Slide 1:

Some non-standard methods to perform calculations with Riemann's zeta function

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These are annotated slides of the talk given by the author on November 22, 2012 at join session of international meeting "Zeta functions" (http://www.mccme.ru/poncelet/2012zeta/) and seminar "Globus" (http://www.mccme.ru/ium/globus.html). The latter seminar is devoted to mathematics in general, and by this reason the talk included a short introduction to the history of the study of Riemann's zeta function giving necessary (and, hopefully, sufficient) background for understanding the rest of the talk.

Experts may skip the first 21 slides and jump to Slide 22.

Other materials related to this talk, including video, can be found at http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod/talks/poncelet_2012.

Slide 2:

Blowing the Trumpet

"The physicist George Darwin used to say that every once in a while one should do a completely crazy experiment, like blowing the trumpet to the tulips every morning for a month. Probably nothing will happen, but if something did happen, that would be a stupendous discovery."

Ian Hacking
Representing and Intervening
Cambridge University Press
p.154, 1983

This research started as a kind of "mathematical blowing the trumpet to the tulips" but the outcome seems to be striking and deserving further investigation.

George Darwin was a son of famous naturalist Charles Darwin.

Slide 3:

Riemann's zeta function

Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \ldots \quad (1)$$

The series converges for $s > 1$ and diverges at $s = 1$:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \quad (2)$$

The zeta function is named after Georg Friedrich Bernhard Riemann but it was first studied by Leonhard Euler.

Slide 4:

Basel Problem (Pietro Mengoli, 1644)

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=?$$

Leonhard Euler:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=\zeta(2) = 1.64493406684822644.. \quad (3)$$

$$\frac{\pi^2}{6} = 1.64493406684822644.. \quad (4)$$

$$\zeta(2)=\frac{\pi^2}{6} \quad (5)$$

Pietro Mengoli asked in 1644: *What is the value of this sum?*

Euler calculated more than a dozen decimal digits of the sum, which wasn't an easy exercise because the series in (3) converges very slowly. Luckily, Euler invented what is nowadays called *Euler–Maclaurin summation.*

Knowing the value (4), Euler conjectured the equality (5). His first "proof" given in 1735 was not rigorous by today's standards but later he returned to this problem several times and gave a number of quite rigorous proofs.

Slide 5:

Other values of $\zeta(s)$ found by EULER

$$\zeta(2) = \frac{1}{6}\pi^2 \quad (6)$$

$$\zeta(4) = \frac{1}{90}\pi^4 \quad (7)$$

$$\zeta(6) = \frac{1}{945}\pi^6 \quad (8)$$

$$\zeta(8) = \frac{1}{9450}\pi^8 \quad (9)$$

$$\zeta(10) = \frac{691}{638512875}\pi^{10} \quad (10)$$

$$\zeta(12) = \frac{2}{18243225}\pi^{12} \quad (11)$$

$$\zeta(14) = \frac{3617}{325641566250}\pi^{14} \quad (12)$$

Euler didn't stop by merely answering Mengoli's question and found many other values of $\zeta(s)$. But why such strange numerators of $\zeta(10)$ and $\zeta(14)$?

Slide 6:

Bernoulli numbers

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{\mathrm{B}_k x^k}{k!} \tag{13}$$

$$\mathrm{B}_0 = 1, \quad \mathrm{B}_2 = \frac{1}{6}, \quad \mathrm{B}_4 = -\frac{1}{30}, \quad \mathrm{B}_6 = \frac{1}{42}, \quad \mathrm{B}_8 = -\frac{1}{30}, \tag{14}$$

$$\mathrm{B}_{10} = \frac{5}{66}, \quad \mathrm{B}_{12} = -\frac{691}{2730}, \quad \mathrm{B}_{14} = \frac{7}{6}, \quad \mathrm{B}_{16} = -\frac{3617}{510} \tag{15}$$

$$\mathrm{B}_1 = -\frac{1}{2}, \quad \mathrm{B}_3 = \mathrm{B}_5 = \mathrm{B}_7 = \mathrm{B}_9 = \mathrm{B}_{11} = \cdots = 0 \tag{16}$$

No doubt that Euler immediately recognized 691 and 3617 as the numerators of the so-called *Bernoulli numbers.* Named after Jacob Bernoulli, these numbers can be defined in many ways, in particular, from the coefficients in the Taylor expansion (13).

Notice that starting from 3, all Bernoulli numbers with odd indices are equal to zero.

