Tensor Term Logic for categorial grammars

Sergey Slavnov

HSE University
sslavnov@yandex.ru

Lambek grammars

- **Types:** $Tp ::= Prop|Tp\backslash Tp|Tp/Tp|Tp \bullet TP$.
- **Interpretation:**
  - Type is a set of words.
  - $w \in A\backslash B \Leftrightarrow \forall u \in A \; uw \in B$;
  - $w \in B/A \Leftrightarrow \forall u \in A \; wu \in B$;
  - $u \in A, v \in B \Rightarrow uv \in A \bullet B$.
    More generally:
    for any type C if $\forall u \in A, v \in B$ it holds that $uv \in C$ then $A \bullet B \subseteq C$.
- **Example:**
  - John $\in NP$, loves $\in (NP\backslash S)/NP$, Mary $\in NP \Rightarrow$
    loves Mary $\in NP\backslash S \Rightarrow$ John loves Mary $\in S$.
- **What else?**

Example: relativization

- **Lexicon:**
  - John $\in NP$, loves $\in (NP\backslash S)/NP$, Mary $\in NP$,
  - whom $\in (NP\backslash NP)/(S/NP)$.
- **Derivation:**
  1. $\forall x \in NP$ loves $x \in NP\backslash S$,
  2. $\forall x \in NP$ John loves $x \in S$,
  3. John loves $\in S/NP$;
  4. whom John loves $\in NP\backslash NP$;
  5. Mary whom John loves $\in NP$.

Example: coordination

- **Lexicon:**
  - John, Jim, Mary $\in NP$, loves, hates $\in (NP\backslash S)/NP$,
  - and $\in (X\backslash X)/X$,
  - where $X = S/NP$.
- **Derivation:**
  1. $\forall x \in NP$ loves $x \in NP\backslash S$,
  2. $\forall x \in NP$ John loves $x \in S$,
  3. John loves $\in S/NP$;
  4. John loves $\in X$;
  5. $\forall x \in NP$ hates $x \in NP\backslash S$,
  6. $\forall x \in NP$ Jim hates $x \in S$,
  7. Jim hates $\in S/NP$;
  8. Jim hates $\in X$;
  9. and Jim hates $\in X\backslash X$;
  10. John loves and Jim hates $\in X$;
  11. John loves and Jim hates $\in S/NP$;
  12. John loves and Jim hates Mary $\in S$;

Lambek Calculus

- **Sequents:**
  - $A_1, \ldots, A_n \vdash B$, where $A_1, \ldots, A_n, B$ are types.
  - Meaning: if $x_1 \in A_1, \ldots, x_n \in A_n$ then $x_1 \ldots x_n \in B$.
- **Natural deduction rules (without $\bullet$):**

$$A \vdash A (\mathrm{Id})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}(/\mathrm{I}), \quad \frac{\Gamma \vdash B/A \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}(/\mathrm{E}),$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A\backslash B}(\backslash \mathrm{I}), \quad \frac{\Delta \vdash A \quad \Gamma \vdash A\backslash B}{\Delta, \Gamma \vdash B}(\backslash \mathrm{E}),$$

- **Example:**

$$\frac{A \vdash A \qquad \dfrac{(A\backslash B)/C \vdash (A\backslash B)/C \quad C \vdash C}{(A\backslash B)/C, C \vdash A\backslash B}(/\mathrm{E})}{A, (A\backslash B)/C, C \vdash B}(\backslash \mathrm{E})$$

Encoding grammars

- **Non-logical axioms:**
  - Treat each terminal symbol as a type.
  - Encode the typing judgement $x \in A$ as the sequent $x \vdash A$.
- **Example:**
  - Lexicon:
    $$\text{John} \in NP, \quad \text{loves} \in (NP\backslash S)/NP, \quad \text{Mary} \in NP.$$
  - Encodes as
    $$\text{John} \vdash NP, \quad \text{loves} \vdash (NP\backslash S)/NP, \quad \text{Mary} \vdash NP.$$
  - Derivation:

$$\dfrac{\text{John} \vdash NP \qquad \dfrac{\text{loves} \vdash (NP\backslash S)/NP \qquad \text{Mary} \vdash NP}{\text{loves Mary} \vdash NP\backslash S}(/\mathrm{E})}{\text{John loves Mary} \vdash S}(\backslash\mathrm{E})$$

