m} \# (\overline{\mathbb{C}P}^2)^{\# n}, \;\; m = \frac{1}{3}(d^3 - 6d^2 + 11d - 3),$$

$$n = \frac{1}{3}(d-1)(2d^2 - 4d + 3).$$

If d is even, M_d^2 is spin and it is or.-preserving homeomorphic to

$$(\mathrm{K3})^{\# m} \# (\mathbb{C}P^1 \times \mathbb{C}P^1)^{\# n}, \;\; m = \frac{d(d^2-4)}{48},$$

$$n = \frac{1}{48}(d-4)(13d^2 - 44d + 12).$$

The Problem of

$$\Omega_{\mathrm{SO}}^{-4} = \langle [\mathbb{C}P^2] \rangle_{\mathbb{Z}}$$
$$\Omega_{\mathrm{U}}^{-4} = \langle [\mathbb{C}P^1]^2, [\mathbb{C}P^2] \rangle_{\mathbb{Z}}$$

From the [Buc70, Theorem 4.11], we obtain:

Theorem

For each $d \geq 1$, we have in Ω_{U}:

$$[M_d^2] = \frac{d(d-1)^2}{2}[\mathbb{C}P^1]^2 + \frac{d(4-d^2)}{3}[\mathbb{C}P^2].$$

We see that $[M_d^2]_{\Omega_{\mathrm{U}}} \mapsto \frac{d(4-d^2)}{3}[\mathbb{C}P^2]_{\Omega_{\mathrm{SO}}}$ under the natural homomorphism

$$\Omega_U^{-4} \longrightarrow \Omega_{\mathrm{SO}}^{-4} .$$

The Problem of

So, in Ω_{U}, we have the identification:

$$[M_d^2] = \begin{cases} (m-n)[\mathbb{C}P^2] + \dfrac{d(d-1)^2[\mathbb{C}P^1]^2}{2}, & d \text{ is odd,} \\ m[\mathrm{K3}] + n[\mathbb{C}P^1]^2 - \dfrac{(d-4)(d-2)(7d-6)[\mathbb{C}P^1]^2}{48}, & d \text{ is even} \end{cases}$$

The terms in blue correspond to the "naïve" case of complex structures glued along connected sums: $[M_1 \# M_2] = [M_1] + [M_2]$.

The terms in magenta describe the discrepancy provided by the complex structure of M_d^2.

Classification of 6-Manifolds

Let M be a closed oriented simply connected 6-manifold with

$$H^3(M, \mathbb{Z}) = 0 \text{ and } \operatorname{Tors} H^*(M, \mathbb{Z}) = 0.$$

We have the following data:

- a free abelian group $H^2(M)$,
- a symmetric trilinear map $\mu : \operatorname{Sym}^3(H^2(M)) \longrightarrow \mathbb{Z}$, $\mu(x, y, z) = \langle x \cup y \cup z, [M] \rangle$,
- a linear form $p : H^2(M) \longrightarrow \mathbb{Z}$, $p(x) = \langle p_1(M) \cup x, [M] \rangle$, where $p_1(M)$ is the first Pontryagin class,
- the second Stiefel–Whitney class $w_2(M) \in H^2(M, \mathbb{Z}/2)$.

Classification of 6-Manifolds

Theorem [Wal66, Jup73, Zhu80]

There exists an orientation-preserving diffeomorphism $f : M' \to M$ if and only if there is an isomorphism

$$\varphi : H^2(M, \mathbb{Z}) \longrightarrow H^2(M', \mathbb{Z})$$

such that

$$\mu'(\varphi(x), \varphi(y), \varphi(z)) = \mu(x, y, z), \quad p'(\varphi(x)) = p(x), \quad \overline{\varphi}(w_2) = w_2'.$$

In the case of quasitoric manifolds, under certain assumptions on the moment polytopes(see [BEM+17]), f is a diffeomorphism iff there exists a graded ring isomorphism

$$\psi : H^*(M, \mathbb{Z}) \to H^*(M', \mathbb{Z}).$$

The Milnor–Hirzebruch Problem and M^3_d

The Milnor–Hirzebruch problem was first posed by Hirzebruch in his ICM 1958 talk [Hir58]. Its algebraic version can be formulated as follows:

Which sets of $p(n)$ characteristic numbers

$$c_\lambda, \ \lambda \in \mathscr{P}(n)$$

can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth irreducible complex algebraic variety M^n?

