A new proof of Milnor-Wood inequality

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Circle bundles, Euler class, and Milnor-Wood theorem

1. A *circle bundle* is a locally trivial (topological) oriented circle bundle over a closed oriented surface S_g.
2. Its Euler class: $\mathcal{E}(E \xrightarrow{\pi} S_g) \in \mathbb{Z}$.
3. Existence of a continuous section $\Leftrightarrow \quad \mathcal{E} = 0$.
4. A *transverse foliation* of a circle bundle is a smooth foliation of E whose 2-dimensional *leaves* are transverse to the fibers. Each transverse foliation comes from a connection form with zero curvature, and vice versa.

Theorem

If an oriented circle bundle $E \xrightarrow{\pi} S_g$ has a smooth transverse foliation, then the Euler class of the bundle satisfies

$$|\mathcal{E}| \leq 2g - 2.$$

Classical proof

1. Observe that foliated bundles are related to homomorphisms
$$\phi : \pi_1 \to Homeo^+(S^1).$$
2. Express $\mathcal{E}$ via ϕ.
3. Use the properties of Poincarè rotation number of products and commutators.
4. Win!

Our proof

1. Take (a part of) a single leaf. It looks like a section, but it is NOT a section. It is a *quasisection*.
2. Develop a generic theory of quasisections. Derive a local formula which expresses the Euler class in terms of singularities of quasisections. (Done by P., Shamazov, Turevskii)
3. Apply the formula to the part of the leaf.
4. Win!
5. Observe that we proved more.

Quasisection instead of a section

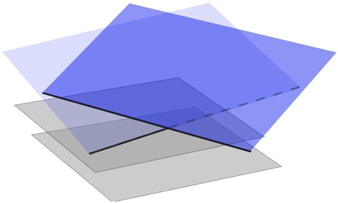
Def.

A quasisection of $E \xrightarrow{\pi} S_g$ *is an (either bordered or closed) smooth surface* Q *and a smooth map* $q : Q \to E$ *such that* $\pi \circ q(Q) = S_g$.

To compute the Euler class, we need the singularities of a quasisection (self-crossings, folds, Whitney umbrellas, pleats, etc.).

For Milnor-Wood, the quasisection is embedded disk $\mathcal{Q}$ with no self-crossings, no folds, no Whitney umbrellas, no pleats, etc. The only type of singularities are **transverse self-crossings of** $\pi(\partial \mathcal{Q})$**.**

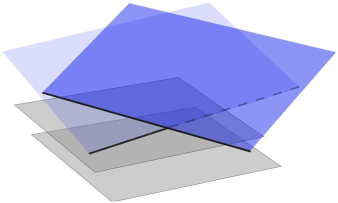
Let $x \in S_g$ be a self-crossing point of $\pi(\partial \mathcal{Q})$. We call x a *singular vertex* of the quasisection $\mathcal{Q}$. Locally $\mathcal{Q}$ consists of two bordered sheets, and a non-zero number of regular sheets.



Let C_x be a small circle embracing x. Imagine a point y goes along the circle C_x in the ccw direction, starting from a place with no fold in the preimage, that is, with the minimal number of points in the preimage $\pi^{-1}(y) \cap \mathcal{Q}$. Let us **order the two bordered sheets** as follows: the preimage $\pi^{-1}(y)$ meets the first border line first. The other border line is the second one.

Define two numbers, n and k:

1. $n(x)$ is the number of regular sheets of $\mathcal{Q}$ lying between the first and the second border lines, if one counts from the **first** border line in the direction of the fiber.
2. $k(x)$ is the number of regular sheets lying between the border lines, if one counts from the **second** border line in the direction of the fiber.



(Here we have $k = 0$, and $n = 2$.)
A singular vertex x is assigned a *weight* :

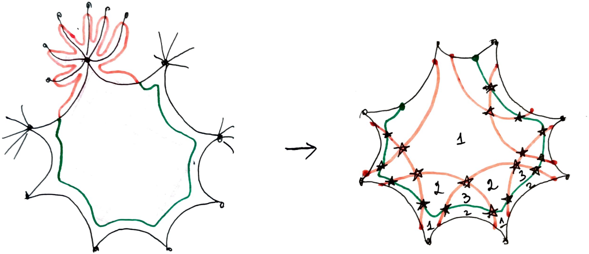
$$\mathcal{W}_{bb}(x) = \frac{(n-k)}{(n+k)(n+k+1)(n+k+2)}.$$

