Concept of space adapted to modern physics

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The Development of modern Geometry that influences our concept of space

§1. Introduction: the development of modern geometry that influences our concept of space Bernhard Riemann (1826-1866) and his teacher C.F. Gauss (1777-1855) are no doubt the two great geometers who founded modern geometry.

Riemann

Gauss

The beautiful theory of Riemannian geometry has in effect changed our views of the concept of space which was introduced by the ancient Greek geometers.

It is fair to say that without this development, it would had taken many more years for Einstein (1879-1955) with helps from Grossman (1878-1936) and Hilbert to accomplish the great theory of general relativity.

Riemann initiated the concept of modern geometry through the following paper in 1854:

Über die Hypothesen, welche der Geometrie zu Grunde liegen [The Hypotheses on which Geometry is Based].

This paper is truly a spectacular work: Riemann had few works prior to inspire him or to provide guidance, with the exception of some bits of work of Gauss and some philosophical work of Herbart.

He felt that *"the theorems of geometry cannot be deduced from the general notion of magnitude alone, but only from those properties which distinguished space from other conceivable entities, and these properties can only be found experimentally."*

"We can only investigate their probability, and therefore a judgment as to the admissibility of extending them outside the limits of observation, in the realms of both the immeasurably great and the immeasurably small.

Either the physical reality on which space is founded must be a discrete variety, or else the foundation of its metric relation must be sought from outside source in the forces which bind together its elements . . . .

This takes us into the realm of another science – physics."

Geometry of singular spaces

I would like to reflect upon how we may think about geometry as a whole and what we can do in the future. The subject is connected with Geometry, Analysis and Mathematical Physics, and this is exactly what Riemann had in mind about 160 years ago when he thought about geometry.

He was very much concerned about the role of space in **physics**. And we saw in above, he questioned what kind of concepts of space can be drawn from physics. One may note that his discussion of the heat conduction motivated him to give the definition of the curvature tensor.

Hence I think any sensible motivation on the fundamental concept of space should be linked to the intuitions from physics of nature. We are facing a great challenge in this century on how to work out a concept of geometry that is capable to understand **general relativity in the large** and **quantum physics in the small**.

There have been proposals on such geometry. The most outstanding one being the noncommutative geometry of Alain Connes.

I am not expert of his work, as my taste in geometry is largely traditional geometry over complex or real number field, as is motivated from intuitions coming from curvature, topology, physics and analysis, especially from the point of view of differential equations. However, what I said here may be considered as my first primitive step towards understanding of a suitable version of quantum geometry. My goal is to understand geometry through operator theory where classical spacetime may disappear altogether. At this stage of development, I shall not consider geometry over noncommutative fields, as it is a bit too complicated.

It is generally accepted that quantum mechanics should take effect in the very small scale of space while general relativity should take effect in the large scale. Hence space may be discrete in the small and one can then study quantum effects. The serious question is **how to join this small quantum space to large scale spacetime**.

In any case, we may need to study the geometry of discrete space time and develop properties that may exhibit similarity with continuous spacetime. But we need to review what we know about continuous spacetime first. Much of the continuous geometry has been developed since the time of Riemann and we like to preserve their key properties.

§2. Symmetry and singular Spaces
In the hands of **S. Lie, Klein, E. Cartan and Hermann Weyl**, we see the very powerful tools based on symmetry which strongly influenced the development of geometry for more than a century.

Consideration of symmetry reveals deep facts of nature through general relativity and gauge theory which dictates forces of elementary particles.

In the attempt to unify gravity with electromagnetism, many interested theories were developed.Given a metric, one can construct a concept of parallel transportation which preserves the metric. This is called metric connection. When the connection has no torsion, it is called Levi-Civita connection and it is unique.

Levi-Civita suggested to allow the connection to have torsion and the correction is no more unique and it becomes an independent object.

In the hands of E. Cartan, Weyl and Einstein, the theory of connections was generalized to fiber bundles and became one of the most powerful tools in geometry and physics. The theory of characteristic classes due to Stiefel, Whitney, Pontryagin and Chern are important for quantizations of geometry and gauge theory. In physics, Weyl's applied the Abelian gauge theory to electromagnetism. The application of nonabelian gauge theory to particle physics was due to Yang-Mills and it dominates the modern theory of interactions of elementary particles.

The principle of using symmetry to dictate geometry is usually referred to as Klein's Erlangen program. Such a principle eventually led Minkowski to discover Minkowski spacetime and also Einstein to find general relativity, where the equivalence principle requires the field equation for gravity to be independent of the choice of coordinates (i.e. the symmetry group is the group of diffeomorphisms). In the past thirty years, the concept of symmetry has been enlarged to include supersymmetry by physicist. I believe that uplift the concept of space to more general level may give us more freedom to include more symmetrics, which should play important role, even for classical geometry.

Geometry for Einstein equations and special holonomy group

§3. Geometry for Einstein equations and special holonomy group

In order to demonstrate on what I propose. I shall focus on the theory of Einstein manifolds. The construction of the Riemannian version of a vacuum Einstein equation with a possible cosmology constant is still the most challenging problem in geometry and analysis. It is a problem in analysis as it provides a nice **elliptic system** in a suitable gauge. This system is nonlinear and a good definition of weak solution of Einstein equation is needed. We shall find such a definition.

Only when the manifold has either a large group of symmetries or with special internal symmetry (or special holonomy group) do we know how to construct such Einstein manifolds.

In general, Levi-Civita connections give rise to a group which is defined by parallel transportation of a reference frame along all loops at one point. This group is called holonomy group and provides internal symmetries for the manifold.

Many complete and compact manifolds with special holonomy groups have now been constructed, thanks to the works of many geometers. It is remarkable that they are Einstein manifolds, i.e., their curvature tensor satisfies the equation of Einstein in the Riemannian setting.