This version of the problem remains largely open, although some arithmetic restrictions are known from the theorem of Grothendieck–Riemann–Roch–Hirzebruch.

The Milnor–Hirzebruch Problem and M_d^3

Theorem [Wal66]

Let M be a closed smooth 1-connected 6-manifold.
Then we can write M as a connected sum (up to diifeomorphism)

$$M \cong M_1 \# M_2,$$

where $H_3(M_1)$ is finite and M_2 is diffeomorphic to a connected sum of copies of $\mathbb{S}^3 \times \mathbb{S}^3$.

From this result, we get the orientation preserving diffeomorphism

$$M_d^3 \cong N_d^3 \# (\mathbb{S}^3 \times \mathbb{S}^3)^{b_3(M_d^3)/2}.$$

We have $H^*(N_d^3) \cong H^*(\mathbb{C}P^3)$ additively, $p_1(N_d^3) = (5 - d^2)x^2$, $p_1(\mathbb{C}P^3) = 4h^2$, where $H^2(N_d) = \mathbb{Z}x,\ H^2(\mathbb{C}P^3) = \mathbb{Z}h$.
So, N_d^3 is $\mathbb{C}P^3$-like.

The Milnor–Hirzebruch Problem and M_d^3

Corollary

For any $d \geq 3$ there exists a smooth 6-dimensional homology projective space N_d^3 (i.e. $H^*(N_d^3) \cong H^*(\mathbb{C}P^3)$ additively) such that

$$p_1(N_d^3) = (5 - d^2)x^2,$$

where x is the generator of $H^2(N_d^3, \mathbb{Z})$.

The Milnor–Hirzebruch Problem and M_d^3

Theorem

The manifold N_d^3 is projective algebraic if and only if $d \in \{1, 2\}$.

$M_1^3 = N_1^3 \cong \mathbb{C}P^3$.

$M_2^3 = N_2^3 \cong \mathrm{SO}(5)/(\mathrm{SO}(2) \times \mathrm{SO}(3))$ is an algebraic variety.

There exists an additive isomorphism

$$H^*(N_d^3, \mathbb{Z}) \cong H^*(\mathbb{C}P^3, \mathbb{Z})$$

which is *not* a ring isomorphism for $d > 1$.

The Milnor–Hirzebruch Problem and M_d^3

Once again, from [Buc70, Theorem 4.11], we obtain:

Theorem

Let $A = [\mathbb{C}P^1]$, $B = [\mathbb{C}P^2]$, and $C = [\mathbb{C}P^3]$ in Ω_{U}^{-6}. Then we get

$$24[M_d^3] = -d\left(15d(d-1)^2A^3 - 20(d-1)^2(d+1)AB \right.$$
$$\left. -6(5-d^3)C\right).$$

For $d = 1$: $[M_1^3] = [\mathbb{C}P^3]$.

For $d = 2$: $2[M_2^3] = -5A^3 + 10AB - 3C$.

For $d = 5$: $[M_5^3] = -50(5A^3 - 8AB + 3C)$.

