Algebraic topology of C^*-algebras

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Noncommutative geometry and topology

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Theorem

Pavlov, Troisky Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p : \mathcal{Y} \to \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action

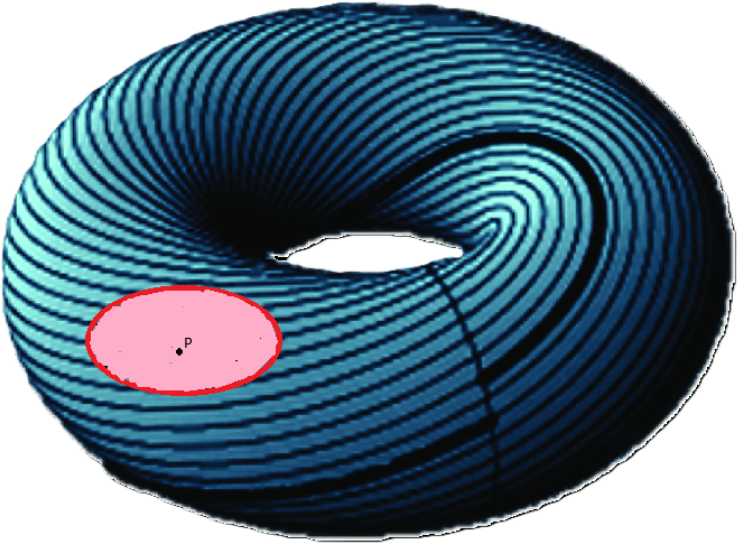
$$(f\xi)(y) = f(y)\xi(p(y)), \quad f \in C(\mathcal{Y}), \quad \xi \in C(\mathcal{X}),$$

then p is a finite-fold covering.

There is a counterexample to the Theorem. Alexandru Chirvasitu. *Non-commutative branched covers and bundle unitarizability*, arXiv:2409.03531v1, 2024.

Development of the (co)homology theory of C^*-algebras such that:

- for any commutative C^*-algebra $C(\mathcal{X})$ it coincides with (co)homology theory of $\mathcal{X}$.
- the theory is not trivial even for algebras having one-point spectrum.



Irrational rotation C^*-algebra $C_r^*(M, \mathcal{F})$ and its one-sided ideal.

Definition

Let A be a partially ordered set, S a subset of A. We say an element $a \in A$ is a *meet* (or *greatest lower bound*) for S, and write $a = \wedge S$, if

(a) a is an lower bound for S, i.e. $a \leq s$ for all $s \in S$, and

(b) if b satisfies $\forall \in S \quad b \leq s$, then $b \leq a$.

The antisymmetry axiom ensures that the join of S, if it exists, is unique. If S is a two-element set $\{s, t\}$, we write $s \wedge t$ for $\wedge\{s, t\}$ and if S is the empty set $\emptyset$, we write 0 for $\wedge\emptyset$.

Definition

A *meet-semilattice* is a partially ordered set which supports for any finite set the greatest lower bound.

Definition

A subset $\mathfrak{F}$ of a meet-semilattice A is said to be an *filter* if

(a) $\mathfrak{F}$ is a sub-meet-semilattice of A; i.e. $1 \in A$, and $a, b \in \mathfrak{F}$ imply $a \wedge b \in \mathfrak{F}$; and

(b) $\mathfrak{F}$ is a lower set; i.e. $a \in \mathfrak{F}$ and $a \leq b$ imply $b \in \mathfrak{F}$.

Any homomorphism $\phi : L' \to L''$ of semi-lattices yields a map of filers

$$\{I_\lambda\}_{\lambda\in\Lambda} \mapsto \text{ minimal filter containing } \{\phi(I_\lambda)\}_{\lambda\in\Lambda} \tag{1}$$

Definition

A maximal filter is an *ultrafilter*.

Example

If $\mathcal{X}$ is a topological space then the $\mathcal{X}$-*semi-lattice* is a meet-semilattice $\mathfrak{Lattice}(\mathcal{X})$ such that elements of $\mathfrak{Lattice}(\mathcal{X})$ are open subsets of $\mathcal{X}$ and one has

$$\begin{aligned} \mathcal{U}' \wedge \mathcal{U}'' &\stackrel{\text{def}}{=} \mathcal{U}' \cap \mathcal{U}'', \\ \mathcal{U}' \leq \mathcal{U}'' \quad \Leftrightarrow \quad &\mathcal{U}'' \subset \mathcal{U}', \\ 0 &\stackrel{\text{def}}{=} \emptyset. \end{aligned} \tag{2}$$

A set of neighborhoods of a point of Hausdorff space is an ultrafilter.