Slide 7:

Other values of $\zeta(s)$ found by EULER

$$\zeta(6) = \frac{1}{945}\pi^6 = \frac{2^5}{6!}\mathrm{B}_6\pi^6 \tag{17}$$

$$\zeta(8) = \frac{1}{9450}\pi^8 = -\frac{2^7}{8!}\mathrm{B}_8\pi^8 \tag{18}$$

$$\zeta(10) = \frac{691}{638512875}\pi^{10} = \frac{2^9}{10!}\mathrm{B}_{10}\pi^{10} \tag{19}$$

$$\zeta(12) = \frac{2}{18243225}\pi^{12} = -\frac{2^{11}}{12!}\mathrm{B}_{12}\pi^{12} \tag{20}$$

$$\zeta(14) = \frac{3617}{325641566250}\pi^{14} = \frac{2^{13}}{14!}\mathrm{B}_{14}\pi^{14} \tag{21}$$

$$\zeta(2k) = (-1)^{k+1}\frac{2^{2k-1}}{(2k)!}\mathrm{B}_{2k}\pi^{2k} \qquad k = 1, 2, \ldots \tag{22}$$

After substituting (14)–(15) into (6)–(12), it is easy to guess the form of the other factors, and Euler gave the general formula (22).

We still know very little about the value of $\zeta(s)$ for odd positive values of s.

One more value of $\zeta(s)$ given by Euler

$$\zeta(0) = 1^0 + 2^0 + 3^0 + \cdots = 1 + 1 + 1 + \ldots = -\frac{1}{2} \tag{23}$$

$$\begin{aligned}
\eta(s) &= (1 - 2\cdot 2^{-s})\zeta(s) && (24)\\
&= (1 - 2\cdot 2^{-s})(1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \ldots) && (25)\\
&= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \ldots\\
&\quad - 2\cdot 2^{-s} \qquad - 2\cdot 4^{-s} - \ldots && (26)\\
&= 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \ldots && (27)
\end{aligned}$$

The alternating Dirichlet series

$$1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \ldots \tag{28}$$

converges for $s > 0$.

$$1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2} \tag{29}$$

Euler proceeded by giving a striking value (23) for $\zeta(0)$. His argumentation is very important for understanding the main part of this talk.

Namely, Euler considered the function (24) which is also defined by the alternating Dirichlet series (27). For $s = 0$ the partial sums of this series oscillate around $\frac{1}{2}$ and plugging this number as the value of $\eta(0)$ in (24), one comes to (23).

Other values of $\zeta(s)$ given by EULER

$$\zeta(0) = 1^0 + 2^0 + 3^0 + \cdots = 1 + 1 + 1 + \cdots = -\frac{1}{2} \tag{30}$$

$$\zeta(-1) = 1^1 + 2^1 + 3^1 + \cdots = 1 + 2 + 3 + \ldots = -\frac{1}{12} \tag{31}$$

$$\zeta(-2) = 1^2 + 2^2 + 3^2 + \cdots = 1 + 4 + 9 + \cdots = 0 \tag{32}$$

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + \cdots = 1 + 8 + 27 + \ldots = \frac{1}{120} \tag{33}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\zeta(-11) = 1^{11} + 2^{11} + \cdots = 1 + 2048 + \ldots = \frac{691}{32760} \tag{34}$$

$$\zeta(-m) = -\frac{\mathrm{B}_{m+1}}{m+1} \qquad m = 0, 1, \ldots \tag{35}$$

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0 \tag{36}$$

In a similar manner Euler obtained other values for negative integer argument. The appearance of 691 in the numerator of $\zeta(-11)$ suggests that Bernoulli numbers are again involved, and Euler gave the general formula (35).

In particular, recalling (16) he got (36).

Slide 10:

The functional equation

$$\zeta(2k) = (-1)^{k+1}\frac{2^{2k-1}}{(2k)!}\mathrm{B}_{2k}\pi^{2k} \qquad k = 1, 2, \ldots \tag{37}$$

$$\zeta(-m) = -\frac{\mathrm{B}_{m+1}}{m+1} \qquad m = 0, 1, \ldots \tag{38}$$

$$m = 2k - 1 \tag{39}$$

$$\zeta(1-2k) = (-1)^k 2^{1-2k}\pi^{-2k}(2k-1)!\zeta(2k) \qquad k = 1, 2, \ldots \tag{40}$$

Selecting m according to (39), one can eliminate Bernoulli numbers from (37) and (38) and get (40).