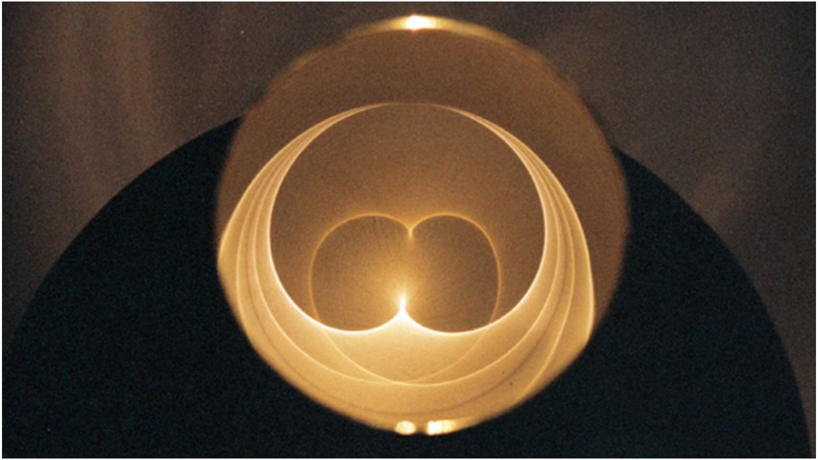
Recall that $\Omega_{\mathrm{SO}}^{-6} = 0$.

Basics of Projective Duality

Consider a projective space $\mathbb{P} = \mathbb{P}(V_{\mathbb{C}})$. Hyperplanes in $\mathbb{P}$ form the dual projective space $\mathbb{P}^* = \mathbb{P}(V^*)$.

Let $X \subset \mathbb{P}$ be a closed *irreducible* algebraic subvariety.
A hyperplane $H \subset \mathbb{P}$ is said to be tangent to X if there exists a smooth point $x \in X$ such that $x \in H$ and the tangent space to H at x contains the tangent space to X at x, i.e. $\mathcal{T}_x X \subset \mathcal{T}_x H$.

Denote by $X^\vee \subset \mathbb{P}^*$ the closure of the set of all hyperplanes tangent to X. The variety $X^\vee$ is called *projectively dual* to X.



The Coffee Cup Caustic.
Grace Weir, 2005
Reproduced from graceweir.com/page31/page31.html

Basics of Projective Duality

- Let $H = \{\ell = 0\}$. Then

  x is a singular point of $X \cap H \Leftrightarrow d\ell|_{\mathcal{T}_x X} = 0 \Leftrightarrow \mathcal{T}_x X \subset \mathcal{T}_x H.$

  If X is smooth and does not lie in any hyperplane, then

  $H \in X^\vee \;\Leftrightarrow\; \exists x \in X$ with $\mathcal{T}_x X \subset \mathcal{T}_x H \;\Leftrightarrow\; H \cap X$ is singular.

- For any projective variety $X \subset \mathbb{P}$ we have $(X^\vee)^\vee = X$.

- If $z \in X$ and $H \in X^\vee$ are smooth points, then

  H is tangent to X at $z \Leftrightarrow$ $z \subset \mathbb{P}^*$ is tangent to $X^\vee$ at H.

- If X is irreducible then $X^\vee$ is irreducible.

Basics of Projective Duality

For any smooth curve $X \subset \mathbb{P}^2$ of degree d, according to Plücker formulas [GKZ94, Proposition 2.4], we have $\deg X^\vee = d(d-1)$.

Hence, $X^\vee$ cannot be smooth for $\deg X > 3$ because $(X^\vee)^\vee = X$.

Basics of Projective Duality

Suppose that X is a hypersurface (not a hyperplane) in $\mathbb{P}(V)$. Then $X^\vee = \{\Delta_X = 0\}$ is a hypersurface in $\mathbb{P}(V^*)$. We shall call the *X-discriminant* the defining polynomial Δ_X of $X^\vee$.

Example. For $M_2^1 = \{x^2 + y^2 = z^2\}$, we have

$$(M_2^1)^\vee = \{u^2 + v^2 + w^2 = 0\}.$$

Example [GKZ94]. For the curve $M_3^1 = \{x^3 + y^3 = z^3\}$ we have

$$(M_3^1)^\vee = \{u^6 + v^6 + w^6 - 2u^3v^3 - 2u^3w^3 - 2v^3w^3 = 0\}.$$

The curve $(M_3^1)^\vee$ is not smooth since M_3^1 is smooth and $\deg M_3^1 = 3 > 2$.

n-Valued Groups: a Brief History

In 1971, V. M. Buchstaber and S. P. Novikov proposed a construction *motivated by the theory of characteristic classes* [BN71].