From the Zorn's lemma it follows that any filter is a subset of an ultrafilter.

Definition

If A be a C^*-algebra then *A-semi-lattice* is a meet-semilattice of closed left ideals such that

$$\begin{aligned} L' \wedge L'' &\stackrel{\text{def}}{=} L' \cap L'', \\ L' \le L'' \quad \Leftrightarrow \quad & L' \subset L'', \\ 0 &\stackrel{\text{def}}{=} \{0\} \subset A \end{aligned} \tag{3}$$

We denote this semilattice by $\mathfrak{Lattice}\left(A\right)$ and denote by $\mathfrak{Filters}\left(A\right)$ a set of filers.

Definition

If $\mathfrak{Ultrafilters}\left(A\right)$ is a set of ultrafilters of the meet-semilattice $\mathfrak{Lattice}\left(A\right)$ then for any closed left ideal $L \subset A$ denote by

$$\mathfrak{Ultrafilters}\left(A\right)_L \stackrel{\text{def}}{=} \left\{ x \in \mathfrak{Ultrafilters}\left(A\right) \middle| \, \exists L' \in x \quad L' \cap L = \{0\} \right\}$$

The *space of A-ultrafilters* is a set $\mathfrak{Ultrafilters}\left(A\right)$ with a the smallest topology o such that all sets $\mathfrak{Ultrafilters}\left(A\right)_L$ are open.

Remark

If $A = C_0(\mathcal{X})$ *then* $\mathfrak{Lattice}(A) \cong \mathfrak{Lattice}(\mathcal{X})$, $\mathfrak{Filters}(A) \cong \mathfrak{Filters}(\mathcal{X})$ *and* $\mathfrak{Ultrafilters}(A) \cong \mathfrak{Ultrafilters}(\mathcal{X})$.

Theorem

One has:

(a) *A topological space* $\mathcal{X}$ *is Hausdorff if and only if every ultrafilter on* $\mathcal{X}$ *has at most one limit.*

(b) *A topological space* $\mathcal{X}$ *is compact if and only if every ultrafilter has at least one limit.*

Remark

If sX is locally compact then there are "infinite ultrafilters" which do not correspond to points of $\mathcal{X}$.

Here we would like exclude "infinite ultrafilters".

Theorem

For each C^*-algebra A there is a dense hereditary ideal $K(A)$, which is minimal among dense ideals.

Definition

The ideal $K(A)$ is said to be the *Pedersen's ideal of A*.

One has $K(C_0(\mathcal{X})) = C_c(\mathcal{X})$

Definition

(*Ivankov*) An ultrafilter $x \in \mathfrak{Ultrafilters}(A)$ is a *finite point* if there is a nontrivial element $a \in K(A) \setminus \{0\}$ such that

$$Aa \in x.$$

The *Gelfand space* $\mathfrak{Gelfand}(A)$ of C^*-algebra A is a topological subspace of the space of A-ultrafilters (cf. Definition 8)

$$\mathfrak{Gelfand}(A) \overset{\text{def}}{=} \{x \in \mathfrak{Ultrafilters}(A) | \exists L \in x \quad L \text{ is a finite point}\}.$$

Theorem

(Commutative Gelfand-Naĭmark theorem). Let A be a commutative C^*-algebra and let $\mathcal{X}$ be the spectrum of A. There is the natural $*$-isomorphism $\gamma : A \xrightarrow{\cong} C_0(\mathcal{X})$.

Lemma

(Ivankov) (Generalized commutative Gelfand theorem). If $\mathcal{X}$ is a locally compact, Hausdorff space then is a natural homeomorphism $\mathfrak{Gelfand}_{\mathcal{X}} : \mathfrak{Gelfand}\left(C_0\left(\mathcal{X}\right)\right) \cong \mathcal{X}$.

Definition

If A is a C^*-algebra then a linear map $\lambda : A \to A$ is said to be a *left centralizer* if

$$\lambda(ab) = \lambda(a)\, b \quad \forall a, b \in A. \tag{4}$$

Similarly one defines a *right* centralizer. Denote the spaces of left and right centralizers by $\mathbf{LC}(A)$ and $\mathbf{RC}(A)$.