Relativization once more

- **Axioms:**
  - John $\vdash NP$, loves $\vdash (NP\backslash S)/NP$, Mary $\vdash NP$,
  - whom $\vdash (NP\backslash NP)/(S/NP)$.
- **Derivation:**
  - $$\dfrac{\text{John} \vdash NP \quad \dfrac{\text{loves} \vdash (NP\backslash S)/NP \quad NP \vdash NP}{\text{loves}, NP \vdash NP\backslash S}(/E)}{\dfrac{\text{John loves}, NP \vdash S}{\text{John loves} \vdash S/NP}(/I)}(\backslash E)$$
  - $$\dfrac{\text{Mary} \vdash NP \quad \dfrac{\text{whom} \vdash (NP\backslash NP)/(S/NP) \quad \text{John loves} \vdash S/NP}{\text{whom John loves} \vdash NP\backslash NP}}{\text{Mary whom John loves} \vdash NP}(\backslash E)$$

Lambek Calculus Gentzen style

- **Sequent calculus rules:**

$$\frac{A \in Prop}{A \vdash A}(\text{Axiom}) \qquad \frac{\Gamma \vdash A \quad \Theta_1, A, \Theta_2 \vdash B}{\Theta_1, \Gamma, \Theta_2 \vdash B}(\text{Cut}),$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}(/\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, B/A, \Gamma, \Theta_2 \vdash C}(/\text{L}),$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A\backslash B}(\backslash\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, \Gamma, A\backslash B, \Theta_2 \vdash C}(\backslash\text{L}),$$

$$\frac{\Theta_1 \vdash A \quad \Theta_2 \vdash B}{\Theta_1, \Theta_2 \vdash A \bullet B}(\bullet\text{R}), \qquad \frac{\Theta_1, A, B, \Theta_2 \vdash C}{\Theta_1, A \bullet B, \Theta_2 \vdash C}(\bullet\text{L})$$

‘

Lambek Calculus Gentzen style

- **Sequent calculus rules:**

$$\frac{A \in Prop}{A \vdash A}(\text{Axiom}) \qquad \frac{\Gamma \vdash A \quad \Theta_1, A, \Theta_2 \vdash B}{\Theta_1, \Gamma, \Theta_2 \vdash B}(\text{Cut}),$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}(/\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, B/A, \Gamma, \Theta_2 \vdash C}(/\text{L}),$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \backslash B}(\backslash\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, \Gamma, A \backslash B, \Theta_2 \vdash C}(\backslash\text{L}),$$

$$\frac{\Theta_1 \vdash A \quad \Theta_2 \vdash B}{\Theta_1, \Theta_2 \vdash A \bullet B}(\bullet\text{R}), \qquad \frac{\Theta_1, A, B, \Theta_2 \vdash C}{\Theta_1, A \bullet B, \Theta_2 \vdash C}(\bullet\text{L})$$

- **Cut-elimination:** the Cut rule is redundant!

‘

Lambek Calculus Gentzen style

- **Sequent calculus rules:**

$$\frac{A \in Prop}{A \vdash A}(\text{Axiom}) \qquad \frac{\Gamma \vdash A \quad \Theta_1, A, \Theta_2 \vdash B}{\Theta_1, \Gamma, \Theta_2 \vdash B}(\text{Cut}),$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}(/\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, B/A, \Gamma, \Theta_2 \vdash C}(/\text{L}),$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \backslash B}(\backslash\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, \Gamma, A \backslash B, \Theta_2 \vdash C}(\backslash\text{L}),$$

$$\frac{\Theta_1 \vdash A \quad \Theta_2 \vdash B}{\Theta_1, \Theta_2 \vdash A \bullet B}(\bullet\text{R}), \qquad \frac{\Theta_1, A, B, \Theta_2 \vdash C}{\Theta_1, A \bullet B, \Theta_2 \vdash C}(\bullet\text{L})$$

‘

Lambek Calculus Gentzen style

- **Sequent calculus rules:**

$$\frac{A \in Prop}{A \vdash A}(\text{Axiom}) \qquad \frac{\Gamma \vdash A \quad \Theta_1, A, \Theta_2 \vdash B}{\Theta_1, \Gamma, \Theta_2 \vdash B}(\text{Cut}),$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}(/\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, B/A, \Gamma, \Theta_2 \vdash C}(/\text{L}),$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A\backslash B}(\backslash\text{R}), \qquad \frac{\Gamma \vdash A \quad \Theta_1, B, \Theta_2 \vdash C}{\Theta_1, \Gamma, A\backslash B, \Theta_2 \vdash C}(\backslash\text{L}),$$

$$\frac{\Theta_1 \vdash A \quad \Theta_2 \vdash B}{\Theta_1, \Theta_2 \vdash A \bullet B}(\bullet\text{R}), \qquad \frac{\Theta_1, A, B, \Theta_2 \vdash C}{\Theta_1, A \bullet B, \Theta_2 \vdash C}(\bullet\text{L})$$

- **Example:**

$$\frac{C \vdash C \qquad \dfrac{A \vdash A \quad B \vdash B}{A, A \setminus B \vdash B}(\backslash\text{L})}{A, (A \setminus B)/C, C \vdash B}(/\text{L})$$

‘

Problems with LC

- **Lexicon:**
  - John $\in NP$, loves $\in (NP\backslash S)/NP$, Mary $\in NP$,
  - whom $\in (NP\backslash NP)/(S/NP)$,
  - madly $\in (NP\backslash S)\backslash(NP\backslash S)$.