This construction describes a multiplication in which *the product of any pair of elements is a multiset of n points* [Buc06].

An axiomatic definition of n-valued groups and the first results of their algebraic theory were obtained in a subsequent series of works by V. M. Buchstaber.

Currently, the theory of n-valued (formal, finite, discrete, topological, and algebro-geometric) groups and their applications in various areas of mathematics and mathematical physics are being developed by a number of authors.

n-Valued Monoids

An *n-valued monoid* is a space X equipped with an operation

$$* : X \times X \to \mathrm{Sym}^n(X)$$

where $\mathrm{Sym}^n(X) = X^{\times n}/S_n$ is the space of unordered n-tuples of elements of X:

- *Associativity.* The n^2-multisets

$$[x * w \mid w \in y * z],$$
$$[w * z \mid w \in x * y]$$

  coincide.
- *Unit.* There exists an element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

  for every $x \in X$.

n-Valued Groups

An *n-valued group* X is an n-valued monoid equipped with an inverse map

$$\mathrm{inv} : X \to X$$

that is, a map such that for every $x \in X$

$$x * \mathrm{inv}(x) \ni e, \quad \mathrm{inv}(x) * x \ni e.$$

The notions of homomorphisms, commutativity, kernels, and related concepts admit natural generalizations from the context of 1-valued groups to that of n-valued groups.

Algebraic n-Valued Monoids and Groups

An *algebraic n-valued monoid* is an algebraic variety X equipped with an associative n-valued multiplication given by a *rational morphism* $X \times X \to \mathrm{Sym}^n(X)$ with a neutral element $e \in X$ such that

$$x * e \;=\; e * x \;=\; [x, x, \dots, x] \quad \text{for every } x \in X.$$

An *algebraic n-valued group* is an algebraic n-valued monoid on X together with a *regular morphism* $\mathrm{inv} : X \to X$ such that for any $x \in X$ the following two conditions hold:

$$e \in x * \mathrm{inv}(x), \qquad x * \mathrm{inv}(x) = \mathrm{inv}(x) * x.$$

Polynomials $p_d(z; x, y)$

Introduce the following symmetric polynomials:

$$p_d(z; x, y) = \prod_{r,s=1}^{d} \left(\sqrt[d]{z} + \varepsilon^r \sqrt[d]{x} + \varepsilon^s \sqrt[d]{y} \right),$$

where $\varepsilon = e^{2\pi i/d}$ and $\sqrt[d]{-}$ denotes some branch of the root.

$$\begin{aligned}
p_1 &= \sigma_1, \\
p_2 &= \sigma_1^2 - 4\sigma_2, \\
p_3 &= \sigma_1^3 - 27\sigma_3, \\
p_4 &= \sigma_1^4 - 2^3\sigma_1^2\sigma_2 + 2^4\sigma_2^2 - 2^7\sigma_1\sigma_3, \\
p_5 &= \sigma_1^5 - 5^4\sigma_1^2\sigma_3 - 5^5\sigma_2\sigma_3, \\
p_6 &= \sigma_1^6 - 2^2 \cdot 3\sigma_1^4\sigma_2 - 2 \cdot 3^4 \cdot 17\sigma_1^3\sigma_3 + 2^4 \cdot 3\sigma_1^2\sigma_2^2 \\
&\quad - 2^3 \cdot 3^4 \cdot 19\sigma_1\sigma_2\sigma_3 - 2^6\sigma_2^3 + 3^3 \cdot 19^3\sigma_3^2
\end{aligned}$$

where $\sigma_j' s$ are elementary symmetric polynomials in x, y, z.