If both A and $\widetilde{A}$ are C^*-algebra then any injective homomorphism

$$\varphi_R : A \hookrightarrow \mathbf{RC}\left(\widetilde{A}\right)$$

of $\mathbb{C}$-algebras yields a homomorphism of lattices

$$\begin{gathered}\mathfrak{Lattice}\left(\widetilde{A}\right) \xrightarrow{\mathfrak{Lattice}(\varphi_R)} \mathfrak{Lattice}(A)\,,\\ \widetilde{L} \mapsto \bigcap\left\{L \subset A \middle| \widetilde{L} \subset \widetilde{A}\varphi_R(L)\right\}.\end{gathered} \tag{5}$$

On the other hand the homomorphism $\mathfrak{Lattice}\left(\varphi_R\right)$ yields a mapping of filters

$$\begin{aligned} \mathfrak{Filters}\left(\widetilde{A}\right) &\xrightarrow{\mathfrak{Filters}(\varphi_R)} \mathfrak{Filters}\left(A\right), \\ \widetilde{x} \mapsto \text{ the filter generated by } &\left\{\mathfrak{Lattice}\left(\varphi_R\right)\left(\widetilde{L}\right) \middle| \widetilde{L} \in \widetilde{x}\right\}. \end{aligned} \tag{6}$$

Lemma

If the homomorphism φ_R is injective then the map (6) *induces a continuous mapping*

$$\mathfrak{Ultrafilters}\left(\widetilde{A}\right) \xrightarrow{\mathfrak{Ultrafilters}(\varphi_R)} \mathfrak{Ultrafilters}\left(A\right) \tag{7}$$

Definition

The injective homomorphism φ_R is *good* if the given by (7) mapping $\mathfrak{Ultrafilters}\left(\varphi_R\right)$ yields the natural continuous map

$$\mathfrak{Gelfand}\left(\widetilde{A}\right) \xrightarrow{\mathfrak{Gelfand}(\varphi_R)} \mathfrak{Gelfand}\left(A\right).$$

Lemma

Ivankov Any injective homomorphism

$$\varphi_R : A \hookrightarrow \mathbf{RC}\left(\widetilde{A}\right)$$

of $\mathbb{C}$*-algebras naturally yields a continuous mapping*

$$\mathfrak{Gelfand}\left(\widetilde{A}\right) \xrightarrow{\mathfrak{Gelfand}(\varphi_R)} \mathfrak{Gelfand}\left(A\right) \tag{8}$$

If $\mathcal{H}$ is a Hilbert space and $A = \mathcal{K}(\mathcal{H})$ then for any left ideal $L \subset$ there is a closed $\mathbb{C}$-linear subspace $V_L \subset \mathcal{K}(\mathcal{H})$ such that

$$L = \{ a \in \mathcal{K}(\mathcal{H}) | \, aV_L = \{0\}\}.$$

If L is maximal then V_L is a one dimensional space. Any ultrafilter is principal, generated by maximal ideal. The set of one-dimensional subspaces of $\mathcal{H}$ is a complex projective space $\mathbb{C}P(\mathcal{H})$, i.e. there is the natural set theoretic bijective map $\mathfrak{Gelfand}(\mathcal{K}(\mathcal{H})) \cong \mathbb{C}P(\mathcal{H})$. The topology of $\mathfrak{Gelfand}(\mathcal{K}(\mathcal{H}))$ contains all sets $\mathbb{C}P(\mathcal{H}) \setminus V$ where V is a linear projective $\mathbb{C}$ subspace of $\mathbb{C}P(\mathcal{H})$. There identity map yields the continuous map

$$\phi_{\mathcal{H}} : \mathbb{C}P(\mathcal{H})_{\mathfrak{Hausdorff}} \to \mathbb{C}P(\mathcal{H})_{\mathfrak{Gelfand}}$$

where $\mathbb{C}P_{\mathfrak{Hausdorff}}(\mathcal{H})$ is the projective space with Hausdorff topology and $\mathbb{C}P(\mathcal{H})_{\mathfrak{Gelfand}}$ is the same set with the topology of the Gelfand space.

If $\dim \mathcal{H} = n < \infty$ then $\mathbb{C}P(\mathcal{H}) = \mathbb{C}P^n$ and there is the Zariski topology on $\mathbb{C}P^n$ such that closed subsets are given by polynomial equations. The closed sets of the Gelfand space are given by linear equations, so the Zariski topology is finer then Gelfand one. There is a composition of continuous maps

$$\mathbb{C}P^n_{\mathfrak{Hausdorff}} \to \mathbb{C}P^n_{\mathfrak{Zariski}} \to \mathbb{C}P^n_{\mathfrak{Gelfand}}$$

where the subscript $_{\mathfrak{Zariski}}$ means the Zariski topology. If $\dim \mathcal{H} = \infty$ and $\{L_0, L_1, ...\}$ is a set of mutually orthogonal codimension one projective subspaces of $\mathbb{C}P(\mathcal{H})$ then

$$\mathbb{C}P(\mathcal{H}) = \bigcup_{j=0}^{\infty} (\mathbb{C}P(\mathcal{H}) \setminus L_j)$$