Euler Identity ≡ Fundamental Theorem of Arithmetic

Theorem [Euler].

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \ldots \quad (41)$$

$$= \prod_{p \text{ is a prime}} \frac{1}{1 - p^{-s}} \quad (42)$$

Proof.

$$\prod_{p \text{ is a prime}} \frac{1}{1 - p^{-s}} = \prod_{p \text{ is a prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right) \quad (43)$$

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \ldots \quad (44)$$

$$= \zeta(s) \quad (45)$$

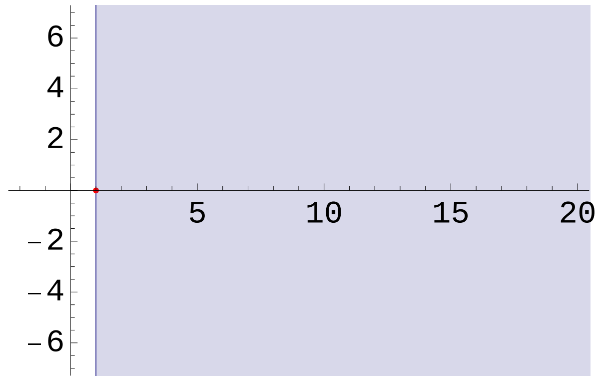
In order to obtain (43) from the right-hand side, one needs just to sum up the geometrical progression. Obtaining (43) from (44) is based on the Fundamental Theorem of Arithmetic which states that every natural numbers can be represented as the product of powers of different primes, and this can be done in a unique way (up to the order of factors).

Riemann's zeta function

Dirichlet series:

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

$s = \sigma + \mathrm{i}t$



The series converges in the semiplane $\mathrm{Re}(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.

Riemann began to study the zeta-function for complex arguments and established the tradition of writing this complex argument as $s = \sigma + \mathrm{i}t$.

Riemann's zeta function

Euler's values of $\zeta(s)$ for negative integer s were correct:

$$\zeta(-m) = -\frac{\mathrm{B}_{m+1}}{m+1} \qquad m = 0, 1, \ldots \tag{46}$$

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0 \tag{47}$$

Points $s_1 = -2,\ s_2 = -4,\ \ldots,\ s_k = -2k, \ldots$ are called the trivial zeroes of the zeta function, and it has no other real zeroes.

Great Euler was right!

Slide 14:

The functional equation

Euler:

$$\zeta(1-2k) = (-1)^k 2^{1-2k}\pi^{-2k}(2k-1)!\zeta(2k) \qquad k = 1, 2, \ldots \quad (48)$$

Riemann:

$$\zeta(1-s) = \cos\left(\frac{\pi s}{2}\right) 2^{1-s}\pi^{-s}\Gamma(s)\zeta(s) \qquad s = \sigma + \mathrm{i}t \quad (49)$$

To pass from (48) to (49) one had to guess the correct counterpart of $(-1)^k$.

Function $\xi(s)$

$$\zeta(1-s) \quad = \quad \cos\left(\frac{\pi s}{2}\right) 2^{1-s}\pi^{-s}\Gamma(s)\zeta(s) \tag{50}$$

$$\cos\left(\frac{\pi s}{2}\right) = \frac{\pi}{\Gamma(-\frac{s}{2}+\frac{1}{2})\Gamma(\frac{s}{2}+\frac{1}{2})} \qquad \Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \tag{51}$$

$$\underbrace{\pi^{-\frac{1-s}{2}}(-s)\Gamma\left(\frac{1-s}{2}+1\right)\zeta(1-s)}_{\xi(1-s)} = \underbrace{\pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2}+1\right)\zeta(s)}_{\xi(s)} \tag{52}$$

$$\xi(1-s) \quad = \quad \xi(s) \tag{53}$$

Function $\xi(s)$ is entire, its zeroes are exactly non-trivial (i.e., non-real) zeroes of $\zeta(s)$.

In the right-hand side of (52), the pole of the zeta-function is cancelled by the factor $s-1$, and the poles of the Γ-factor are cancelled by the trivial zeroes of the zeta-function.