Problems with LC

- **Lexicon:**
  - John $\in NP$, loves $\in (NP\backslash S)/NP$, Mary $\in NP$,
  - whom $\in (NP\backslash NP)/(S/NP)$,
  - madly $\in (NP\backslash S)\backslash(NP\backslash S)$.
- **Try:**
  - $\forall x \in NP$ loves $x \in NP\backslash S$,
  - $\forall x \in NP$ loves x madly $\in NP\backslash S$,
  - $\forall x \in NP$ John loves x madly $\in S$,
  - $\cdots$

Problems with LC

- **Lexicon:**
  - John $\in NP$, loves $\in (NP\backslash S)/NP$, Mary $\in NP$,
  - whom $\in (NP\backslash NP)/(S/NP)$,
  - madly $\in (NP\backslash S)\backslash(NP\backslash S)$.
- **Try:**
  - $\forall x \in NP$ loves $x \in NP\backslash S$,
  - $\forall x \in NP$ loves x madly $\in NP\backslash S$,
  - $\forall x \in NP$ John loves x madly $\in S$,
  - $\ldots$
- That's it... :(
- "Mary whom John loves madly" **underivable**.

Tensor terms

- **We have:** *Ind* — *infinite* set of indices, T — terminal alphabeth.
- **Tensor term:**
  - formal product $[w_1]_{j_1}^{i_1} \cdots [w_n]_{j_n}^{i_n}$,
  - where $w_1, \ldots, w_n \in T^*$, $i_1, j_1, \ldots, i_n, j_n \in Ind$,
  - $i_1 \neq j_1, \ldots, i_n \neq j_n$,
  - there is **at most one upper and one lower occurrence of any index**.
- **Multiplication:** for $t = [u_1]_{j_1}^{i_1} \cdots [u_k]_{j_k}^{i_k}$, $s = [v_1]_{\beta_1}^{\alpha_1} \cdots [v_m]_{\beta_m}^{\alpha_m}$

$$ts = [u_1]_{j_1}^{i_1} \cdots [u_k]_{j_k}^{i_k} [v_1]_{\beta_1}^{\alpha_1} \cdots [v_m]_{\beta_m}^{\alpha_m}.$$

- **Relations:**

$$ts = st, \quad [u]_\alpha^x [v]_y^\alpha = [uv]_y^x.$$

Tensor terms

- **We have:** *Ind* — *infinite* set of indices, T — terminal alphabeth.
- **Tensor term:**
  - formal product $[w_1]^{i_1}_{j_1} \cdots [w_n]^{i_n}_{j_n}$,
  - where $w_1, \ldots, w_n \in T^*$, $i_1, j_1, \ldots, i_n, j_n \in Ind$,
  - $i_1 \neq j_1, \ldots, i_n \neq j_n$,
  - there is **at most one upper and one lower occurrence of any index**.
- **Multiplication:** for $t = [u_1]^{i_1}_{j_1} \cdots [u_k]^{i_k}_{j_k}$, $s = [v_1]^{\alpha_1}_{\beta_1} \cdots [v_m]^{\alpha_m}_{\beta_m}$

$$ts = [u_1]^{i_1}_{j_1} \cdots [u_k]^{i_k}_{j_k} [v_1]^{\alpha_1}_{\beta_1} \cdots [v_m]^{\alpha_m}_{\beta_m}.$$

- **Relations:**

$$ts = st, \quad [u]^x_\alpha [v]^\alpha_y = [uv]^x_y.$$

- **Example:**
  - $t = [\text{John loves}]^\alpha_\beta \cdot [\text{madly}]^\mu_\nu$, $s = [\text{Mary}]^\beta_\mu$,
  - $ts = [\text{John loves}]^\alpha_\beta [\text{madly}]^\mu_\nu [\text{Mary}]^\beta_\mu = [\text{John loves}]^\alpha_\beta [\text{Mary}]^\beta_\mu [\text{madly}]^\mu_\nu =$
    $= [\text{John loves Mary}]^\alpha_\mu [\text{madly}]^\mu_\nu = [\text{John loves Mary madly}]^\alpha_\nu$.