Similarly

$$\mathbb{C}P^n = \bigcup_{j=0}^{n} \left(\mathbb{C}P^n \setminus L_j \right).$$

where $\{L_0, ..., L_n\}$ is a set of mutually orthogonal codimension one subspaces of $\mathbb{C}P^n$. where $\{L_0, ..., L_n\}$ is a set of mutually orthogonal codimension one projective subspaces of $\mathbb{C}P^n$.

Definition

Given a set X and a collection $\mathscr{W} = \{W\}$ of subsets of X, the *nerve* of $\mathscr{W}$ denoted by $K(\mathscr{W})$, is the simplicial complex whose simplexes are finite nonempty subsets of $\mathscr{W}$ with nonempty intersections. Thus the vertices of $K(\mathscr{W})$ are nonempty elements of $\mathscr{W}$.

Theorem

Let $\mathscr{A}$ be a sheaf of Abelian groups on $\mathcal{X}$ and let $\mathscr{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ an open covering of $\mathcal{X}$ having the property that $H^p(\mathcal{U}_\sigma; \mathscr{A}) = 0$ for $p > 0$ and all $\sigma \in K(\mathscr{U})$ in the nerve of coverin. Then there is a canonical isomorphism

$$H^*(\mathcal{X}, \mathscr{A}) \cong \check{H}^*(\mathscr{U}; \mathscr{A}). \tag{9}$$

If F is an Abelian group and $\mathscr{F}$ is a corresponding constant sheaf on $\mathbb{C}P(\mathcal{H})_{\mathfrak{Hausdorff}}$ and $\mathscr{W} = \{\mathbb{C}P(\mathcal{H}) \setminus L_j\}_{j=0,1,\ldots}$ or $\mathscr{W} = \{\mathbb{C}P^n \setminus L_j\}_{j=0,\ldots,n}$ then

$$\forall \sigma \in K(\mathscr{W}) \quad \mathcal{U}_\sigma = \mathcal{H} \setminus \bigcup_{j=0}^{n_\sigma} L_j^\sigma, \tag{10}$$

$$\forall j = 1, \ldots, n \quad L_j^\sigma \text{ is a linear subspace of } \mathcal{H} \quad \operatorname{codim}_{\mathbb{R}} L_j^\sigma \geq 2,$$

From the equation (10) it follows that

$$\mathfrak{Lattice}\left(\mathcal{U}_\sigma\right) \cong \mathfrak{Lattice}\left(\mathcal{H}'\right), \quad \forall p > 0 \quad \forall \sigma \in K\left(\mathscr{W}\right) \quad H^p\left(\mathcal{U}_\sigma; \mathscr{A}\right) = 0. \tag{11}$$

where $\mathcal{H}'$ is a Hilbert space and the isomorphism of semi-lattices comes from the inclusion $\mathcal{U}_\sigma \subset \mathcal{H}'$. If $\mathscr{F}$ is a corresponding to F constant sheaf on $\mathbb{C}P\left(\mathcal{H}\right)_{\mathfrak{Hausdorff}}$ and $\mathscr{W} = \left\{\mathbb{C}P\left(\mathcal{H}\right) \setminus L_j\right\}_{j=0,1,\dots}$ or $\mathscr{W} = \left\{\mathbb{C}P^n \setminus L_j\right\}_{j=0,\dots,n}$ then $H^p\left(\mathcal{U}_\sigma; \mathscr{A}\right) = 0$ for $p > 0$ and all $\sigma \in K\left(\mathscr{W}\right)$ where n $K\left(\mathscr{W}\right)$ is the nerve of $\mathscr{W}$. From the above theorem *turns out that*

$$H^*\left(\mathbb{C}P^n_{\mathfrak{Hausdorff}}, \mathscr{A}\right) \cong \check{H}^*\left(\mathscr{W}; \mathscr{A}\right)$$

Taking into account that $\mathbb{C}P^n \setminus L_j$ *is open in* $\mathbb{C}P\left(\mathcal{H}\right)_{\mathfrak{Gelfand}}$ *for any* j *one has*

$$H^*\left(\mathbb{C}P\left(\mathcal{H}\right)_{\mathfrak{Hausdorff}}, \mathscr{F}_{\mathfrak{Hausdorff}}\right) \cong H^*\left(\mathbb{C}P\left(\mathcal{H}\right)_{\mathfrak{Gelfand}}, \mathscr{F}_{\mathfrak{Gelfand}}\right)$$