Slide 16:

Function $\Xi(t)$

$$\xi(s) = \underbrace{\pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2}+1\right)}_{g(s)}\zeta(s) \tag{54}$$

$$s = \frac{1}{2} + \mathrm{i}t \tag{55}$$

$$\Xi(t) = \xi\left(\tfrac{1}{2} + \mathrm{i}t\right) = \xi\left(1 - \left(\tfrac{1}{2} + \mathrm{i}t\right)\right) = \Xi(-t) \tag{56}$$

$\Xi(t)$ is an even entire function with real value for real values of its argument t.

The Riemann Hypothesis. *All zeroes of $\Xi(t)$ are real numbers.*

Making the change of variable (55) is again a tradition due to Riemann. However, for the functions $\xi(s)$ and $\Xi(t)$ we use modern notation introduced by E. Landau, Riemann himself called the latter function $\xi(t)$.

There is no standard notation for the important factor in (54) that we'll be denoting $g(s)$.

The Riemann Hypothesis was originally stated in terms of the function $\Xi(t)$, not the function $\zeta(s)$.

Slide 17:

Chebyshev function $\psi(x)$

$$\psi(x) = \ln(\mathrm{LCM}(1, 2, ..., \lfloor x \rfloor)) \tag{57}$$

$$= \sum_{\substack{q \text{ is a power} \\ \text{of prime } p}}^{q \le x} \ln(p) \tag{58}$$

$$= \sum_{q=1}^{q \le x} \Lambda(q) \tag{59}$$

Von Mangoldt function

$$\Lambda(q) = \begin{cases} \ln(p), & q = p^k \text{ for some prime } p \\ 0, & \text{otherwise} \end{cases} \tag{60}$$

Riemann established a relationship between prime numbers and the zeroes of the zeta function by giving a (somewhat complicated) expression for $\pi(x)$ – the number of primes below x – via certain sums over these zeroes. Chebyshev's function $\psi(x)$ turned out to be more convenient for studying the distribution of prime numbers.

Slide 18:

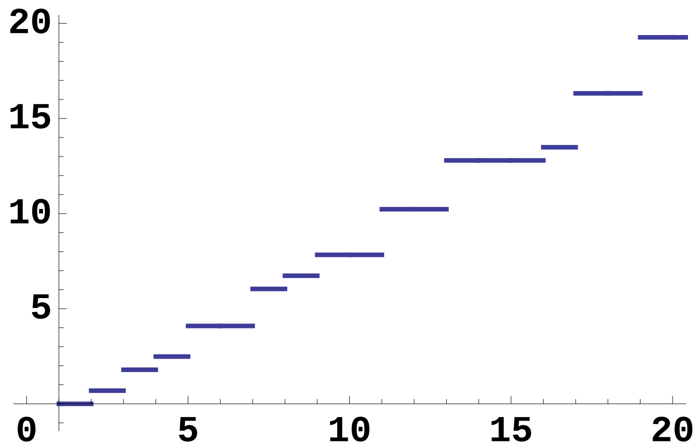
Logarithmic derivative of $\zeta(s)$

$$\zeta(s) = \prod_{p \text{ is a prime}} \frac{1}{1 - p^{-s}} \tag{61}$$

$$\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} &= \sum_{p \text{ is a prime}} \left(\ln \left(\frac{1}{1 - p^{-s}} \right) \right)' && (62)\\
&= \sum_{p \text{ is a prime}} \frac{\ln(p)}{1 - p^{s}} && (63)\\
&= -\sum_{p \text{ is a prime}} \ln(p) \sum_{k=1}^{\infty} p^{-ks} && (64)\\
&= -\sum_{n=1}^{\infty} \Lambda(n) n^{-s} && (65)
\end{aligned}$$

Von Mangoldt's function plays an important role in the theory of the zeta-function because its values are nothing else but (negations of) the coefficients in the Dirichlet series of the logarithmic derivative of the zeta function.

Chebyshev function $\psi(x)$



The jumps occur at primes and prime powers.

Slide 20:

Von Mangoldt Theorem

$$\psi(x) \quad = \quad \ln(\mathrm{LCM}(1, 2, ..., \lfloor x \rfloor)) = \sum_{q=1}^{x} \Lambda(q) \tag{66}$$

Theorem (Hans Carl Fridrich von Mangoldt [1895]). *For any non-integer $x > 1$*

$$\psi(x) = x - \sum_{\xi(\rho)=0} \frac{x^\rho}{\rho} - \sum_{n} \frac{x^{-2n}}{-2n} - \ln(2\pi) \tag{67}$$

Theorem (Jacques Salomon Hadamard [1893]).