Notation and conventions

- Repeated index in a term is **bound**. Otherwise it is **free**.
  - $FI^{\bullet}(t)$ — free *upper* indices of t,
  - $FI_{\bullet}(t)$ — free *lower* indices of t.
  - $FI(t)$ free indices of t.
- **Notation:**
  - if $\alpha \in FI^{\bullet}(t)$, β fresh, then $t^{[\beta/\alpha]} = t$ with β substituted for α.
  - if $\alpha \in FI_{\bullet}(t)$, β fresh, then $t_{[\beta/\alpha]} = t$ with β substituted for α.
- **Kronecker deltas:**
  - $\delta_{\alpha}^{\beta} = [\epsilon]_{\alpha}^{\beta}$, where ϵ is the empty word.

Kronecker deltas

- **Kronecker deltas:**
  - $\delta^{\beta}_{\alpha} = [\epsilon]^{\beta}_{\alpha}$, where ϵ is the empty word.
- **Gluing**:
  - $\delta^{\alpha}_{\beta} \cdot ([u]^{x}_{\alpha}[v]^{\beta}_{y}) = [u\epsilon v]^{x}_{y} = [uv]^{x}_{y}$.
- **Index renaming:**
  - $\delta^{\alpha'}_{\alpha} \cdot [w]^{\alpha}_{\beta} = [\epsilon w]^{\alpha'}_{\beta} = [w]^{\alpha'}_{\beta}$,
  - $[w]^{\alpha}_{\beta} \cdot \delta^{\beta}_{\beta'} = [w\epsilon]^{\alpha}_{\beta'} = [w]^{\alpha}_{\beta'}$.
  - For $\alpha \in FI_{\bullet}(t), \beta$ fresh $\quad \delta^{\alpha}_{\beta} t = t_{[\beta/\alpha]}$.
  - For $\alpha \in FI^{\bullet}(t), \beta$ fresh $\quad \delta^{\beta}_{\alpha} t = t^{[\beta/\alpha]}$.

Intuitionistic tensor types

- **Literals** are assigned integer **valencies:**
- **Atomic Pseudoformulas** ($\widetilde{At}$):
  - $p^{i_1\ldots i_n}_{j_1\ldots j_n}$, where p is a literal of valency n, or $\mathbf{1}^i_j$.
- **Pseudoformulas:** $\widetilde{Fm} ::= \widetilde{At} | (\widetilde{Fm} \otimes \widetilde{Fm}) | (\widetilde{Fm} \multimap \widetilde{Fm})$.
- **Indices** in a pseudoformula have **upper** or **lower polarity**:

$$
\begin{aligned}
I^\bullet(p^{i_1,\ldots,i_n}_{j_1,\ldots,j_n}) &= \{i_1,\ldots,i_n\}, & I_\bullet(p^{i_1,\ldots,i_n}_{j_1,\ldots,j_n}) &= \{j_1,\ldots,j_n\} \\
I^\bullet(A \otimes B) &= I^\bullet(A) \cup I^\bullet(B), & I_\bullet(A \otimes B) &= I_\bullet(A) \cup I_\bullet(B), \\
I^\bullet(A \multimap B) &= I^\bullet(B) \cup I_\bullet(A), & I_\bullet(A \multimap B) &= I_\bullet(B) \cup I^\bullet(A).
\end{aligned}
$$

- **Example:**
  - $X = A^i_j \otimes B^k_l$: $I^\bullet(X) = \{i,k\}$, $I_\bullet(X) = \{j,l\}$.
  - $X = A^i_j \multimap B^k_l$: $I^\bullet(X) = \{j,k\}$, $I_\bullet(X) = \{i,l\}$.
  - $X = (A^i_j \multimap B^k_l) \multimap C^s_t$: $I^\bullet(X) = \{i,l,s\}$, $I_\bullet(X) = \{j,k,t\}$.

Intuitionistic tensor types continued

- **Atomic tensor formula (type):**
  - atomic pseudoformula with no repeated indices.
- **General tensor formula (type):**
  - pseudoformula built from atomic formulas that has **at most one upper and one lower polarity occurrence of any index**.
- Repeated index in a formula is **bound**. Otherwise it is **free**.
  - $FI^{\bullet}(A)$ — free *upper polarity* indices of A,
  - $FI_{\bullet}(A)$ — free *lower polarity* indices of A,
  - $FI(A) = FI^{\bullet}(A) \cup FI_{\bullet}(A)$.
- **Examples of formulas:**
  - $X = A^{\alpha}_{\beta} \otimes B^{\beta}_{\gamma}$, $FI^{\bullet}(X) = \{\alpha\}$, $FI_{\bullet}(X) = \{\gamma\}$.
  - $X = A^{\alpha}_{\beta} \multimap B^{\alpha}_{\gamma}$, $FI^{\bullet}(X) = \{\beta\}$, $FI_{\bullet}(X) = \{\gamma\}$.
  - $X = A^{\beta}_{\gamma} \multimap B^{\alpha}_{\gamma}$, $FI^{\bullet}(X) = \{\alpha\}$, $FI_{\bullet}(X) = \{\beta\}$.

