Sequences of partial sums of multiple Fourier sums

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Convergence to a finite function

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let $L(\mathbb{T})$ be the set of integrable functions $f : \mathbb{T} \to \mathbb{C}$. We associate with any function $f \in L(\mathbb{T})$ its trigonometric Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}, \quad \hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-ikx}dx.$$

For $n \in \mathbb{N}$ we define the n-th partial sum of f as

$$S_n(f;x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}.$$

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We know from a famous theorem of Kolmogoroff (1925) that the trigonometric Fourier sums of any integrable function f converge to f in measure. Therefore, if any subsequence of partial Fourier sums converges on a set of positive measure, it can converge to f only.

Let $d \in \mathbb{N}$. Let mes_d be the standard Lebesgue measure on $\mathbb{T}^d$ and μ_d be the normalized measure $\mu_d = mes_d/(2\pi)^d$.
We consider L^p norms with respect to the measure μ_d.
Moreover, for any $p > 0$ we denote

$$\|f\|_p = \left(\int_{\mathbb{T}^d} |f|^p d\mu_d \right)^{1/p}.$$

For $p \in (0,1)$ this functional is not a norm. Also, we denote $L(\mathbb{T}^d) = L^1(\mathbb{T}^d)$.

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For a function $f \in L(\mathbb{T}^d)$ we define its expansion into the trigonometric Fourier series

$$f \sim \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}}, \quad \mathbf{k} = (k_1, \ldots, k_d), \mathbf{x} = (x_1, \ldots, x_d),$$

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} d\mu_d.$$

We define Pringsheim (or rectangular) partial sums of f as

$$S_{\mathbf{n}}(f;\mathbf{x}) = \sum_{|\mathbf{k}|\leqslant\mathbf{n}} \hat{f}(\mathbf{k})e^{i\mathbf{k}\mathbf{x}},$$

where $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$, $|\mathbf{k}| = (|k_1|, \dots, |k_d|)$, and the inequality $\mathbf{m} \leqslant \mathbf{n}$ means that $m_j \leqslant n_j$ for $j = 1, \dots, d$. The convergence means convergence for $\min(n_1, \dots, n_d) \to \infty$. The cubic sums are Pringsheim sums with $n_1 = \dots = n_d$.

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However, the speaker (2022) proved that if a subsequence of Pringsheim sums of a function $f \in L(\mathbb{T}^d)$ with $\min(n_1, \dots, n_d) \to \infty$ converges to a finite function on a set of positive measure, then the limit must coincide with f almost everywhere on this set. My interest to this problem was inspired by the lecture of M.G. Grigoryan on universal Fourier series at the conference Stechnkin–100. The idea for the proof was caught from the lecture of R.M. Trigub at the same conference.

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In the same paper the speaker proved that for any function $f \in L(\mathbb{T}^d)$ there is a subsequence of Pringsheim sums with $\min(n_1, \dots, n_d) \to \infty$ converging to f in any L^p, $0 < p < 1$. Therefore, we can take a subsequene of this subsequence converging to f almost everywhere.

Convergence to infinity

The situation changes drastically if we allow partial sums to cnverge to infinity. The speaker (2023) proved that for $d > 1$ and for any sequence of Pringsheim sums $S_{\mathbf{n}^{(j)}}$ with $\lim_{j\to\infty} \min(n_1^{(j)}, \dots, n_d^{(j)}) = \infty$ there exist a function $f \in L(\mathbb{T}^d)$ and a sequence $\{j_\nu\}$ such that

$$\lim_{\nu\to\infty} |S_{\mathbf{n}^{(j_\nu)}}(f)| = \infty$$

almost everywhere. We will consider that $d \geqslant 2$.

For any bounded set $A \subset \mathbb{R}^d$ and a function $f \in L(\mathbb{T}^d)$ we define a partial sum of f related to A as

$$S_A(f)(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d \cap A} \hat{f}(\mathbf{k}) e^{i\mathbf{kx}}.$$

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In this talk I will discuss the following question. We have a sequence $\{A_j\}$ of bounded subsets of $\mathbb{R}^d$. Do there exist a function $f \in L(\mathbb{T}^d)$ and a sequence $\{j_\nu\}$ such that

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almost everywhere?
If this holds, then we say that the sequence $\{A_j\}$ satisfies the condition $(*)$.

The speaker (2023) found a large class of sequences $\{A_j\}$ satisfying (*). In particular, (*) holds for partial sums over balls, i.e.

$$A_j = \{\mathbf{x} : \sum_{\mu=1}^{d} x_\mu^2 \leqslant j\}$$

and for hyperbolic sums

$$A_j = \{\mathbf{x} : \prod_{\mu=1}^{d} \max(1, |x_\mu|) \leqslant j\}.$$

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We say that a sequence $\{A_j\}_{j\geqslant 1}$ of subsets of $\mathbb{R}^d$ is absorbing if for any $\mathbf{k} \in \mathbb{Z}^d$ and any $j \geqslant j(\mathbf{k})$ we have $\mathbf{k} \in A_j$.

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In the most recent paper (2025) we have proved that any absorbing sequence of bounded convex subsets of $\mathbb{R}^d$ satisfies (*).

Necessary and sufficient conditions for $d = 2$

Theorem 1

Let $p \in (0,1)$ and $\{A_j\}$ be a sequence of bounded convex subsets of $\mathbb{R}^2$. Then the following conditions are equivalent:
1) there is $r \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ the set $\mathbb{Z}^2 \cap A_j$ is covered by r parallel lines;
2) there is $r \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ the set $\mathbb{Z}^2 \cap A_j$ is covered by r lines;
3) there is $c > 0$ such that for any j there is a line l_j such that $|\mathbb{Z}^2 \cap l_j \cap A_j| \geqslant c|\mathbb{Z}^2 \cap A_j|$;
4) there is a constant $C > 0$ such that for any $f \in L(\mathbb{T}^d)$ the inequality $\|S_{A_j}(f)\|_p \leqslant C\|f\|_1$ holds;
5) the sequence $\{A_j\}$ does not satisfy (*).

Theorem 1 is proven in our paper (2025) where the case $d > 2$ is also discussed.

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Some implications take place for any $d > 1$. The implications 1) $\to$ 2) $\to$ 3) and 4) $\to$ 5) are obvious. Also, it is not difficult to prove the implication 2) $\to$ 4).
Also, we prove that for $d > 2$ the implications 3) $\to$ 2) $\to$ 1) and 3) $\to$ 5) do not hold and conjecture that 5) $\to$ 2) holds for any $d \geqslant 2$. Thus, this conjecture means that the condition (*) for convex sets is equivalent to 2).

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Some implications take place for any $d > 1$. The implications 1) $\to$ 2) $\to$ 3) and 4) $\to$ 5) are obvious. Also, it is not difficult to prove the implication 2) $\to$ 4).
Also, we prove that for $d > 2$ the implications 3) $\to$ 2) $\to$ 1) and 3) $\to$ 5) do not hold and conjecture that 5) $\to$ 2) holds for any $d \geqslant 2$. Thus, this conjecture means that the condition (*) for convex sets is equivalent to 2).

THANK YOU!