Groups $\mathbb{G}_d(\mathbb{C})$

The polynomial

$$p_d(z; (-1)^d x, (-1)^d y)$$

defines a commutative algebraic d-valued group $\mathbb{G}_d(\mathbb{C})$ on $\mathbb{C}$ with multiplication

$$x * y = [z \mid p_d(z; x, y) = 0],$$

neutral element 0 and inverse $\mathrm{inv}(x) = (-1)^d x$.

θ-Circulants

For any elements $a_0, \ldots, a_{n-1} \in \mathbb{k}$, introduce a *$\theta$-circulant matrix*:

$$\mathrm{Circ}_\theta(a_0, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & \theta a_1 & \theta a_2 & \cdots & \theta a_{n-1} \\ a_{n-1} & a_0 & \theta a_1 & \ddots & \vdots \\ a_{n-2} & a_{n-1} & a_0 & \ddots & \theta a_2 \\ \vdots & \ddots & \ddots & \ddots & \theta a_1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

Wendt Matrices

Theorem (Buchstaber, Kornev, 2025)

The polynomial $p_d(z; x, y)$ defining the d-valued multiplication is the determinant of a y-circulant $d \times d$ matrix:

$$\mathrm{Circ}_y \left(w^d + (-1)^{d+1} x + y, \binom{d}{1} w, \dots, \binom{d}{n-1} w^{d-1} \right)$$

where $w^d = z$.

These matrices generalize the Wendt matrices (the substitution $x = (-1)^d$, $y = z = 1$).

$p_d(z; x, y)$ and Wendt Matrices

$$p_2 = \begin{vmatrix} w^2 - x + y & 2wy \\ 2w & w^2 - x + y \end{vmatrix}$$

$$p_3 = \begin{vmatrix} w^3 + x + y & 3yw & 3yw^2 \\ 3w^2 & w^3 + x + y & 3yw \\ 3w & 3w^2 & w^3 + x + y \end{vmatrix}$$

$$p_4 = \begin{vmatrix} w^4 - x + y & 4yw & 6yw^2 & 4yw^3 \\ 4w^3 & w^4 - x + y & 4yw & 6yw^2 \\ 6w^2 & 4w^3 & w^4 - x + y & 4yw \\ 4w & 6w^2 & 4w^3 & w^4 - x + y \end{vmatrix}$$

Polynomials $p_d(z; x, y)$

Theorem (Buchstaber, Kornev, 2025)

For prime $d \geqslant 5$, the polynomial

$$p_d(z; x, y) - (x + y + z)^d$$

is divisible by $d^4 xyz$.

This result follows from the above observations and from the following:

Theorem (Wolstenholme, [Wol62])

$$\binom{2d-1}{d-1} - 1$$

is divisible by d^3 for primes $d \geqslant 5$.

Fermat Curves and $\mathbb{G}_d(\mathbb{C})$

Theorem (Buchstaber, Kornev, 2025)

Consider the Fermat curve ($d \geq 2$)

$$M_d^1 = \{x^d + y^d = z^d\}.$$

Then the dual curve $(M_d^1)^\vee \subset (\mathbb{P}^2)^*$ is given by the equation

$$p_{d-1}(w^d; u^d, v^d) = 0.$$

Algebraic Monoids $\mathbb{M}_d(\mathbb{C}P^1)$

Theorem (Buchstaber, Kornev, 2025)

The group $\mathbb{G}_d(\mathbb{C})$ extends (only) to an algebraic d-valued coset monoid $\mathbb{M}_d(\mathbb{C}P^1)$ on $\mathbb{C}P^1$ with

$$* : \; \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$$

$$(x_1 : x_0) * (y_1 : y_0) = (b_d : b_{d-1} : \cdots : b_0),$$

where $b_j = b_j(x, y)$ is the coefficient of $z_1^{d-j} z_0^j$ in the homogeneous polynomial

$$(x_0 y_0 z_0)^d \; p_d\left(\frac{z_1}{z_0};\; (-1)^d \frac{x_1}{x_0},\; (-1)^d \frac{y_1}{y_0}\right)$$

whenever $(x_1 : x_0)$ and $(y_1 : y_0)$ are not *both* equal to $(1 : 0)$.