Interpretation of tensor types

- Tensor type is a set of tensor terms.
- If $t \in A$ then $FI^{\bullet}(t) = FI^{\bullet}(A)$, $FI_{\bullet}(t) = FI_{\bullet}(A)$.
- If $t \in A$ then $t^{[\alpha/\beta]} \in A^{[\alpha/\beta]}$.
- If $t \in A$ then $t_{[\alpha/\beta]} \in A_{[\alpha/\beta]}$.
- $t \in A \multimap B \Leftrightarrow \forall s \in A\ ts \in B$.
- $t \in A, s \in B \Rightarrow ts \in A \otimes B$.
  More generally:
  for any type C if $\forall t \in A, s \in B$ it holds that $ts \in C$ then $A \otimes B \subseteq C$.
- $\delta_\beta^\alpha \in \mathbf{1}_\beta^\alpha$.
  More generally: for any type C if $\delta_\beta^\alpha \in C$ then $\mathbf{1}_\beta^\alpha \subseteq C$.

Examples

- **Lexicon:** $[\text{John}]^i_j \in np^i_j$, $[\text{sleeps}]^j_k \in np^i_j \multimap s^i_k$.
- **Derivation:**
  - $[\text{sleeps}]^j_k[\text{John}]^i_j = [\text{John}]^i_j[\text{sleeps}]^j_k = [\text{John sleeps}]^i_k \in s^i_k$.

Examples

- **Lexicon:** $[\text{John}]^i_j \in np^i_j, \quad [\text{sleeps}]^j_k \in np^i_j \multimap s^i_k.$
- **Derivation:**
  - $[\text{sleeps}]^j_k[\text{John}]^i_j = [\text{John}]^i_j[\text{sleeps}]^j_k = [\text{John sleeps}]^i_k \in s^i_k.$
- **Lexicon2:** $[\text{John}]^i_j, [\text{Mary}]^i_j \in np^i_j, \quad [\text{loves}]^i_j \in np^j_y \multimap (np^x_i \multimap s^x_y).$

Examples

- **Lexicon:** $[\text{John}]^i_j \in np^i_j, \quad [\text{sleeps}]^j_k \in np^i_j \multimap s^i_k.$
- **Derivation:**
  - $[\text{sleeps}]^j_k[\text{John}]^i_j = [\text{John}]^i_j[\text{sleeps}]^j_k = [\text{John sleeps}]^i_k \in s^i_k.$
- **Lexicon2:** $[\text{John}]^i_j, [\text{Mary}]^i_j \in np^i_j, \quad [\text{loves}]^i_j \in np^j_y \multimap (np^x_i \multimap s^x_y).$
- **Derivation2:**
  - $[\text{Mary}]^j_y \in np^j_y; \quad [\text{loves}]^i_j[\text{Mary}]^j_y = [\text{loves Mary}]^i_y \in np^x_i \multimap s^x_y;$
  - $[\text{John}]^x_i \in np^x_i; \quad [\text{loves Mary}]^i_y[\text{John}]^x_i = [\text{John loves Mary}]^x_y \in s^x_y.$

Examples

- **Lexicon:** $[\text{John}]^i_j \in np^i_j, \quad [\text{sleeps}]^j_k \in np^i_j \multimap s^i_k.$
- **Derivation:**
  - $[\mathrm{sleeps}]^j_k[\mathrm{John}]^i_j = [\mathrm{John}]^i_j[\mathrm{sleeps}]^j_k = [\mathrm{John\ sleeps}]^i_k \in s^i_k.$
- **Lexicon2:** $[\text{John}]^i_j, [\text{Mary}]^i_j \in np^i_j, \quad [\text{loves}]^i_j \in np^j_y \multimap (np^x_i \multimap s^x_y).$
- **Derivation2:**
  - $[\text{Mary}]^j_y \in np^j_y; \quad [\text{loves}]^i_j[\mathrm{Mary}]^j_y = [\mathrm{loves\ Mary}]^i_y \in np^x_i \multimap s^x_y;$
  - $[\text{John}]^x_i \in np^x_i; \quad [\mathrm{loves\ Mary}]^i_y[\text{John}]^x_i = [\mathrm{John\ loves\ Mary}]^x_y \in s^x_y.$
- **Translating Lambek types:**
  - Lambek type $F \mapsto$ tensor type $\|F\|^x_y$.
  - p — literal $\Rightarrow \|p\|^x_y = p^x_y$.
  - $\|A/B\|^x_y = \|B\|^y_z \multimap \|A\|^x_z, \ \|B \setminus A\|^x_y = \|B\|^z_x \multimap \|A\|^z_y,$
  - $\|A \bullet B\|^x_y = \|A\|^x_z \otimes \|B\|^z_y.$