Among manifolds with special holonomy group, we have a reasonable understanding of **Kähler manifolds**, **Calabi-Yau manifolds** and **HyperKähler manifolds**. However, we do not have good control of manifolds with holonomic group equal to G_2, $Spin(7)$, and $Sp(1)Sp(n)$. They are all Einstein manifolds with Ricci curvature equal to zero.

We need a theorem similar to the Calabi-Yau theorem which reduce problems in manifolds to algebraic geometric problems which can be solved by algebraic means.

Besides the importance of minifolds with special holonomy group in geometry and algebraic geometry, these manifolds play important roles in string theory and M-theory. It would therefore be of great importance to gain a deep understanding of these Einstein manifolds.

In classical general relativity, we are more interested in metrics with Lorentzian signature. In some stationary spacetimes such as the one described by the Kerr metric, there is a procedure called Wick rotation that can "analytically" continue the Einstein metric with Lorentzian signature to one with Riemannian signature. In an amazing manner, the singularities of spacetime disappear after Wick rotation. These manifolds play important role in Gibbons-Hawking theory of quantum gravity.

Although the Wick rotation construction is done in an ad hoc manner, but it is worth white to point out that the Wick rotated Kerr metric admits a nontrivial second order differential operator that commutes with Laplacian. I propose this to be a concept that generalizes manifold with special holonomy group.
It is still not known how to classify all Lorentzian manifolds with special holonomy. They may be important for general relativity. I shall not discuss manifolds with Lorentzian signature here.

When the holonomic group is proper subgroup of the orthogonal group, these are special subspaces of the tensor product of tangent bundle, and the associated projection operators commute with the Laplacian that acts on functions and forms.

In particular, the eigenforms of the Laplacian have natural splitting coming from the projections operators. The theory of Hodge made use of this powerful and natural splitting on harmonic forms, which account for topology of the manifold. It builds a bridge between topology and analysis.

I shall discuss how to generalize the concept of Riemannian geometry by using the Laplace operator. In this new setting, the holonomy group will be replaced by the graded ring of local operators that commute with the Laplacian.
Although it will be more natural to start with Dirac operator, but the analysis is more different and will be pursued later.

Laplacian and construction of generalized Riemannian geometry in terms of operators

§4. Laplacian and construction of generalized Riemannian geometry in terms of operators

Most of the known Einstein manifolds are obtained by reduction of variables by group actions or by constructing manifolds with special holonomic group, or by combining such constructions.

In such a process, we may have to handle spaces which have singularities. The most common singularities that we can handle are orbifold singularities. But their structures are not rich enough to describe problem in modern physics.

We need to enlarge the category of manifolds to allow manifolds with general singularities. But at the same time, we would like to keep the natural geometric operators to be well-defined on such singular spaces.

We propose to formulate a theory that replaces Riemannian manifolds by operators acting on a Hilbert space.

(i) On a compact manifold M, the Riemannian metric gives rise to a measure. If we normalize the total volume to be one, all these measures are equivalent to each other by a volume preserving diffeomorphism. Hence we have a Hilbert space $H = L^2(M)$ and an algebra A of unitary operators defined by the group of measures preserving diffeomorphisms.(if we want to avoid the use of the measure, we can replace functions by half densities.)

(ii) Within the Hilbert space H, we have a subalgebra C of smooth functions which determines the differential structure of M. The Laplacian L is a self-adjoint operator defined on C which is local in the sense that for any $\varphi_1, \varphi_2 \in C$, $\varphi_1 \varphi_2 = 0 \Rightarrow \varphi_1 L(\varphi_2) = 0$.

(iii) The inner product $\langle \varphi, (-L)^s \varphi \rangle$ is positive on $\{\varphi \in C : \langle \varphi, 1 \rangle = 0\}$. Its completions are Hilbert spaces that will be called H_s, and L is an isomorphism from H_{s+2} to H_s.

The space of Riemannian metrics can be considered as the orbit space of the space of the triples (H, C, L) mod A, the group of unitary operators defined by the group of measure preserving diffeomorphisms. Note that the algebra A is a subalgebra of the endomorphism ring of C. We would like to make sure this orbit space is Hausdorff and the concept of stable manifold in the sense of geometric invariant theory may be needed. In principle, we can therefore replace a Riemannian manifold by (H, C, L) which satisfies the above properties.

In order for the triple to recover standard properties of Riemannian geometry, we shall make several assumptions.

(1) Compatibility of multiplication with inner product:
there exists constants $a, b > 0$ such that

$$a \parallel f \parallel^2 \leq \langle f^2, 1\rangle \leq b \parallel f \parallel^2, \quad \langle fg, h\rangle = \langle f, gh\rangle.$$

(2) The Cone of positive functions:
The Cone defined by taking $H-$ closure of $\left\{\sum_{i=1}^k \rho_i^2 : \rho_i \in H\right\}$ will be called H^+.

Then for any element $\rho \in H$, there is a unique element $\rho^+ \in H^+$ and ρ^- so that

$$\rho = \rho^+ + \rho^-$$

and $\langle \rho^-, g\rangle \leq 0$ for all $g \in H^+$.
If we define $\overline{H^+} = \{h \in H : \langle h, g\rangle \geq 0,$ for all $g \in H^+\}$,
Then $H^+ \subseteq \overline{H^+}$

It is easy to prove that

$$\| \rho^+ \|^2 \leq 2 \| \rho \|^2$$

and

$$\| \rho^- \|^2 \leq \| \rho \|^2 .$$

If $\overline{H^+} = H^+$ then $-\rho^- \in H^+$
In any case, we shall assume that $L(\rho^2) - 2\rho L(\rho) \in \overline{H^+}$. We need this for proving that $exp(tL)$ preserves $\overline{H^+}$. In order to define an inner product on the space of differentials, we assume further that for any set of elements $\{f_i, g_i\}$ in C, we have

$$\sum g_i g_j [L(f_i f_j) - 2 f_j L(f_i)] \in \overline{H^+}$$

(3) The embedding from H_s to H_{s-1} are compact operators for all s. One can then show that the spectrum of L is discrete and that it tends to infinity when H is infinite-dimensional.

