On a mechanism of diffusion in Hamiltonian systems

Valery V. Kozlov
Steklov Mathematical Institute of Russian Academy of Sciences

The main problem of dynamics according to Poincaré

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}; \quad 1 \leqslant k \leqslant n$$

$$H = H_0(y) + \varepsilon H_1(x,y) + \varepsilon^2 H_2(x,y) + \dots$$

$x = (x_1, \dots, x_n) \mod 2\pi$ are the angular variables
$y = (y_1, \dots, y_n)$ are the momenta, ε is a small parameter
$\Gamma = D \times \mathbb{T}^n$ is the phase space, $D \subset \mathbb{R}^n = \{y\}$.
For $\varepsilon = 0$ we have a completely integrable system: $y = y^0$, $x = \omega t + x^0$; $\omega = (\omega_1, \dots, \omega_n)$,
$\omega_k = \frac{\partial H}{\partial y_k}\Big|_{y^0}$, $\{y = y^0, x \mod 2\pi\}$ are invariant tori

Non-degenerate system:

$$\left|\frac{\partial^2 H_0}{\partial y^2}\right| \neq 0$$

Isoenergetically non-degenerate system:

$$\begin{vmatrix} & & & \omega_1 \\ & \dfrac{\partial^2 H_0}{\partial y_i \partial y_j} & & \vdots \\ & & & \omega_n \\ \omega_1 & \dots & \omega_n & 0 \end{vmatrix} \neq 0$$

Theorem of A.N. Kolmogorov, KAM theory

An example by V.I. Arnold (1964):

$$H = \frac{1}{2}(y_1^2 + y_2^2) + y_3 + \varepsilon(\cos x_1 - 1)(1 + \mu \sin x_2 + \mu \cos x_3).$$

Let $0 < A < B$. Then for any $\varepsilon > 0$ there exists a $\mu_0(A, B, \varepsilon) > 0$ such that for any $0 < \mu < \mu_0$ the system has a solution such that $y_2(0) < A$, $y_2(t) > B$ for some t.

$$t \sim \mu^{-1} e^{c/\sqrt{\varepsilon}}, \quad c = const, \quad t > 0.$$

'Arnold diffusion' (B.V. Chirikov)

Conjecture. Typically for $n \geqslant 3$, the system is *topologically unstable*: in any arbitrarily small neighborhood of any point of the phase space there is a phase trajectory along which the slow variables change by an amount of order 1.

'The details of the proof must be monstrous, although ... the idea is outlined very clearly.' J. Moser, Math. Reviews.

Hamilton equations

Again, $\Gamma = D \times \mathbb{T}^3$ and $H = y_1 y_3 + y_2 + \varepsilon H_1(x_1, x_2)$, $H_1 \colon \mathbb{T}^2 \to \mathbb{R}$ is an analytic function.

The system with the Hamiltonian $H_0 = y_1 y_3 + y_2$ is degenerate, yet isoenergetically non-degenerate.

Explicit form of the Hamilton equations:

$$\dot{x}_1 = y_3, \quad \dot{x}_2 = 1, \quad \dot{x}_3 = y_1; \quad \dot{y}_1 = \varepsilon f, \dot{y}_2 = \varepsilon g, \quad \dot{y}_3 = 0$$

$$f = -\frac{\partial H_1}{\partial x_1}, \quad g = -\frac{\partial H_1}{\partial x_2}$$

The momentum $F = y_3$ is a first integral; there is another first integral, namely

$$\Phi = y_3 x_2 - x_1.$$

However, this first integral is a multivalued function on Γ; at the same time, $\{\Gamma, \Phi\} = 0$.

The Hamilton equations are integrable *by quadratures.*

The proof of ‘Kolmogorov’s theorem’.

Let us fix the constant $y_3 = c$. We obtain the *reduced* Hamiltonian system with the following Hamiltonian

$$H_c = cy_1 + y_2 + \varepsilon H_1(x_1, x_2).$$

Let us expand the function f into a Fourier series:

$$f = \sum{}' f_{m_1 m_2} e^{i(m_1 x_1 + m_2 x_2)}.$$

Let us formally integrate $y_1 = \varepsilon f$. We obtain $y_1 = \alpha_1 + \varepsilon F(x_1, x_2)$, $\alpha_1 = const$,

$$F = \sum{}' \tfrac{f_{m_1 m_2}}{i(m_1 c + m_2)} e^{i(m_1 x_1 + m_2 x_2)}.$$

Then, for almost all $c \in \mathbb{R}$ we have $|m_1 c + m_2| \geqslant \frac{C}{|m_1|^\nu}$ for some ν and C and for all integer $m_1 \neq 0$.

Therefore, F is an analytic function on $\mathbb{T}^2$.

Similarly, $y_2 = \alpha_2 + \varepsilon G(x_1, x_2)$, $\alpha_2 = const$.

In order to obtain *three-dimensional* Kolmogorov tori, we have to solve the equation $\dot{x}_3 = y_1 = \alpha_1 + F(x_1, x_2)$. The equation $\dot{x}' = F(x_1, x_2)$ can be solved similarly to the above ones. Therefore, we obtain the third angular variable $x_3 - x'$ on the three-dimensional invariant torus, which changes with the constant frequency α_1.