Example with relativization

- **Lexicon:**
  - $[\mathrm{John}]_j^i, [\mathrm{Mary}]_j^i \in np_j^i$, $\quad [\mathrm{loves}]_j^i \in np_y^j \multimap (np_i^x \multimap s_y^x)$,
  - $[\mathrm{madly}]_k^y \in (np_i^x \multimap s_y^x) \multimap (np_i^z \multimap s_k^z)$,
  - $[\mathrm{who}]_z^t \in (\mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)) \multimap (np_t^u \multimap np_k^u)$.

Example with relativization

- **Lexicon:**
  - $[\text{John}]_j^i, [\text{Mary}]_j^i \in np_j^i$, $[\text{loves}]_j^i \in np_y^j \multimap (np_i^x \multimap s_y^x)$,
  - $[\text{madly}]_k^y \in (np_i^x \multimap s_y^x) \multimap (np_i^z \multimap s_k^z)$,
  - $[\text{who}]_z^t \in (\mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)) \multimap (np_t^u \multimap np_k^u)$.
- **Derivation:**
  1. $\forall [X]_y^j \in np_y^j \ [\text{loves}]_j^i[X]_y^j = [\text{loves}X]_y^i \in np_i^x \multimap s_y^x$;
  2. $\forall [X]_y^j \in np_y^j \ [\text{madly}]_k^y[\text{loves}X]_y^i = [\text{loves}X\text{madly}]_k^i \in np_i^z \multimap s_k^z$;
  3. $[\text{John}]_i^z \in np_i^z$;
  4. $\forall [X]_y^j \in np_y^j \ [\text{loves}X\text{madly}]_k^i[\text{John}]_i^z = [\text{John loves}X\text{madly}]_k^z \in s_k^z$;
  5. $\forall [X]_y^j \in np_y^j \ [\text{John loves}X\text{madly}]_k^z = ([\text{John loves}]_j^z[\text{madly}]_k^y)[X]_y^j \in s_k^z$;
  6. $[\text{John loves}]_j^z[\text{madly}]_k^y \in np_y^j \multimap s_k^z$;
  7. $\delta_y^j([\text{John loves}]_j^z[\text{madly}]_k^y) = [\text{John loves madly}]_k^z \in \mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)$;
  8. $[\text{who}]_z^t[\text{John loves madly}]_k^z = [\text{who John loves madly}]_k^t \in np_t^u \multimap np_k^u$;
  9. $[\text{Mary}]_t^u \in np_t^u$;
  10. $[\text{who John loves madly}]_k^t[\text{Mary}]_t^u = [\text{Mary who John loves madly}]_k^u \in np_k^u$.

Tensor sequents

- **Tensor sequents:**
  - Expression $A_{(1)}, \ldots, A_{(n)} \vdash B$,
  - where $A_{(1)}, \ldots, A_{(n)}, B$ are tensor types,
  - for $k \neq l$ $FI^{\bullet}(A_{(k)}) \cap FI^{\bullet}(A_{(l)}) = FI_{\bullet}(A_{(k)}) \cap FI_{\bullet}(A_{(l)}) = \emptyset$.
- **Meaning:**
  - if $t_{(1)} \in A_{(1)}, \ldots, t_{(n)} \in A_{(n)}$ then $t_{(1)} \ldots t_{(n)} \in B$.

Intuitionistic Tensor Term Logic

- Intuitionistic Tensor Term Logic (**ITTL):**

$$\frac{A \in At}{A \vdash A}(\mathrm{Id}) \quad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}(\mathrm{Cut}) \quad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}(\mathrm{Ex})$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}(\mathrm{L} \multimap) \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}(\mathrm{R} \multimap)$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}(\mathrm{L}\otimes) \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}(\mathrm{R}\otimes)$$

$$\frac{\Gamma, A \vdash B \; j \in FI_{\bullet}(A)}{\Gamma, A_{[i/j]}, \mathbf{1}_j^i \vdash B}(\mathrm{L}\mathbf{1}_{\rightarrow}) \quad \frac{\Gamma \vdash A \; j \in FI^{\bullet}(A)}{\mathbf{1}_j^i, \Gamma \vdash A^{[i/j]}}(\mathrm{R}\mathbf{1}^{\rightarrow})$$

$$\frac{\Gamma, \mathbf{1}_j^i, A \vdash B \; j \in FI^{\bullet}(A)}{\Gamma, A^{[i/j]} \vdash B}(\mathrm{L}\mathbf{1}^{\leftarrow}) \quad \frac{\Gamma, \mathbf{1}_j^i \vdash A \; j \in FI_{\bullet}(A)}{\Gamma \vdash A_{[i/j]}}(\mathrm{R}\mathbf{1}_{\leftarrow})$$

- This is **cut-free** and **decidable**.