(4) If λ_k are the eigenvalues of $-L$, we assume that $\lambda_k \geq 0$, $\lim_{k\to\infty} k^{-\frac{2}{n}}\lambda_k$ exists and depends only on $Vol(M)$. This is Weyl's law in Riemannian geometry.

Tauberian theorems say that the Weyl law is equivalent to the statement

$$\lim_{t \to 0} t^{\frac{n}{2}} tr\, e^{tL}$$

exists, and is equal to a number a_0 depending only on $Vol(M)$. We shall also assume the existence of

$$a_1 = \lim_{t \to 0} t^{-1} \left(t^{\frac{n}{2}} tr\, e^{tL} - a_0 \right)$$

Note that in Riemannian geometry, a_1 is the total scalar curvature of the manifold M.The number n seen above will be defined to be the dimension of the manifold. Hence we shall consider a_1 as an action defined on the space of (H, C, L) mod A. In this way, our generalized manifolds is Einstein if they are critical points of this functional.

An important example is a metric with g_{ij} non-smooth but positive definite. We assume that g_{ij} is bounded above and below by smooth Riemannian metrics. With low regularity on g_{ij} one can make sense of a_1 and hence weak solution of Einstein equation.

Differential topology of the operator geometry

§5. Differential Topology of the operator geometry
With the algebra C, we can define the tangent bundle to be the space of derivation of C. They are also bounded local operators defined on H_1. The tangent bundle is a module over C. The wedge product of the tangent vectors can be formed in the usual way.

The dual spaces are differential forms. Given a function f in C, we can define a differential form by $df(X) = X(f)$.
If $f_1, f_2, \cdots, f_m$ are linear independent over C, then we expect that polynomials of them will produce enough functions to prove that the k-th eigenvalue of L is not greater than $Ck^{\frac{2}{m}}$ if certain scaling properties hold for L. In such cases, $m \leq n$.

It is quite likely that for an n-dimensional manifold, the differential of any n distinct eigenfunctions with distinct non-zero eigenvalues are independent over C.
It will be interesting to find conditions such that the analogue statement for the case of an operator manifold. That will mean that such a manifold has dimension n iff the maximal m such that $df_1 \wedge \cdots \wedge df_m \neq 0$ is equal to n.

On a manifold, the set of functions vanish at one point form a maximal ideal m in C.
The space of maximal ideals can be identified with the manifold.
Each element f in C can be considered as a function defines on the space of maximal ideals by simply assign the value $f(m)$ to m where $f(m)$ is the unique scalar multiple of the constant function 1 such that $f - f(m)$ belongs to m. There is a topology and a masure defined on the space of maximal ideals by requiring all the functions come from C to be continuous. The inner product of H defines integration on this space. For any openset Θ containing m, we assume that $\cap m' \subset m^i$ for $m' \subset \Theta$ for any $i > 0$.

The space m/m^2 can be considered as the cotangent space at the point represented by m.
The space m^2 is in general not closed in H_s topology unless s is large. We shall assume such an s exists such that m^2 is a proper closed subideal of m. The inner product of Hs give rise to an inner product on m and hence on m/m^2, as quotient space of m by a closed subspace m^2. This inner product will depend on s. However, we can compare this inner product with the L^2 inner product on H, and it will be given by a self adjoint operator Q We can take s th root of Q and take limit as $s \to \infty$. Hence we obtain an inner product independent of s.

Note that the elements in the tangent bundle, namely the derivations of C, defines linear maps on m/m^2. In fact, if f_i g_i, h_i are in m, then for any derivation X, $X(f + \sum_i g_i h_i)$ mod m gives a real number $X(f)$ independent of choice of g_i or h_i. This is because $\sum_i g_i X(h_i) + \sum_i h_i X(g_i)$ is an element of m. Hence $X(f)$ is well-defined and linear. Conversely, if for each m, we have a linear functional l_m on m/m^2, we can define a derivation x by mapping each $f \in C$ to $X(f) = l_m(f - f(m))$ where $f(m)$ is the unique scalar multiple of the constant function I so that $f - f(m) \in m$.

Since $(f - f(m))(g - g(m)) \in m^2$, $X\big[(f - f((m))\big(g - g(m)\big] = 0$ and one proves easily that $X(fg) = f(m)X(g) + g(m)X(f)$.
We claim that the derivation X defined in this way is a local operator, i.e., if $f, g \in C$ and $fg = 0$, then $gX(f) = 0$.
This can be seen as follows:

Suppose $g \notin m$, then by continuity, $g \notin m'$ for all maximal ideal in a neighborhood of m.
Since $fg \in m'$, $f \in m'$ for all m' near m. This implies that $f \in m^2$.
Hence $l_m(f) = 0$ and $gX(f) = 0$.
In order for X to be a bounded operator, we need to assume

$$\mid l_m\big(f - f(m)\big) \mid \leq C \parallel f \parallel_{H_1}$$

Where C is independent of f.

Note that for $f \in C$, we can project f to m/m^2 by taking away the constant term $f(m)$. This element defines df as a bounded linear functional on the space of derivation.
For any f_i and g_i in C, $\sum_i g_i df_i$ defines a map from m to m/m^2. It maps m to $\sum g(m)\big(f_i - f_i(m)\big)$. Hence it has a norm $\| \sum g_i \cdot df_i \|_m^2$ and we can integrate it over the space of m. In this way, we have a positive inner product in the space of differentials.

Since we assume that for any $f \in C$, $Xf \in C$, we can define the Lie bracket $[X, Y]$ in the space of derivations. The exterior algebra of cotangent bundle, which is the dual of the space of derivations, admits exterior differentiations in the standard manner. They are differential forms. Its cohomology can be considered as de Rham cohomology of the manifold.