Diffusion

$$H_1 = \sum{}' h_{m_1 m_2} e^{i(m_1 x_1 + m_2 x_2)}; \quad f_{m_1 m_2} = i m_1 h_{m_1 m_2}.$$

Theorem 1. Let $h_{m_1 m_2} \neq 0$ for all $|m_1| + |m_2| \neq 0$. If $\varepsilon \neq 0$, then in arbitrarily small neighborhood of any point of the phase space $\Gamma = \mathbb{R}^3 \times \mathbb{T}^3$ there is a continuum of phase trajectories such that all these trajectories are

1. unbounded
2. Poisson stable

The outline of the proof.

$$y_1(t) = \varepsilon \int_0^t f(cs + x_1^0, s + x_2^0) ds, \quad c = y_3.$$

Then $y_1(t) = \alpha_1 + \varepsilon F(t)$, $F = \sum{}' \frac{f_{m_1 m_2}}{i(m_1 c + m_2)} e^{i(m_1 c + m_2)t} e^{i(m_1 x_1^0 + m_2 x_2^0)}$.

If $F(t)$ is bounded then $\sum{}' \left|\frac{f_{m_1 m_2}}{m_1 c + m_2}\right|^2 < \infty$.

Let us consider the set $M \subset \mathbb{R} = \{c\}$ for which

$$|m_1 c + m_2| < |f_{m_1 m_2}|$$

is satisfied for infinitely many integer pairs m_1, m_2.

For such c, the function $y_1(t)$ is unbounded.

M is everywhere dense in $\mathbb{R}$ and has the cardinality of the continuum. The second property follows from the property of recurrence of integrals of quasiperiodic functions with zero mean value (V.V.K. Applied Math. and Mech. 40:2 (1976), 352-355).

What is the speed of diffusion?

Theorem 2. There exists $\gamma = const > 0$ (which depends on the phase trajectory) and monotonously increasing $t_m \to \infty$ such that

$$\max_{k=1,2} \max_{1 \leqslant t \leqslant t_m} \max_{x(0)} |y_k(t) - y_k(0)| < \varepsilon\gamma \ln t_m$$

for all m. The difference $y(t) - y(0)$ for fixed t depends only on the initial values $x(0)$.

This theorem was proved by V.V.K. and N.G. Moshchevitin (Moscow University Bulletin, Math. and Mech. 1997, №5, 49–52).

Let us put $\varepsilon\gamma \ln t \sim 1$. Then $t \sim e^{c/\varepsilon}$, $c = \gamma^{-1}$.

Remark. For *some* analytic function H_1 there exists a set $M \subset \mathbb{R} = \{c\}$ which is everywhere dense in $\mathbb{R}$ and has the cardinality of the continuum such that for $c \in M$ the reduced Hamiltonian system with the Hamiltonian H_c is *transitive* on every three-dimensional isoenergetic surface $\{H_c = const\}$.

Non-Kolmogorov invariant tori

Theorem 3. For some analytic function $H_1 \colon \mathbb{T}^2 \to \mathbb{R}$ there exist sets M_ω, M_∞, ..., M_k, ... M_0, $M_\varnothing \subset \mathbb{R} = \{c\}$. All these sets are everywhere dense in $\mathbb{R}$ and have the cardinality of the continuum. If $\varepsilon \neq 0$ and

- $c \in M_\omega$ then the manifolds $\{H_c = const\}$ are foliated into analytic invariant tori,
- $c \in M_\infty$ then the manifolds $\{H_c = const\}$ are foliated into C^∞ yet non-analytic invariant tori,
- $c \in M_k$ then the manifolds $\{H_c = const\}$ are foliated into C^k yet not C^{k+1} invariant tori,
- $c \in M_0$ then the manifolds $\{H_c = const\}$ are foliated into continuous yet non-differentiable invariant tori,
- $c \in M_\varnothing$ then the reduced system is topologically transitive on the isoenergetic manifolds $\{H_c = const\}$.

The invariant tori of the reduced system are defined by the equalities

$$y_1 = \alpha_1 + \varepsilon F(x_1, x_2), \quad y_2 = \alpha_2 + \varepsilon G(x_1, x_2).$$

There are two subsets M_i and M_m in the set $M_\varnothing$, which are everywhere dense in $M_\varnothing$ (therefore, they are everywhere dense in $\mathbb{R} = \{c\}$).

If $c \in M_i$ then F and G are Lebesgue integrable, yet discontinuous; if $c \in M_m$ then F and G are measurable yet not integrable. The case $c \in M_i$ leads to the invariant *Aubry-Mather sets*.

The possibility of the second case follows from the result by D.V. Anosov on the solutions of an analytic homological equation (Izvestiya: Mathematics, 1973, Volume 37, Issue 6, Pages 1259–1274).

Nonintegrability

Theorem 4. Let $h_{m_1 m_2} \neq 0$ for all $|m_1| + |m_2| \neq 0$. Then the Hamilton equations do not have an additional single-valued integral $\Phi(x, y, \varepsilon)$, which is analytic in canonical variables and in the small parameter ε, which is independent of the integrals $H = H_0 + \varepsilon H_1$ and $F = y_3$, and which is in involution with F.

This statement is proved using the Poincaré method.

Theorem 5. There exists an analytic function $H_1 \colon \mathbb{T}^2 \to \mathbb{R}$ such that for any given $\varepsilon \neq 0$ the Hamiltonian system does not have an additional continuous integral.

One should use here the function H_1 from Theorem 3.

We say that a *continuous* function $\Phi \colon \Gamma \to \mathbb{R}$ is an additional integral if this function is

1. constant along any phase trajectory,
2. independent of the angular variable x_3,
3. a non-constant function after being restricted on the three-dimensional manifold $\{H_c = const\}$.

Thank you for your attention