Intuitionistic Tensor Term Logic continued

- **Natural deduction-style rules** are **admissible:**

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}(\multimap \mathrm{I}) \qquad \frac{\Gamma \vdash \mathrm{A} \quad \Delta \vdash \mathrm{A} \multimap \mathrm{B}}{\Gamma, \Delta \vdash \mathrm{B}}(\multimap \mathrm{E})$$

- **Encoding typing judgements:**
  - terminal symbols $\mapsto$ literals of valency 1;
  - terms $\mapsto$ sequences of atomic formulas:
  - $[a]_y^x \mapsto \mathbf{a}_y^x; \quad \delta_y^x \mapsto \mathbf{1}_y^x;$
  - $[abc]_y^x = [a]_\alpha^x [b]_\beta^\alpha [c]_y^\beta \mapsto \mathbf{a}_\alpha^x, \mathbf{b}_\beta^\alpha, \mathbf{c}_y^\beta;$
  - $[pq]_t^z = [p]_\gamma^x [b]_t^\gamma \mapsto \mathbf{p}_\gamma^z, \mathbf{t}_t^\gamma;$
  - $[abc]_y^x [pq]_t^z = [a]_\alpha^x [b]_\beta^\alpha [c]_y^\beta \mapsto \mathbf{a}_\alpha^x, \mathbf{b}_\beta^\alpha, \mathbf{c}_y^\beta, \mathbf{p}_\gamma^z, \mathbf{t}_t^\gamma;$
  - $[abc]_y^x [pq]_t^z \delta_j^i \in A_y^x \mapsto \mathbf{a}_\alpha^x, \mathbf{b}_\beta^\alpha, \mathbf{c}_y^\beta, \mathbf{p}_\gamma^z, \mathbf{t}_t^\gamma, \mathbf{1}_j^i \vdash A_y^x.$

Relativization once more

$$\textbf{Lexicon}:\ \textbf{Mary}^i_j \vdash np^i_j,\quad \textbf{John}^i_j \vdash np^i_j,\quad \textbf{loves}^i_j \vdash np^j_y \multimap (np^x_i \multimap s^x_y),$$
$$\textbf{madly}^y_k \vdash (np^x_i \multimap s^x_y) \multimap (np^z_i \multimap s^z_k),$$
$$\textbf{who}^t_z \vdash (\mathbf{1}^j_y \otimes (np^j_y \multimap s^z_k)) \multimap (np^u_t \multimap np^u_k)$$

Relativization once more

$$\textbf{Lexicon}: \textbf{Mary}_j^i \vdash np_j^i,\quad \textbf{John}_j^i \vdash np_j^i,\quad \textbf{loves}_j^i \vdash np_y^j \multimap (np_i^x \multimap s_y^x),$$
$$\textbf{madly}_k^y \vdash (np_i^x \multimap s_y^x) \multimap (np_i^z \multimap s_k^z),$$
$$\textbf{who}_z^t \vdash (\mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)) \multimap (np_t^u \multimap np_k^u)$$

$$\dfrac{\dfrac{np_y^j \vdash np_y^j \quad \textbf{loves}_j^i \vdash np_y^j \multimap (np_i^x \multimap s_y^x)}{np_y^j, \textbf{loves}_j^i \vdash np_i^x \multimap s_y^x}(\multimap \mathrm{E}) \quad \textbf{madly}_k^y \vdash (np_i^x \multimap s_y^x) \multimap (np_i^z \multimap s_k^z)}{\dfrac{np_y^j, \textbf{loves}_j^i, \textbf{madly}_k^y \vdash np_i^z \multimap s_k^z}{\textbf{loves}_j^i, np_y^j, \textbf{madly}_k^y \vdash np_i^z \multimap s_k^z}(\mathrm{Ex})}$$