The space of linear functional defined on the exterior algebra of cotangent bundle will be defined as currents in our geometry. It has a boundary operator dual to the exterior differentiation.We can then define the homology of the manifold.
We can also define the de Rham forms by the C-modules of $df_1 \wedge \cdots \wedge df_m$ and there is a natural exterior differentiation. The invariants associated to this complex should be interesting for singular manifolds. Since it is not clear that the dual of vector fields are spanned by the differential of functions, the cohomology defined by the exterior algebra of differential forms may be different from those defined by differentials of functions.

Inner product on tangent space

§6. Inner product on tangent space
The space spanned by differential of functions can be written as $\sum_i f_i dg_i$. Its inner product $\sum_{i,j}\langle f_i dg_i, f_j dg_j\rangle$ can be defined to be

$$\frac{1}{2}\sum_{i,j}\langle f_i f_j, L(g_i g_j)\rangle - \langle \sum f_i f_j, \sum g_i L(g_j)\rangle.$$

In section 5, we assume this defines a non-degenerate inner product on the space of differentials. This inner product is different from the one defined in the previous section and is preferable. One can then define an inner product on the tangent bundle by

$$\| X \|^2 = \sup_f |\langle X, W\rangle|,$$

where w is any differential of norm one.

Since we can define the norm of the differential of a function, We can define the distance between any two maximal ideals m_1 and m_2 by

$$d(m_1, m_2) = sup\{|f(m_1) - f(m_2)| :\| df(m) \| \leq 1 \text{ for all } m\}$$

The inner product on the tangent bundle and cotangent bundle gives natural inner product on its exterior product. The exterior differentiation has an adjoint operator. In a standard manner, we can define the Hodge Laplacian L_i acting on the space of i forms. Hence starting from the operator L, we have constructed operators L_i acting on spaces that come from the tangent bundle. Classically, the kernel of these operators gives information of the topology of the manifold.

In order to prove that the kernel and the cokernel of these operators are finite dimensional, we need some type of Bochner argument to establish an operator R_i so that

$$L\langle \omega, \omega \rangle - 2\langle \omega, L_i \omega \rangle + \langle R_i \omega, \omega \rangle$$

defines an inner product on the space of i-forms ω such that the completion of the space with respect to this inner product gives a compact embedding into the L_2 space of ω. When ω is one-form, R_i is the Ricci curvature. The kernel of this inner product is given by parallel forms, whose dimension is not greater that the space of constant coefficient i-forms in $\mathbb{R}^n$. They are harmonic forms also.

Note that for the above geometrically defined elliptic operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$, $\mathrm{tr}[\exp(t\mathcal{L}) - \exp(t\mathcal{L}^*)]$ is constant in t and gives rise to the index of $\mathcal{L}$ when $t \to \infty$. In classical geometry, they can be expressed as integrals of local differential forms defined by the curvature when $t \to 0$. This is the local index formula of Atiyah-Bott-Patodi. It is therefore important to find good conditions for the existence of

$$\lim_{t\to 0} tr[exp(t\mathcal{L}) - exp(t\mathcal{L}^*)].$$

Perhaps they can be expressed in terms of the above operators R_i.

Gauge groups, convergence of operator manifolds and Yang-Mills theory

§7. Gauge groups, convergence of operator manifolds and Yang-Mills theory

Given a vector field X, there is a function $\mathrm{div}\ X$ defined by

$$\langle X(f), 1\rangle = -\tfrac{1}{2}\langle \mathrm{div}\ X, f\rangle,$$

for all $f \in H$. Then

$$\Big\langle \big(X + \tfrac{1}{4}\mathrm{div}\ X\big)\, f, g \Big\rangle = -\Big\langle f, (X + \tfrac{1}{4}\mathrm{div}\ X)g \Big\rangle.$$

Therefore

$$X + \tfrac{1}{4}\mathrm{div}\ X$$

is a skew adjoint operator and $\exp(X + \frac{1}{4}\mathrm{div}\ X)$ defines a unitary operator on H. It generates a gauge group acting on (H, C, L). We can replace the group A by this gauge group.

Finite dimensional vector bundles are projective modules over C with finite rank. A metric on the vector bundle V is simply a positive definite symmetric pairing $\langle , \rangle$ on the projective module which is linear over C. A Connection is a map ∇ from the tensor product of the tangent bundle with the projective module to the projective module itself. It is linear over both variables, but linear over C for the first variable.

(i) For $\rho \in C$ and $W \in V$,

$$\nabla_x(\rho W) = X(\rho)W + \rho\nabla_x w$$

(ii) $\nabla_x\langle\ w_1, w_2\rangle = \langle\nabla_x w_1, w_2\rangle + \langle w_1, \nabla_x w_2\rangle$

For each vector field X, ∇_x defines an operator from the vector bundle into itself. It has an adjoint ∇_x^*. Hence we can define an operator

$$\Delta = \sum_{ei} \nabla_{ei}^* \nabla_{ei}$$

Where $\{e_\imath\}$ form an orthonormal basic for the space of vector fields. The operator Δ is independent of the choice of the basis, but depends on the connection.

We shall assume that $t^{\frac{n}{2}} \exp(t\Delta)$ has an expansion $a_0 + a_1 t + a_2 t^2 + \circ(t^2)$ when t is small.

The number a_2 can be considered as Yang-Mills action on the space of connection.

Hence we can define Yang-Mills connections for vector bundles.

In order to define distance between (H, C, L_1) and (H, C, L_2), we replace the triple by (H, C, e^{tL_1}) and (H, C, e^{tL_2}) respectively and we define their distance by

$$\int t^{\frac{n}{2}} tr\,(e^{tL_1} - e^{tL_2})^2 dt.$$

The distance between (H, C, e^{tL_i}) mod A is obtained by taking the distance between the orbits of A acting on e^{tL_i} . The limiting element can be considered as a singular Riemannian manifold. The advantage of the definition of such a singular manifold is that we have naturally defined geometric operators associated to them.

If we change the algebra C, we need a distance between the algebra C_1 and C_2. We shall do the following.