The spectrum of $\mathcal{K}\left(\mathcal{H}\right)$ *has the single point but cohomology of* $\mathfrak{Gelfand}\left(\mathcal{K}\left(\mathcal{H}\right)\right)$ *are not trivial.*

Definition

If A is a C^*-algebra then an inclusion $C_0(\mathcal{Y}) \subset M(A)$ is *Hausdorff blowing-up* of A if both sets

$$C_c(\mathcal{Y})\, A \stackrel{\text{def}}{=} \{fa | f \in C_c(\mathcal{Y}) \quad a \in A\}, \\ AC_c(\mathcal{Y}) \stackrel{\text{def}}{=} \{af | f \in C_c(\mathcal{Y}) \quad a \in A\} \tag{12}$$

are dense in A.

Remark

$C_c(\mathcal{Y})\, A$ is dense in A if and only if $AC_c(\mathcal{Y})$ is dense in A (cf. equations (12)*), i.e. both equations* (12) *are equivalent.*

The "blowing-up" word is inspired by following reasons.

- Sometimes there is the natural partially defined surjective map from Hausdorff blowing-up to the spectrum.
- In the algebraic geometry "blowing-up" means excluding of singular points.

Definition

Let $C_0(\mathcal{Y}) \subset M(A)$ be Hausdorff blowing-up of A (cf. Definition 18), and let $\mathcal{U} \subset \mathcal{Y}$ be an open subset. Both left and right closed ideals $A_{\mathcal{U}}$ and ${}_{\mathcal{U}}A$ of A generated by sets $AC_0(\mathcal{U})$ and $C_0(\mathcal{U})A$ are the *left $\mathcal{U}$-ideal* and the *right $\mathcal{U}$-ideal* respectively. A hereditary C^*-subalgebra of A

$$ {}_{\mathcal{U}}A_{\mathcal{U}} \stackrel{\text{def}}{=} {}_{\mathcal{U}}A \cap A_{\mathcal{U}} = A^*_{\mathcal{U}} \cap A_{\mathcal{U}} \tag{13} $$

is the *$\mathcal{U}$-subalgebra*.

Definition

If $C_0(\mathcal{Y}) \subset M(A)$ is Hausdorff blowing-up of A, $a \in A$ and $\mathcal{U}_a \stackrel{\text{def}}{=} \bigcap \{\mathcal{U} \subset \mathcal{X} | \, a \in {}_{\mathcal{U}}A_{\mathcal{U}}\}$ then the closure $\mathcal{V}_a$ of $\mathcal{U}_a$ is said to be the *support* of a. We write $\operatorname{supp} a \stackrel{\text{def}}{=} \mathcal{V}_a$.

Lemma

If $C_0(\mathcal{Y}) \hookrightarrow M(A)$ is Hausdorff blowing-up and $a \in A$ belongs to the Pedersen's ideal $K(A)$ then the support of a is compact.

From the above Lemma it follows that the injective homomorphism $\varphi_{\mathcal{X}} : C_0(\mathcal{X}) \hookrightarrow M(A)$ is good 14). So there is the natural surjective continuous map

$$\phi_A : \mathfrak{Gelfand}(A) \to \mathcal{X}. \tag{14}$$

So one has from the Remark (14) it follows that

(i) if $\mathscr{A}$ is sheaf of Abelian group in $\mathcal{Y}$ then we have a homomorphism $H^q(\mathcal{Y}, \mathscr{A}) \to H^q(\mathfrak{Gelfand}(A), \phi_A^* \mathscr{A})$ for each q which is functorial in f and natural in $\mathscr{A}$.

(ii) If $\mathscr{B}$ is a sheaf of Abelian groups in $\mathcal{X}$ then we have a spectral sequence (Leray spectral sequence) $H^p(\mathfrak{Gelfand}(A), R^q(\phi_A)_*(\mathscr{B})) \Rightarrow H^{p+q}(\mathcal{X}, \mathscr{B})$ which is natural in $\mathscr{B}$.

Definition

A positive element in C^* - algebra A is *Abelian* if subalgebra $xAx \subset A$ is commutative.

Definition

A *continuous-trace* C^**-algebra* is a C^*-algebra A with Hausdorff spectrum $\mathcal{X}$ such that, for each $x_0 \in \mathcal{X}$ there are a neighbourhood $\mathcal{U}$ of x_0 and $a \in A$ such that $\mathfrak{rep}_x(a)$ is a rank-one projection for all $x \in \mathcal{U}$.