$$\xi(s) = \xi(0) \prod_{\xi(\rho)=0} \left(1 - \frac{s}{\rho}\right) \tag{68}$$

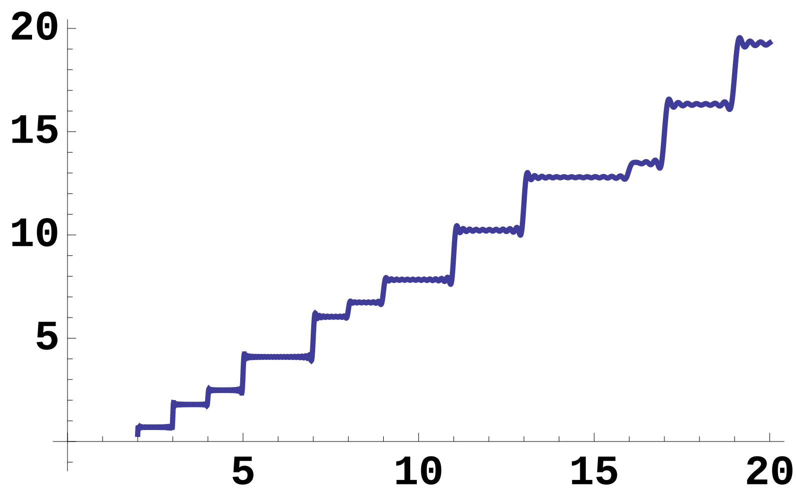
$$\xi(s) = g(s)\zeta(s) = g(s) \prod_{p \text{ is a prime}} \frac{1}{1 - p^{-s}} \tag{69}$$

The proof of von Mangoldt theorem is based on the presence of the two products in (68) and (69).

Slide 21:

Von Mangoldt Theorem

$$\psi(x) \approx x - \sum_{\xi(\rho)=0}^{|\rho|<400} \frac{x^\rho}{\rho} - \sum_n \frac{x^{-2n}}{-2n} - \ln(2\pi) \tag{73}$$

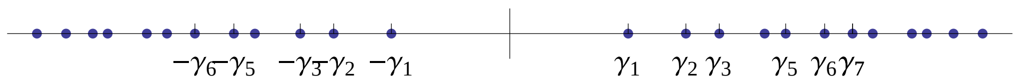


Traditionally, (67) is viewed as an identity between well-defined left and right hand sides, but it can be interpreted in a different way. Imagine that we know nothing about prime numbers, even their definition, but have at our disposal sufficiently many initial zeroes of the zeta function. Then we could "discover" prime numbers just by looking at the plot of the truncated right hand side of (67) as in (73). The prime numbers can be revealed from such a plot either by looking at integers in the vicinity of which the function has a big jump (they will be powers of primes) or by looking at the sizes of jumps (they will be close to natural logarithms of primes). To reveal more and more primes we would need to use more and more zeroes of the zeta function.

Slide 22:

Blowing the Trumpet

Assuming that all zeroes of $\Xi(t)$ are real and simple, let them be denoted $\pm\gamma_1, \pm\gamma_2, \ldots$ with $0 < \gamma_1 < \gamma_2 < \ldots$, thus the non-trivial zeroes of $\zeta(s)$ are $\frac{1}{2} \pm \mathrm{i}\gamma_1, \frac{1}{2} \pm \mathrm{i}\gamma_2, \ldots$



$$\gamma_1 = 14.1347\ldots \qquad \gamma_2 = 21.0220\ldots \tag{74}$$

$$\gamma_3 = 25.0109\ldots \qquad \gamma_4 = 30.4249\ldots \tag{75}$$

Suppose that we have found $\gamma_1, \gamma_2, \ldots, \gamma_{N-1}$; *how could these numbers be used for calculating an (approximate) value of the next zero* γ_N*?*

It was a rather strange idea to seek an answer to this question because the known initial zeroes are distributed rather irregularly.

Slide 23:

Interpolating determinant

Let us try to approximate $\Xi(t)$ by some simpler function also having zeroes at the points $\pm\gamma_1, \dots, \pm\gamma_{N-1}$.

Consider an interpolating determinant with even functions $f_1, f_2, \dots$

$$\begin{vmatrix} f_1(\gamma_1) & \dots & f_1(\gamma_{N-1}) & f_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(\gamma_1) & \dots & f_N(\gamma_{N-1}) & f_N(t) \end{vmatrix} \tag{76}$$

Selecting $f_n(t) = t^{2(n-1)}$ we would obtain just an interpolating polynomial

$$C \prod_{n=1}^{N-1} (t^2 - {\gamma_n}^2) \tag{77}$$

having no other zeros.