Take any two elements φ_1, φ_2 in C_1 with H_1 -norm equal to one, project it into two elements $\overline{\varphi_1}$ and $\overline{\varphi_2}$ in C_2. The projection uses the inner product of H. Then the algebra norm of the projection P in Hom (C_1, C_2) can be defined to be
$sup\{\| \varphi_1\varphi_2 - \overline{\varphi_1\varphi_2} \|_H{:}H_1$ norm of φ_1 and $\varphi_2 = 1\}$

In the other direction, we can define the algebra norm of the projection from C_2 to C_1. Adding these two norms together gives rise to a distance betweenC_1 and C_2.

In the above discussion, I did not discuss the Dirac operator as its existence requires vanishing of the second Stiefel Whitney class, which is not defined over real number. An easy way to go around this is to start out from the Dirac operator instead of the Laplacian acting on functions. We shall come back to this topic later.

Generalized manifolds with special holonomy group

§8. Generalized manifolds with special holonomy group
Special holonomy group gives rise to projection operators acting on the tangent bundle or subspaces of tensor product of copies of tangent bundle and cotangent bundles. These operators are local and commute with the Laplacian.

From this point of view, it is therefore natural to generalize the concept of manifolds with special holonomy group to these manifolds whose Laplacian has nontrivial local commuting or anti-commuting local operators.

In this regard, it is natural to ask the following queation: If $\{\varphi_i\}$ is the orthonomal basis of eigenfunctions of L, then for a sequence of positive numbers $\{a_i\}$ such that $a_i \sim i^{\frac{m}{n}}$, when will the operator $\sum a_i\varphi_i \otimes \varphi_i$ defines a local operator?
The order of the operator will be called m. These operators from a graded algebra by itself. It will be interesting to develop a theory to understand those manifolds where this graded algebra is large. This questions interesting even when we deal with classical Riemannian geometry. When the manifold is the Riemannian Kerr metric, there is a non-trivial secord order operator commute with the Laplacian.

The generalized manifold is said to have symplectic structure if there is a skew-symmetric pairing $\omega(X, Y)$ on the space of vector fields and that $d\omega = 0$. In this case, for any $f \in C$, we can associate a vector field X_f by

$$df(Y) = \omega(X_f, Y)$$

for all Y.
The Poission bracket between two functions f, g are defined by $\{f, g\}$ so that

$$X_{\{f,g\}} = [X_f, Xg]$$

The cycles defined by an ideal I will be Lagrangian if I is invariant under the Poission bracket.
In this way, we can define Lagrangian Cycles with singularities.
We shall define Kahler manifold to be these manifolds admitting an almost complex structure J which acts on the tangent bundle which satisfies $J^2 = -\mathbb{I}$ and also commute with the action of Laplacian on differentia forms. As a consequence, the de Rham cohomology will have the Hodge structure and most of the standard theory will go through.

Maps, subspaces and sigma moduls

§9. Maps, subspaces and sigma moduls

The idea of using space of maps (worldsheets) from Riemann surfaces to determine structures of manifolds, as was done in string theory, has led to many interesting properties of manifolds. This is the sigma model of the manifold. It gives rise to conformal field theory if spacetime has special property.

The idea of associating a "conformal field theory" to manifolds with special holonomy group has contributed immensely to understanding the study of such manifolds.

One of the major achievements is the discovery of the concept of mirror symmetry, where many interesting questions can be understood through duality. It arises from conformal field theory. Hence it will be interesting to associate a conformal field theory to our singular space with special structure.

Subspaces of our abstract manifold can be defined by closed ideals of C in the Hs topology. Then there is naturally defined induced Laplacian acting on the quotient algebra and we can also define mappings of manifolds. We shall explain this in the following.

A point on the manifold is defined by the maximal ideal of functions in *Hs* which vanishes at that point. Here *Hs* are the closure of the algebra C in the norm $\langle \varphi, -L^s \varphi \rangle$. A closed subset E is defined by some closed ideal I of functions Vanishing on this subset. The quotient space H/I and H_s/I admit natural inner products inherited from H and H_s. They give rise to a new triple (H', H'_s, L') because the inner product on H'_s/I, when compared with the inner product on H', defines a self-adjoint operator L'_s. We can consider the lim $(L'_s)^{\frac{1}{s}}$ when $s \to \infty$ and define it to be our Laplacian L'. Hence we have a new triple $(H', \bigcap_s H'_s, L')$.

The new triple can be considered as the triple associated to the closed subset E. Note that the ideal I carries more information then the set E itself. The set E may be zero set of different ideals and the geometry to the ideals. It would be useful to understand the spectral resolution of L' when the closed set is complicated. An important question is when the dimension of this closed subset, by looking at the trace of $\exp(tL')$, is the same as the Hausdorff dimension of the subset or some other related definitions of dimension.

There are natural morphisms between triples which can be considered as generalization of maps from manifolds to each other. An important consideration is the sigma model where we consider maps from two dimensional surfaces to the manifold.

Two dimensional surfaces are those triples where the spectrum of the operator grows linearly. It will be interesting to prove the following uniformization theorem:
Two dimensional triple is conformally isomorphic to a triple formed by a compact surface whose metric has constant curvature.
Conformal means that the operator is the same as the Laplacian of the metric with constant curvature, up to multiplication by a function.

Sigma model consider the space of maps from our given triple to space of all triples defined by compact surfaces. It is a homomorphism mapping the H_1 algebra from one to another. The energy of the map is obtained by comparing the H_1 inner product of the surface with the H_1 algebra of the original manifold. The quotient of H_1 by this ideal has a natural inner product. Comparing this inner product with the original H_1 inner product on the surface give rise to a self-adjoint operator. The trace of this operator can be considered as the energy of the map. Harmomic maps are critical points of this energy.

Discrete spaces

§10. Discrete spaces

As pointed out by Riemann, the basic concept of space may consist of discrete objects. We hope that the formulation discussed in the first part can work for discrete space also.

A very important discrete space is simply a graph which consists of a bunch of vertices and edges joining them. We can use graphs to approximate singular spaces.