Computation of the Euler class

For a quasisection which is a disk with no folds, no self-crossing, etc., the local formula for $\mathcal{E}$ implies:

$$\mathcal{E} = \sum_{x_i} \mathcal{W}_{bb}(x_i).$$

S_g is the standard patch of a regular $4g$-gon; its universal cover $U \xrightarrow{pr} S_g$ is tiled by fundamental domains. Consider a disc $D \subset U$:



fundamental domain with a disk (pink) (bounded by red-green curve)

S_g with the projection of the boundary of the quasisection

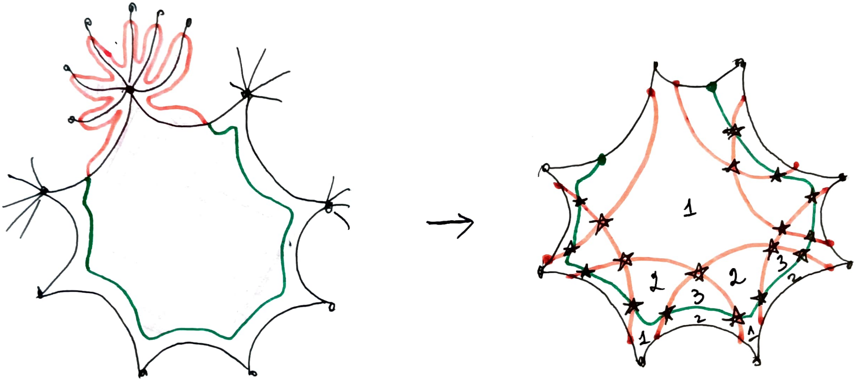
Raise D to the foliation. There exists a map

$$\phi : D \to E \text{ with } pr(x) = \pi(\phi(x)) \ \ \forall x \in D,$$

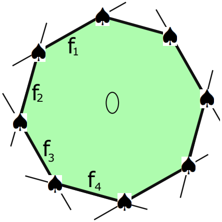
such that $\phi(D)$ lies in a leaf of the foliation.

Applying the local formula

There are $3(4g-2)$ singularities.
Each of them contributes $\pm 1/6$ since $n+k=1$.
Their sum (the Euler class) is at most $3(4g-2)\cdot 1/6 = 2g-1$.
The value $2g-1$ is not achievable.



The value $2g-1$ is not achievable.
Why?
Look "at the other side"!



Computation of $\mathcal{E}$, a reminder.

A way to compute the Euler number:

1. Start with a continuous partial section is defined everywhere except for a finite set of points $x_1, ..., x_m \in S_g$.
2. For each of x_i define an index:
   take a neighborhood U_i bounded by a small circle C_{x_i}. The restriction of the section s defines a map
   $$s_{|C_{x_i}} : C_{x_i} \to S^1.$$
   The source and the target are two oriented circles, so the degree of the map is well-defined. Set
   $$ind_s(x_i) := deg\ s_{|C_{x_i}}.$$
3. Then
   $$\mathcal{E}(E \to S_g) = \sum_i ind_s(x_i).$$

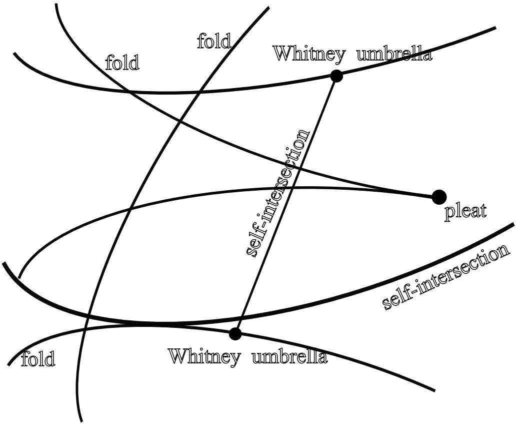
The local formula: M. Kazarian's averaging principle

Assume we have a random section s defined outside some fixed finite set $\{x_i\} \subset B$. Then one can take the expectation:

$$\mathcal{E}(E \to B) = \mathbb{E}\Big(\sum_i ind_s(x_i)\Big) = \sum_i \mathbb{E}(ind_s(x_i)).$$

We will show that a quasisection yields a random section with a finite probability space. In other words, a quasisection gives a finite collection of partial sections, and taking $\mathbb{E}$ amounts to averaging.

The projection of a quasisection:



Thank you for attention!