If A is a continuous trace C^*-algebra and $L \subsetneq A$ is a closed left ideal then there are $\varepsilon > 0$, $a \in A \setminus L$ and an irreducible representation $\rho : A \to B(\mathcal{H})$ with

$$\forall a' \in L \quad \left\|\rho\left(a - a'\right)\right\| > \varepsilon. \tag{15}$$

On the other hand A is a C^*-algebra of type I_0. From this fact one can deduce that a is a satisfying to (15) Abelian element. There is $\xi \in \mathcal{H}$ such that $\rho(a) = \xi\rangle\langle\xi$. One has $\mathcal{H} = \rho(A)\xi$ since ρ is irreducible. If $\xi \in \rho(L)\xi$ then $a \in L$. It is impossible so $\mathbb{C}\xi \cap \rho(L)\xi = \{0\}$. There are $\xi^{\|} \in \rho(L)\xi$ such that if $\xi^{\perp} \stackrel{\mathrm{def}}{=} \xi - \xi^{\|}$ then $\xi^{\perp} \perp \rho(L)\xi$. So for any closed left ideal there is one irreducible representation with $\rho(L)\xi \neq \mathcal{H}$. The spectrum $\mathcal{X}$ of A is Hausdorff. If there are $x', x'' \in \mathcal{X}$ with $x' \neq x''$ and $\rho_{x'}(L)\xi' \neq \mathcal{H}'$, $\rho_{x''}(L)\xi'' \neq \mathcal{H}''$ then there is $f \in C_0(\mathcal{X})$ such that $f(x') = 0$ and $f(x'') = 0$. If $L' \stackrel{\mathrm{def}}{=} L' + Af$ then L' is a closed left ideal such that $L \subsetneq L' \subsetneq A$. So if L is a maximal left ideal then there is a single point with $\rho(L)\xi \neq \mathcal{H}$. If codimension of $\rho(L)\xi \neq \mathcal{H}$ exceeds 1 then there if an Abelian element $a'' \in A$ with

$$\rho(L)\xi \subsetneq \rho(L)\xi + \rho\left(a''\right)\xi \subsetneq \mathcal{H}.$$

Lemma

If A is a continuous trace C^*-algebra then any element of $\mathfrak{Gelfand}(A)$ is given by a pair (x, ξ) where

(i) *x is a point of the spectrum $\mathcal{X}$ of A which corresponds to the irreducible representation $\mathfrak{rep}_x : A \to B(\mathcal{H}_x)$,*

(ii) *$\xi \in \mathbb{C}P(\mathcal{H}_x)$ where $\mathbb{C}P(\mathcal{H}_x)$ is a complex projective space.*

Lemma

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. If A is a stable separable continuous-trace C^*-algebra with spectrum $\mathcal{X}$, there is a locally trivial bundle $(\mathcal{F}, \pi, \mathcal{X})$ with fibre $\mathcal{K}(\mathcal{H})$ and structure group $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$ such that A is $C_0(\mathcal{X})$ -isomorphic to the space of sections $\Gamma_0(\mathcal{X})$

Suppose that A is given by a locally trivial fibre bundle $\mathcal{F}$ with fibre $\mathcal{K} = \mathcal{K}(\mathcal{H})$. There is a subbundle $\mathcal{E} \subset \mathcal{F}$ such that fibers of $\mathcal{E}$ are rank-one positive operators. It givers a bundle $\mathbb{C}P(\mathcal{E})$ with fibre $\mathbb{C}P(\mathcal{H})$. From the Lemma 24 it turns out that there is the natural set theoretic bijective map $\mathfrak{Gelfand}(A) \cong \mathbb{C}P(\mathcal{E})$. If $\mathbb{C}P(\mathcal{E})_{\mathfrak{Gelfand}}$ is $\mathbb{C}P(\mathcal{E})$ supplied with Gelfand topology then there is a bijective continuous map

$$\phi_{\mathcal{E}} : \mathbb{C}P(\mathcal{E})_{\mathfrak{Hausdorff}} \to \mathbb{C}P(\mathcal{E})_{\mathfrak{Gelfand}} \tag{16}$$

Lemma

Let F is an Abelian group. Let $\pi : \mathbb{C}P(\mathcal{E})_{\mathfrak{Hausdorff}} \to \mathcal{X}$ *be the natural surjective mapping If* $\mathcal{X} = \bigcup_{\mathcal{W}\in\mathscr{W}} \mathcal{W}$ *where:*