The study of the geometry of graphs should be fruitful. We need to implement structures over graphs so that it reflects geometry of continuous space. We can construct exactly the same triple as we did in the continuous case.
Which we consider space of functions define on the vertices of the graph, the multiplication of them can be allowed to spread out in a neighbourhood of each vertex. The tangent vectors are linear combinations of edges which act on these functions as derivations. Practically all the structures mentioned above can be carried over.

It turns out that many interesting structures can be defined based only on the combinatorial structure of the graph that resembles continuous geometry.

For example, A. Grigor'yan, Yong Lin, Y. Muranov and myself have found a certain type of graph cohomology that resembles de Rham cohomology which need not be trivial even when the degree of the cohomology is big.

There is also several definitions of Ricci curvature and we can develop theorems parallel to theorems in Riemannian geometry.

There is a natural operator associated to the graph: **the graph Laplacian acting on a function is obtained by averaging the function in a neighbourhood.**

And there are operators acting naturally on de Rham forms as I mentioned above. They are all important operators that can provide invariants for the graph. We can therefore study Hodge theory on graphs. We may like to look at local operators that commute with such operators as was mentioned in the above. As a result, we can provide special structures over the combinatorial part of the graph.

Conclusion

§11. Conclusion

We have proposed new structures over continuous spaces and discrete spaces. They provide many interesting questions for classical geometry.

On the other hand, there are not enough physical intuitions behind the construction. While many interesting geometrical and combinatorial problems have already appeared, we are still a long way to understanding quantum geometry: a geometry that can incorporate quantum mechanics in the small and general relativity in the large.

Coming back to the question raised at the beginning of this talk, where Riemann was wondering what the concept of space should be.

How to go from discrete structure to continuous structure? We may need to understand local symmetries that have performed fundamental role in general relativity: equivalence principle and Lorentzian symmetry. We shall study Lorentzian geometry in a later paper.

Appendix

§12. Appendix
In this appendix, we shall provide some general conditions for the operators to satisfy our hypothesis.

I. argument based on minimax
We shall see that the asymptotic of eigenvalues of $-L$ so that the Weyl law $\lim_{t\to\infty} k^{-\frac{2}{n}}\lambda_k$ holds is related to covering properties of the space.

we assume that there are plenty of non-negative functions ρ_i such that

$$\sum_{i=1}^{k} \rho_i^2 = 1 \tag{1}$$

and that there are positive constants $\lambda(\rho_i)$ such that the following Poincaré inequality holds

$$\frac{1}{2}\langle \rho_i^2, L(\varphi^2)\rangle - \langle \rho_i^2 \varphi, L\varphi\rangle \geq \lambda(\rho_i^2)\langle \rho_i^2 \varphi, \varphi\rangle \tag{2}$$

for all φ such that $\langle \rho_i^2, \varphi\rangle = 0$. In particular if φ is perpendicular to ρ_i^2 for all $i = 1, \cdots, k$, we can sum the above inequality to obtain. $-\langle \varphi, L\varphi\rangle \geq \min_i \lambda(\rho_i^2)\langle \varphi, \varphi\rangle$.

The max-min principle characterization of eigenvalues of L then says that

$$\lambda_{k+1} \geq \min_i \lambda(\rho_i^2) \ .$$

Since we can very the choice of ρ_i^2, we see that

$$\lambda_{k+1} \geq \bar{\lambda}_{k+1} := \max_{\rho_i} \min_i \lambda(\rho_i^2), \tag{3}$$

where ρ_i satisfies (1) and (2).
In Riemannian geometry, ρ_i^2 can be chosen to be characteristic functions of balls in M which cover M and the number of overlap of the balls are bounded by a constant depending on the dimension. The number of such balls are $k \backsim \left(\frac{1}{r}\right)^{\frac{1}{n}} \mathrm{Vol}(M)$ where $n = \dim\, M$ and $\lambda(\rho_i) \backsim \frac{1}{r^2} \backsim \left(\frac{k}{Vol(M)}\right)^{\frac{2}{n}}$. Note that r^2 reflects the order of L is chosen to be two.

Hence $\lambda_{k+1} \gtrsim \left(\frac{k}{Vol(M)}\right)^{\frac{2}{n}}$.
As for upper bound of λ_{k+1}, we consider ρ_i such that $\langle \rho_i \rho_j \rangle = 0$ for $i \neq j$ and $\langle \rho_i, \rho_i \rangle = 1$. Then for

$$\psi = \sum a_i \rho_i$$

$$\begin{aligned} -\langle \psi, L\psi \rangle &= -\sum a_i^2 \langle \rho_i, L\rho_i \rangle \\ &\geqslant \min_i [-\langle \rho_i, L\rho_i \rangle] \langle \psi, \psi \rangle \ . \end{aligned}$$

Hence $\lambda_{k+1} \leqslant \min_i(-\langle \rho_i, L\rho_i \rangle)$. In the case of Riemannian geometry, we choose ρ_i to be first eigenfunction Dirichlet problem of balls that are disjoint, Hence $-\langle \rho_i, L\rho_i \rangle \sim \left(\frac{1}{r}\right)^2$ and $k \backsim \left(\frac{1}{r}\right)^{\left(\frac{1}{n}\right)} Vol(M)$. Therefore $\lambda_{k+1} \leqslant C' \left(\frac{k}{Vol(M)}\right)^{\frac{2}{n}}$.

In general, we allow choice of ρ_i so that $\langle \rho_i, \rho_j \rangle = \delta_{ij}$, we define

$$\bar{\bar{\lambda}}_{k+1} = \min_{\rho} \min_{i} [-\langle \rho_i, L\rho_i \rangle]$$

Then

$$\bar{\lambda}_{k+1} \leq \lambda_{k+1} \leq \bar{\bar{\lambda}}_{k+1},$$

where $\bar{\lambda}_{k+1}$ is defined in equation (3). If we assume is that

$$\lim_{k\to\infty} k^{-\frac{2}{n}} \bar{\lambda}_k = \lim_{k\to\infty} k^{-\frac{2}{n}} \bar{\bar{\lambda}}_k$$

which depends only on $Vol(M)$, as is similar to the case of smooth manifolds, the Weyl law holds and n would be the dimension of our space.