Algebraic Monoids $\mathbb{M}_d(\mathbb{C}P^1)$

Here the point ∞ is absorbing, i.e.

$$\infty * x = x * \infty = [\infty, \infty, \ldots, \infty]$$

for any $x \in \mathbb{C}P^1 \backslash \{\infty\}$, and the value $\infty * \infty$ is undefined.

Projective Duality is a Shift on $\mathbb{M}_d$'s

For each $d \geq 2$ consider a curve

$$X_d = \{p_d(z; x, y) = 0\}.$$

Theorem (Buchstaber, Kornev, 2025)

Under projective duality the curve X_d ($d \geq 2$) goes to

$$X_d^\vee = \{\, (uvw)^{d-1}\, p_{d-1}(1/w; 1/u, 1/v) = 0 \,\} \subset (\mathbb{C}P^2)^*.$$

The composition of the duality $X_d \mapsto X_d^\vee$ with the subsequent Möbius transformation $(u, v, w) \mapsto (1/u, 1/v, 1/w)$ defines a shift operation

$$\mathbb{M}_d(\mathbb{C}P^1) \mapsto \mathbb{M}_{d-1}(\mathbb{C}P^1)$$

in the family of algebraic d-valued monoids.

Projective Duality is a Shift on $\mathbb{M}_d$'s

Example. For $X_2^\vee$ we have the parametrization

$$(u, v) = \left(-\frac{1}{1+t}, \frac{1}{t}\right) \quad \text{or} \quad \frac{1}{u} + \frac{1}{v} = -1.$$

Taking the projective closure (homogenization), we find that $X_2^\vee$ is given by the zero locus of the polynomial

$$P_1 = uvw\, p_1(w^{-1}; u^{-1}, v^{-1}) = (u+v)w + uv.$$

Example. For $X_3^\vee$:

$$(u, v) = \left(\frac{1}{(1+t)^2}, \frac{1}{t^2} \right) \quad \text{or} \quad \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} = 1.$$

The curve $X_3^\vee$ is given by the polynomial

$$P_2 = (uvw)^2\, p_2(w^{-1}; u^{-1}, v^{-1}) = (uv - w(u+v))^2 - 4uvw^2,$$

$$P_2 = (uv)^2 + (vw)^2 + (uw)^2 - 2u^2vw - 2uv^2w - 2uvw^2.$$

Projective Duality is a Shift on $\mathbb{M}_d$'s

For any smooth curve $X \subset \mathbb{P}^2$ of degree d, according to Plücker formulas [GKZ94, Proposition 2.4], we have $\deg X^\vee = d(d-1)$.

Since $\deg X_d^\vee = (d-1)^2$ for $d \geq 3$, the curve X_d cannot be smooth.

The curve X_2 is smooth.

Discriminants and X_d's

Theorem (Gaiur, Rubtsov, van Straten, 2024)

The discriminant $\Delta_t(P)$ of the polynomial

$$P(t) = (zt^{d-1} + y)(1+t)^{d-1} + (-1)^{d-1} x t^{d-1}$$

with respect to the variable t, which is a polynomial of degree $4d-6$, is related to $p_d(z; x, y)$ by

$$(-1)^d (d-1)^{2d-2} (xyz)^{d-2} p_d(z; x, y) = \Delta_t(P)$$

for each $d \geq 2$.

This result has a nice explanation from the point of view of projective duality.