Clearly, the determinant (76) vanishes for $t = \pm\gamma_1, \dots, \pm\gamma_{N-1}$ because for such values of t the matrix contains two equal columns.

Slide 24:

Modelling situation

If $\gamma_1^*, \gamma_2^*, \ldots$ are zeros of the function

$$\Xi^*(t) = \sum_{k=1}^{N} f_k(t) \tag{78}$$

then the determinant

$$\begin{vmatrix} f_1(\gamma_1^*) & \ldots & f_1(\gamma_{N-1}^*) & f_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(\gamma_1^*) & \ldots & f_N(\gamma_{N-1}^*) & f_N(t) \end{vmatrix} \tag{79}$$

vanishes at every zero of $\Xi^*(t)$.

The determinant (79) vanishes if t is equal to *any* zero of $\Xi^*(t)$ because for such a t the rows of the matrix sum up to the zero row.

Slide 25:

Selecting $f_1, f_2, \ldots$

$$\Xi(t) = \sum_{n=1}^{\infty} \alpha_n(t) \qquad \alpha_n(t) = g\left(\tfrac{1}{2} + \mathrm{i}t\right) n^{-\left(\frac{1}{2}+\mathrm{i}t\right)} \tag{80}$$

$$\Xi(t) = \sum_{n=1}^{\infty} \alpha_n(t) \qquad \text{for } \mathrm{Im}(t) < -\frac{1}{2} \tag{81}$$

$$\Xi(t) = \Xi(-t) = \sum_{n=1}^{\infty} \alpha_n(-t) \qquad \text{for } \mathrm{Im}(t) > \frac{1}{2} \tag{82}$$

$$\Xi(t) = \sum_{n=1}^{\infty} \frac{\alpha_n(t) + \alpha_n(-t)}{2} = \sum_{n=1}^{\infty} \beta_n(t) \tag{83}$$

Our case differs from the modeling situation of the preceding slide in two respects. First, the sum in (80) is infinite. Second, its summands aren't even functions.

To remove the first difference we introduce even functions $\beta_n(t)$. However, the equality (83) is purely symbolic because the half planes of convergence of the two series (81) and (82) don't intersect.

Slide 26:

Our interpolating determinant

$$\Delta_N(t) = \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) & \beta_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) & \beta_N(t) \end{vmatrix} \tag{84}$$

$$\beta_n(t) = \frac{g\left(\frac{1}{2} - \mathrm{i}t\right) n^{-\left(\frac{1}{2} - \mathrm{i}t\right)} + g\left(\frac{1}{2} + \mathrm{i}t\right) n^{-\left(\frac{1}{2} + \mathrm{i}t\right)}}{2} \tag{85}$$

$$g(s) = \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2} + 1\right) \tag{86}$$

$$\Xi(t) = \sum_{n=1}^{\infty} \beta_n(t) \tag{87}$$

For the determinant we need only finitely many functions, so for the moment we can ignore the divergence of (87).

Blowing the trumpet: the outcome

$$\begin{aligned} \Delta_{47}(\mathbf{138.11604}) &= -2.18497\ldots\cdot 10^{-1216} < 0 \\ \gamma_{47} &= \mathbf{138.11604}20545334\ldots \\ \Delta_{47}(\mathbf{138.11605}) &= +4.68242\ldots\cdot 10^{-1216} > 0 \end{aligned}$$

$$\begin{aligned} \Delta_{321}(\mathbf{572.4199841324527640452769 27}) &= -1.75211\ldots\cdot 10^{-34969} < 0 \\ \gamma_{321} &= \mathbf{572.419984132452764045276927}10734686\ldots \\ \Delta_{321}(\mathbf{572.419984132452764045276928}) &= +5.72682\ldots\cdot 10^{-34968} > 0 \end{aligned}$$

$$\Delta_{600}(\mathbf{939.02430089921838218421841870917956106917}\backslash$$
$$\mathbf{858909398785769363406563371} = +3.90403\ldots\cdot 10^{-108715} > 0$$
$$\gamma_{600} = \mathbf{939.02430089921838218421841870917956106917}\backslash$$
$$\mathbf{858909398785769363406563371}13493189\ldots$$
$$\Delta_{600}(\mathbf{939.02430089921838218421841870917956106917}\backslash$$
$$\mathbf{858909398785769363406563372}) = -9.67471\ldots\cdot 10^{-108716} < 0$$

Here are just a few numerical examples.
A zero of $\Delta_{47}(t)$ has 8 decimal digits coinciding with digits of γ_{47}.
For $N = 321$ there are already 15 coinciding decimal digits.
For $N = 600$ the number of common digits increases to 38.