II. Weak Maximum Principle for Heat equation

Given any element $\rho \in H$, $\rho_t = \exp(tL)\rho$ will satisfy the heat equation:

$$\begin{cases} \frac{d\rho_t}{dt} = L(\rho_t) \\ \lim\limits_{t \to 0} \rho_t = \rho \end{cases} \tag{II.1}$$

Since $-L$ is a positive operator, one can easily prove that for all $i > 0$

$$\langle \rho_t, (-L)^i \rho_t \rangle < \infty \tag{II.2}$$

and hence $\rho_t \in H_i$ for all i. In classical manifold theory, $\cap_{i=1}^{\infty} H_i$ are smooth functions. While $C \subset \cap_{i=1}^{\infty} H_i$ in general, it will be useful to find conditions so that $C = \cap_{i=1}^{\infty} H_i$. This is a consequence of Sobolev embedding theorem. In the following we shall relax the

equation (II.1) and derive consequences.

Suppose the initial data $\rho \in H^+$, i.e. $\langle \rho, f^2 \rangle > 0$ for all $f \neq 0 \in H$. We would like to demonstrate that the same inequality holds true for all t. This may be considered as a weak maximum principle.

In order to achieve this, we make the assumption that for any $f \in H_1$ which is not a multiple of 1.

$$0 \neq L(f^2) - 2f \cdot L(f) \in H^+ \tag{II.3}$$

Then

$$\frac{d}{dt}\langle \rho, f^2 \rangle = 2\left\langle \rho, f\left(\frac{df}{dt} + Lf\right)\right\rangle + \langle \rho, g \rangle + \left\langle \frac{d\rho}{dt} - L\rho, f^2 \right\rangle \tag{II.4}$$

for some $g \neq 0 \in H^+$.
At the time T, we are given a function f_0. We then construct $\sigma = (\mathrm{Exp}(T-t)L)\ f_0$. For this f, $\frac{df}{dt} + Lf = 0$. Hence

$$\frac{d}{dt}\langle \rho, f^2 \rangle = \langle \rho, g \rangle \tag{II.5}$$

If $T_0 > 0$ is the first time so that $\langle \rho, g \rangle > 0$ for all $0 \leq t < T_0$ and for all $g \neq 0 \in H^+$. Our assumption says that this is true for $t = 0$, hence $T_0 \geq 0$. On the other hand, we can replace ρ by $\rho + \varepsilon 1$ for small $\varepsilon > 0$. In that case $T_0 > 0$. The above equation shows that T_0 can be prolonged to T.

We conclude with the following theorem:

Theorem II.2 Suppose $\rho_0 \in \overline{H^+}$ and $\frac{d\rho}{dt} - \Delta\rho \in \overline{H^+}$. Then if (II.3) holds, $\rho \in \overline{H^+}$ for all t.

The weak maximum principle allows us to give estimate for the operator $\exp(tL)$. The idea is to find good super or sub solution of the heat equation and as a result, one finds estimate of solutions of the heat equation.

III. Sobolev Inequality and Analytic Dimension

On a more abstract space that we are discussing, perhaps we can define Sobolev inequality by the following inequality:

For all $f \in \overline{H_1^+}$,

$$\|f\|^{\frac{m+2}{m}} \leq c_1 \|f\|_1 \langle f, 1 \rangle^{\frac{2}{m}} + c_2 \|f\| \langle f, 1 \rangle^{\frac{2}{m}} \qquad \text{(III.1)}$$

where m, c_1, and c_2 are positive constants independent of f.

The smallest m satisfying this inequality will be called the *analytic dimension*.

The heat kernel can be formulated in the following way:

Take the tensor product $H \bigotimes H$ where we pick $\{e_i\}$ to be orthonormal basis for H and $\{e_i \otimes e_j\}$ for $H \otimes H$. $\mathrm{Exp}(tL)$ defines an element in $H \bigotimes H$ in the following way:

For any f, $g \in H$, $\langle \mathrm{Exp}(tL)f, g\rangle$ defines a bilinear form and hence a linear functional on $H \bigotimes H$. By duality, it gives rise to an element $\rho \in H \otimes H$ so that

$$\langle \mathrm{Exp}(tL)f, g\rangle = \langle \rho, f \otimes g\rangle \tag{III.2}$$

The element ρ will satisfy the heat equation. The fact that L is self-adjoint implies that ρ is symmetric.

Theorem III.1 Assume Sobolev inequality (III.1) holds, then $\mathrm{Tr}\exp(tL) \leq Ct^{-\frac{m}{2}}$ and the dimension of the space is not greater than m.

Proof:

Let $\{\varphi_\imath\}$ be an orthonormal base of H. Then $\sum_i \varphi_i \otimes \varphi_i$ can be considered as the Delta function.

The trace $Exp(2tL)$ is defined by $\sum_i \langle Exp(2tL)\varphi_i, \varphi_i \rangle$.

It is equal to $\sum_i \langle exp(tL)exp(tL)\varphi_i, \varphi_i \rangle = \sum \parallel exp(tL)\varphi_i \parallel^2$.

Suppose we consider $\rho_\circ = \sum \varphi_i \otimes \varphi_i$ as element in $H \otimes H$, and $Exp(tL)$ acts on $H \otimes H$ through the action on the first factor. Then we have

$$\rho_t = \sum_i Exp(tL)\varphi_i \otimes \varphi_i$$

Hence $\| \rho_t \|^2 = \sum_i \| exp(tL)\varphi_i \|^2$.