- *for any* $\mathcal{W} \in \mathscr{W}$ *one has* $\pi^{-1}(\mathcal{W}) \cong \mathcal{W} \times \mathbb{C}P(\mathcal{H})_{\mathfrak{Hausdorff}}$*,*
- $H^p(\mathcal{U}, \mathscr{F}_{\mathfrak{Hausdorff}}) = 0$ *for all* $\mathcal{U} \in K(\mathscr{W})$ *and* $p > 0$*.*

then there is the natural isomorphism

$$H^*(\mathfrak{Gelfand}(A), \mathscr{F}_{\mathfrak{Gelfand}}) \cong H^*\left(\mathbb{C}P(\mathcal{E})_{\mathfrak{Hausdorff}}, \mathscr{F}_{\mathfrak{Hausdorff}}\right)$$

where both $\mathscr{F}_{\mathfrak{Hausdorff}}$ *and* $\mathscr{F}_{\mathfrak{Gelfang}}$ *are corresponding to F locally constant sheaves.*

If $\mathcal{X}$ is a connected, locally compact, Hausdorff space then there is a homeomorphism $\mathfrak{Gelfand}\left(C_0\left(\mathcal{X}\right)\right) \cong \mathcal{X}$. If $\left(C_0\left(\mathcal{X}\right), \widetilde{A}, G, \mathfrak{lift}\right)$ is a noncommutative covering then the Lemma **??** yield a continuous map

$$\mathfrak{Gelfand}\left(\mathfrak{lift}\right) : \mathfrak{Gelfand}\left(\widetilde{A}\right) \to \mathcal{X}.$$

Lemma

If B is a hereditary C^*-subalgebra of A then the map $t \mapsto t \cap B$ is a homeomorphism between $\check{A} \setminus \mathrm{hull}\left(B\right)$ and $\check{B}$, where

$$\mathrm{hull}\left(B\right) = \left\{x \in \hat{A} \;\middle|\; \mathfrak{rep}_x\left(B\right) = \{0\}\right\}.$$

If $B \subset A$ is a hereditary C^*-subalgebra evenly covered by $\left(C_0\left(\mathcal{X}\right), \widetilde{A}, G, \mathfrak{lift}\right)$ (cf. Definition 31) then there is a connected open subset $\mathcal{U} \subset \mathcal{X}$ with $B \cong \mathcal{U}$. From the Definition 31 it follows that $\mathfrak{Gelfand}\left(\mathfrak{lift}\right)^{-1}\left(\mathcal{U}\right)$ is the disjoint union of homeomorphic to $\mathcal{U}$ connected open subsets of $\mathfrak{Gelfand}\left(\widetilde{A}\right)$, i.e. the map $\mathfrak{Gelfand}\left(\mathfrak{lift}\right)$ is a covering.

If $\widetilde{A}$ is not commutative then there is $\widetilde{x} \in \mathfrak{Gelfand}\left(\mathfrak{lift}\right)$ with

$$\dim \widetilde{A}/\widetilde{x} > 1.$$

. From the condition (b) of the Definition 30 it follows that

$$\dim C_0\left(\mathcal{X}\right)/\mathfrak{Gelfand}\left(\mathfrak{lift}\right)\left(\widetilde{x}\right) > 1.$$

It is impossible so $\widetilde{A} \cong C_0\left(\mathfrak{Gelfand}\left(\widetilde{A}\right)\right)$ is a commutative C^*-algebra. Thus there is an 1-1 correspondence between topological and noncommutative coverings of $C_0\left(\mathcal{X}\right)$. From the Definition **??** that the fundamental group of $C_0\left(\mathcal{X}\right)$ if it exists is isomorphic to $\pi_1\left(\mathcal{X}\right)$.

Definition

Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a continuous map. An open subset $\mathcal{U} \subset \mathcal{X}$ is said to be *evenly covered* by $\widetilde{\pi}$ if $\widetilde{\pi}^{-1}(\mathcal{U})$ is the disjoint union of open subsets of $\widetilde{\mathcal{X}}$ each of which is mapped homeomorphically onto $\mathcal{U}$ by $\widetilde{\pi}$. A continuous map $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ is called a *covering* if each point $x \in \mathcal{X}$ has an open neighbourhood evenly covered by $\widetilde{\pi}$. $\widetilde{\mathcal{X}}$ is called the *covering space* and $\mathcal{X}$ the *base space* of the covering.