Relativization once more

$$\textbf{Lexicon} : \textbf{Mary}^i_j \vdash np^i_j, \quad \textbf{John}^i_j \vdash np^i_j, \quad \textbf{loves}^i_j \vdash np^j_y \multimap (np^x_i \multimap s^x_y),$$
$$\textbf{madly}^y_k \vdash (np^x_i \multimap s^x_y) \multimap (np^z_i \multimap s^z_k),$$
$$\textbf{who}^t_z \vdash (\mathbf{1}^j_y \otimes (np^j_y \multimap s^z_k)) \multimap (np^u_t \multimap np^u_k)$$

$$\dfrac{\dfrac{\dfrac{\textbf{John}^z_i \vdash np^z_i \quad \textbf{loves}^i_j, np^j_y, \textbf{madly}^y_k \vdash np^z_i \multimap s^z_k}{\textbf{John}^z_i, \textbf{loves}^i_j, np^j_y, \textbf{madly}^y_k \vdash s^z_k}(\multimap \mathrm{E})}{\textbf{John}^z_i, \textbf{loves}^i_j, \textbf{madly}^y_k, np^j_y \vdash s^z_k}(\mathrm{Ex})}{\textbf{John}^z_i, \textbf{loves}^i_j, \textbf{madly}^y_k \vdash np^j_y \multimap s^z_k}(\multimap \mathrm{I})$$

Relativization once more

$$\textbf{Lexicon}: \textbf{Mary}^i_j \vdash np^i_j, \quad \textbf{John}^i_j \vdash np^i_j, \quad \textbf{loves}^i_j \vdash np^j_y \multimap (np^x_i \multimap s^x_y),$$
$$\textbf{madly}^y_k \vdash (np^x_i \multimap s^x_y) \multimap (np^z_i \multimap s^z_k),$$
$$\textbf{who}^t_z \vdash (\mathbf{1}^j_y \otimes (np^j_y \multimap s^z_k)) \multimap (np^u_t \multimap np^u_k)$$

$$\dfrac{\dfrac{\dfrac{\mathbf{1}^j_y \vdash \mathbf{1}^j_y \qquad \textbf{John}^z_i, \textbf{loves}^i_j, \textbf{madly}^y_k \vdash np^j_y \multimap s^z_k}{\mathbf{1}^j_y, \textbf{John}^z_i, \textbf{loves}^i_j, \textbf{madly}^y_k \vdash \mathbf{1}^j_y \otimes (np^j_y \multimap s^z_k)}(\otimes\mathrm{R})}{\textbf{John}^z_i, \textbf{loves}^i_j, \mathbf{1}^j_y, \textbf{madly}^y_k \vdash \mathbf{1}^j_y \otimes (np^j_y \multimap s^z_k)}(\mathrm{Ex})}{\textbf{John}^z_i, \textbf{loves}^i_j, \textbf{madly}^j_k \vdash \mathbf{1}^j_y \otimes (np^j_y \multimap s^z_k)}(\mathrm{L}\mathbf{1}^{\leftarrow})$$

Relativization once more

$$\textbf{Lexicon}: \textbf{Mary}_j^i \vdash np_j^i, \quad \textbf{John}_j^i \vdash np_j^i, \quad \textbf{loves}_j^i \vdash np_y^j \multimap (np_i^x \multimap s_y^x),$$
$$\textbf{madly}_k^y \vdash (np_i^x \multimap s_y^x) \multimap (np_i^z \multimap s_k^z),$$
$$\textbf{who}_z^t \vdash (\mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)) \multimap (np_t^u \multimap np_k^u)$$

$$\dfrac{\textbf{John}_i^z, \textbf{loves}_j^i, \textbf{madly}_k^j \vdash \mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z) \qquad \textbf{who}_z^t \vdash (\mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)) \multimap (np_t^u \multimap np_k^u)}{\textbf{who}_z^t, \textbf{John}_i^z, \textbf{loves}_j^i, \textbf{madly}_k^j \vdash np_t^u \multimap np_k^u} (\multimap \mathrm{E})$$

Relativization once more

$$\textbf{Lexicon}: \textbf{Mary}_j^i \vdash np_j^i, \quad \textbf{John}_j^i \vdash np_j^i, \quad \textbf{loves}_j^i \vdash np_y^j \multimap (np_i^x \multimap s_y^x),$$
$$\textbf{madly}_k^y \vdash (np_i^x \multimap s_y^x) \multimap (np_i^z \multimap s_k^z),$$
$$\textbf{who}_z^t \vdash (\mathbf{1}_y^j \otimes (np_y^j \multimap s_k^z)) \multimap (np_t^u \multimap np_k^u)$$

$$\frac{\textbf{Mary}_t^u \vdash np_t^u \quad \textbf{who}_z^t, \textbf{John}_i^z, \textbf{loves}_j^i, \textbf{madly}_k^j \vdash np_t^u \multimap np_k^u}{\textbf{Mary}_t^u, \textbf{who}_z^t, \textbf{John}_i^z, \textbf{loves}_j^i, \textbf{madly}_k^j \vdash np_k^u} (\multimap \mathrm{E})$$