Now

$$\begin{aligned} \frac{d}{dt} \| \rho_t \|^2 &= 2\langle \frac{d\rho_t}{dt}, \rho_t \rangle \\ &= -2\langle -L\rho_t, \rho_t \rangle \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\langle \rho_t, 1 \otimes 1 \rangle &= \langle L\rho_t, 1 \otimes 1 \rangle \\ &= 0 \end{aligned}$$

Hence $\langle \rho_t, 1 \otimes 1 \rangle = \langle \rho_\circ, 1 \otimes 1 \rangle = 1$

By (III.1), we find

$$C_1 \frac{d}{dt} \parallel \rho_t \parallel \leq - \parallel \rho_t \parallel^{\frac{m+2}{m}} + c_2 \parallel \rho_t \parallel$$

.

Hence $\parallel \rho_t \parallel$ decays lite $C\tau^{\frac{-m}{2}}$.
However we need to prove $\rho_t \in \overline{H_1^+}$. But this can be achieved as in (II.2).

For comparison with classical argument, we reproduce the following argument Nash.

inequality says that for any smooth function f,

$$\left(\int f^{\frac{2m}{m-2}}\right)^{\frac{m-2}{2m}} \leq c_1 \left(\int |\nabla f|^2\right)^{\frac{1}{2}} + c_2 \left(\int f^2\right)^{\frac{1}{2}} \tag{III.4}$$

where c_1 and c_2 are constants independent of f.

By applying Holder inequality,
we obtain

$$\left(\int f^2\right) \leq \left(\int f^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m+2}} \left(\int |f|\right)^{\frac{2}{m}} \tag{III.5}$$

Hence

$$\left(\int f^2\right)^{\frac{m+2}{2m}} \leq \left(\int f^{\frac{2m}{m-2}}\right)^{\frac{m-2}{2m}} \left(\int |f|\right)^{\frac{1}{2m}} \tag{III.6}$$

and

$$\left(\int f^2\right)^{\frac{m+2}{2m}} \leq c_1 \left(\int |\nabla f|^2\right)^{\frac{1}{2}} \left(\int |f|\right)^{\frac{2}{m}} + c_2 \left(\int f^2\right)^{\frac{1}{2}} \left(\int |f|\right)^{\frac{2}{m}} \tag{III.7}$$

Let us define the number m that satisfies (III.7) for all $f \in C$ to be analytic dimension of our triple (H, C, L) (Note that $\int |\nabla f|^2 = -\langle f, \Delta f\rangle$).

The semi-group $\exp(-tL)$ acts on $H = L^2(M)$ with a kernel function $h(t,x,y)$ which satisfies the heat equation

$$\begin{cases} \frac{\partial h}{\partial t} = \Delta_x h(t,x,y) \\ \lim_{t\to 0} h(t,x,y) = \delta_y(x) \end{cases} \tag{III.8}$$

The integral $\int h(t,x,y)dy$ is preserved by the first equation. Since $\lim_{t\to 0}\int h(t,x,y)dy = 1$, we conclude that for all $t > 0$,

$$\int h(t,x,y)dy = 1 \tag{III.9}$$

One can also prove that $\frac{d}{dt}\int |h| \leq 0$. Hence $\int |h|(t,x,y) = 1$ which means $h(t,x,y) \geq 0$.

Note that

$$\frac{d}{dt}\int h^2 = 2\int h\Delta h = -2\int |\nabla_x h|^2 \tag{III.10}$$

Hence

$$\frac{d}{dt}\int h^2 \leq -C_3\left(\int h^2\right)^{\frac{m+2}{m}} + C_4\int h^2 \tag{III.11}$$

Since

$$\lim_{t\to 0}\int h^2(t,x,y) = \infty,$$

we conclude that for t small,

$$\int h^2(t,x,y) \leq C'\left(1 - e^{-c_4 t}\right)^{\frac{m}{2}} \tag{III.12}$$

The semi-group property of $\exp(-tL)$ shows

$$\begin{aligned} h(2t,x,x) &= \int h(t,x,y)h(t,y,x) \\ &= \int h^2(t,x,y) \end{aligned} \tag{III.13}$$

Hence

$$h(t,x,y) \leq \overline{C}\left(1-e^{-\frac{c_4}{2}t}\right)^{\frac{m}{2}} \tag{III.14}$$

$$\begin{aligned} \mathrm{Tr}\ \exp(t\Delta) &= \int h(t,x,x)dx \\ &\leq \overline{C}\left(1-e^{-\frac{c_4}{2}t}\right)^{\frac{m}{2}} \mathrm{Vol}(M) \end{aligned} \tag{III.15}$$

Since we assume

$$\mathrm{Tr}\ \exp(t\Delta) \sim ct^{\frac{-n}{2}}, \tag{III.15}$$

we conclude that the analytic dimension m is not less than the dimension of the space.

IV. Compactness Before we discuss compactness, we need to introduce a norm on the algebra H_1, the completion of C by the norm $-\langle \varphi_1, L\varphi \rangle$.
The map

$$H_1 \bigotimes H_1 \longrightarrow H_1$$
$$f \otimes g \longrightarrow fg, \tag{IV.1}$$

is assumed to be continuous bilinear and there is a positive constant C so that

$$\| fg \|_1 \leq C \| f \|_1 \| g \|_1 \tag{IV.2}$$

for all $f, g \in H_1$

The smallest constant C will be the norm of the algebra H_1.

Let us fix one triple (H, C, L) and we assume that we have a family of such triples such that the constant C in (IV,2) is uniform for all triples in this family.

And also that there is a constant D so that the ratio of any two H_1 norms of any X in this family is bounded from above by D.

Then we can choose a fixed orthonormal basis $\{\varphi_\imath\}$ associated to H_1 of one triple. We shall assume $\varphi_0 =$ constant function. Then for each other triple, we can find a new orthonormal basis by Gram-Schmidt process.

By the Sobolev inequality in III, we can prove that each i, the $\imath - th$ element will converge in H to a new element $\psi_\imath \in H$.

For the new sequence $\{\psi_\imath\}$, we can define an algebra structure by taking limit of the multiplication.