Let A be a C^*-algebra. The group $\mathrm{Aut}\left(\widetilde{A}\right)$ of *-automorphisms carries (at least) two different topologies making it into a topological group The most important is *the topology of pointwise norm-convergence* based on the open sets

$$\{\alpha \in \mathrm{Aut}(A) \mid \|\alpha(a) - a\| < 1\}, \quad a \in A.$$

The other topology is the *uniform norm-topology* based on the open sets

$$\left\{\alpha \in \mathrm{Aut}(A) \,\middle|\, \sup_{a \neq 0} \|a\|^{-1} \|\alpha(a) - a\| < \varepsilon \right\}, \quad \varepsilon > 0 \tag{17}$$

which corresponds to following "norm"

$$\|\alpha\|_{\mathrm{Aut}} = \sup_{a \neq 0} \|a\|^{-1} \|\alpha(a) - a\| = \sup_{\|a\|=1} \|\alpha(a) - a\|. \tag{18}$$

Above formula does not really means a norm because $\mathrm{Aut}\,(A)$ is not a vector space. Henceforth the uniform norm-topology will be considered only.

Definition

We say that a C^*-algebra A is *connected* if it cannot be represented as a direct sum $A \cong A' \oplus A''$ of nontrivial C^*-algebras A' and A''.

Definition

Ivankov Let A be an connected C^*-algebra and let $\widetilde{A}$ be connected C^*-algebra , and let $\mathfrak{lift} : A \hookrightarrow M\left(\widetilde{A}\right)$ be an injective $*$-homomorphism of C^*-algebras such that following conditions hold:

(a) if $\mathrm{Aut}\left(\widetilde{A}\right)$ is a group of $*$-automorphisms of $\widetilde{A}$ then the group

$$G \stackrel{\mathrm{def}}{=} \left\{g \in \mathrm{Aut}\left(\widetilde{A}\right) \middle| \forall a \in \mathfrak{lift}\left(A\right) \quad ga = a\right\}$$

is discrete

(b) $A = \widetilde{A}^G \stackrel{\text{def}}{=} \left\{a \in \widetilde{A} \middle| \forall g \in G \quad a = ga\right\}$.

We say that the triple $\left(A, \widetilde{A}, G\right)$ and/or the quadruple $\left(A, \widetilde{A}, G, \mathfrak{lift}\right)$ and/or $*$-homomorphism $\mathfrak{lift} : A \hookrightarrow \widetilde{A}$ is a *noncommutative finite-fold pre-covering*. We write $G\left(\widetilde{A} \middle| A\right) \stackrel{\mathrm{def}}{=} G$.

Definition

Ivankov

Let $\left(A,\widetilde{A},G,\mathfrak{lift}\right)$ be a noncommutative pre-covering. A connected hereditary C^*-subalgebra $B\subset A$ is $\left(A,\widetilde{A},G,\mathfrak{lift}\right)$- *evenly covered by* $\left(A,\widetilde{A},G,\mathfrak{lift}\right)$ if there is a hereditary C^*-subalgebra $\widetilde{B}\subset\widetilde{A}$ with a $*$-isomorphism $\mathfrak{lift}^{\widetilde{B}}: B\cong\widetilde{B}$ such that

$$\forall b\in B\quad \mathfrak{lift}\left(b\right)=\beta\text{-}\sum_{g\in G}g\,\mathfrak{lift}^{\widetilde{B}}\left(b\right) \tag{19}$$

where $\beta\text{-}\sum$ means the convergence with respect to the strict topology of $M\left(\widetilde{A}\right)$

Definition

Ivankov

A noncommutative pre-covering $\left(A, \widetilde{A}, G, \mathfrak{lift}\right)$ with unital A is a *unital noncommutative covering* if for any $x \in \mathfrak{Gelfand}\left(A\right)$ there is a hereditary connected C^*-subalgebra of B evenly covered by $\left(A, \widetilde{A}, G, \mathfrak{lift}\right)$ with $B \in x$

Definition

Ivankov

A noncommutative finite-fold pre-covering $\left(A, \widetilde{A}, G, \mathfrak{lift}\right)$ *noncommutative covering* if there is if there is an unital noncommutative covering $\left(B, \widetilde{B}, G, \widetilde{\mathfrak{lift}}\right)$ with inclusions $A \subset B$ and $\widetilde{A} \subset \widetilde{B}$ such that:

(a) both A and B are essential ideals of B and $\widetilde{B}$,

(b) $\mathfrak{lift} \stackrel{\text{def}}{=} \left.\widetilde{\mathfrak{lift}}\right|_A$,

(c) the action $G \times \widetilde{B} \to \widetilde{B}$ naturally comes from the $G \times \widetilde{A} \to \widetilde{A}$

Thank you