POV: Projective Duality

Consider the curve $X_d^\vee$. Its parametrization in the chart $\{w = 1\}$:

$$(u, v) = \big((-1-t)^{1-d},\ t^{1-d}\big).$$

By the definition of the $X_d^\vee$-discriminant, the curve $X_d^{\vee\vee} = \{p_d(z; x, y) = 0\}$ is an irreducible component of the discriminant of the polynomial of t obtained by restricting the line

$$ux + vy + 1 = 0$$

onto $X_d^\vee$, i.e. the curve $X_d^{\vee\vee}$ is the discriminant in t of $P(t)$, where $P(t)$ is a polynomial obtained from the equation

$$\frac{1}{(-1-t)^{d-1}} \cdot x + \frac{1}{t^{d-1}} \cdot y + 1 = 0$$

after homogenization.

POV: Projective Duality

If $xyz = 0$, then for $d \geq 2$ the polynomial $P(t)$ has a $(d-1)$-fold root $t^* \in \{0, -1, \infty\}$. Thus, $\Delta_t(P)$ is divisible by a certain power of the monomial xyz.

By [GRS24, Theorem 2.2], $\Delta_t(P)$ has no other singular components. The required statement now follows by comparing degrees.

Polynomials $p_{d,n}$

In connection with Bessel kernels for solutions of Picard–Fuchs differential equations for the kernel

$$K_d = \sum_{j,k} \binom{j+k}{k}^d \frac{x^j y^k}{z^{j+k}},$$

the iterated analogue of the polynomials $p_d(z; x, y)$ was considered in [GRS24]:

$$p_{d,n}(z; \boldsymbol{x}) = \prod_{k_1,\dots,k_n=1}^{d} \left(\sqrt[d]{z} + \varepsilon^{k_1} \sqrt[d]{x_1} + \cdots + \varepsilon^{k_n} \sqrt[d]{x_n}\right).$$

Operations $O_{d,n}(\mathbb{C}P^1)$

The polynomial $p_{d,n}(z;\boldsymbol{x})$ defines an n-ary d^{n-1}-valued algebraic operation

$$\mu(x_1, \ldots, x_n) = [z \mid p_{d,n}(z;\boldsymbol{x}) = 0].$$

Denote by $O_{d,n}(\mathbb{C}P^1)$ the variety $\mathbb{C}P^1$ with the operation μ.

Let

$$X_d^{n-1} = \{\, p_{d,n} = 0 \,\}$$

be the hypersurface in $\mathbb{C}P^n$. For integers $d \geq 2$ and $n \geq 2$ define

$$P_{d,n} = (u_1 \cdots u_n w)^{d-1}\, p_{d-1}(w^{-1}; u_1^{-1}, \ldots, u_n^{-1})$$

the polynomial of degree d^{n-1}.

Operations $O_{d,n}(\mathbb{C}P^1)$

Theorem (Buchstaber, Kornev, 2025)

The composition of the duality ($d \geq 2$, $n \geq 2$)

$$X_d^{n-1} \mapsto (X_d^{n-1})^\vee = \{P_{d,n} = 0\} \subset (\mathbb{C}P^n)^*$$

with the subsequent Möbius transformation

$$(u_1, \ldots, u_n, w) \mapsto (1/u_1, \ldots, 1/u_n, 1/w)$$

defines a shift operation

$$O_{d,n}(\mathbb{C}P^1) \mapsto O_{d-1,n}(\mathbb{C}P^1)$$

in the family of n-ary d^{n-1}-valued algebraic structures $O_{d,n}(\mathbb{C}P^1)$.

Fermat Hypersurfaces and $p_{d,n}$'s

Theorem (Buchstaber, Kornev, 2025)

Let M_d^{n-1} be a Fermat hypersurface

$$M_d^{n-1} = \{x_1^d + x_2^d + \ldots + x_n^d = z^d\}$$

in $\mathbb{C}P^n$ with coordinates $x_1, \ldots, x_n, z$. The dual hypersurface is defined by the equation

$$(M_d^{n-1})^\vee = \{p_{d-1,n}(w^d; u_1^d, \ldots, u_n^d) = 0\}$$

in $(\mathbb{C}P^n)^*$ with the dual coordinates $u_1, \ldots, u_n, w$.

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Thank you for attention!

Happy birthday, Andrey!