Slide 28:

Blowing the trumpet: the outcome

$$\Delta_{47}(\mathbf{138.11604}) = -2.18497\ldots \cdot 10^{-1216} < 0$$
$$\gamma_{47} = \mathbf{138.11604}20545334\ldots$$
$$\Delta_{47}(\mathbf{138.11605}) = +4.68242\ldots \cdot 10^{-1216} > 0$$

$$\Delta_{47}(\mathbf{139.7362}) = +1.27744\ldots \cdot 10^{-1216} > 0$$
$$\gamma_{48} = \mathbf{139.7362}08952121\ldots$$
$$\Delta_{47}(\mathbf{139.7363}) = -9.88309\ldots \cdot 10^{-1216} < 0$$

$$\Delta_{47}(\mathbf{141.12370}) = -1.85988\ldots \cdot 10^{-1217} < 0$$
$$\gamma_{49} = \mathbf{141.12370}74040211\ldots$$
$$\Delta_{47}(\mathbf{141.12371}) = +2.40777\ldots \cdot 10^{-1217} > 0$$

$$\Delta_{47}(\mathbf{143.11184}) = +8.00594\ldots \cdot 10^{-1218} > 0$$
$$\gamma_{50} = \mathbf{143.11184}58076206\ldots$$
$$\Delta_{47}(\mathbf{143.11185}) = -8.98353\ldots \cdot 10^{-1218} < 0$$

It turned out that $\Delta_N(t)$ allows us to calculate good approximations not only to the next still unused zero γ_N but to γ_{N+k} as well for values of k that are not too large.

Blowing the trumpet: the outcome

$\Delta_{321}(\mathbf{572.4199841324527640452769\,27})=-1.75211\ldots\cdot 10^{-34969}<0$
$\gamma_{321} = \mathbf{572.419984132452764045276927}10734686\ldots$
$\Delta_{321}(\mathbf{572.419984132452764045276928})=+5.72682\ldots\cdot 10^{-34968}>0$

$\Delta_{321}(\mathbf{573.61461052675812998724603})=+2.27756\ldots\cdot 10^{-34967}>0$
$\gamma_{322} = \mathbf{573.61461052675812998724603}21096433\ldots$
$\Delta_{321}(\mathbf{573.61461052675812998724604})=-9.27805\ldots\cdot 10^{-34967}<0$

$\Delta_{321}(\mathbf{575.093886014494885607457\,17})=-5.83758\ldots\cdot 10^{-34968}<0$
$\gamma_{323} = \mathbf{575.09388601449488560745717}22821982\ldots$
$\Delta_{321}(\mathbf{575.09388601449488560745718})=+2.78750\ldots\cdot 10^{-34967}>0$

....

$\Delta_{321}(\mathbf{585.9845632049883005741\,8})=-5.42229\ldots\cdot 10^{-34968}<0$
$\gamma_{331} = \mathbf{585.98456320498830057418}72589855\ldots$
$\Delta_{321}(\mathbf{585.98456320498830057419})=+8.57284\ldots\cdot 10^{-34969}>0$

These are just a few examples, tables showing the number of digits common to the $(N+k)$th zero of $\Xi(t)$ and a zero of $\Delta_N(t)$ for diverse N and k can be found at

`http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod`.

Slide 30:

Blowing the trumpet: the outcome

$\Delta_{12000}(t)$ has zeroes having more than 2000 common decimal digits with $\gamma_{12000}, \gamma_{12001}, \ldots, \gamma_{12010}$

I don't have a full explanation of the high accuracy of the zeroes of determinants $\Delta_N(t)$ as approximations to zeroes of $\Xi(t)$. Two heuristic hints will be presented now.

Partial explanation. I

$$\Xi(t) = \sum_{n=1}^{\infty} \beta_n(t) \tag{88}$$

$$\Delta_N(t) = \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) & \beta_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) & \beta_N(t) \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n} \beta_n(t) \tag{89}$$

$$\tilde{\delta}_{N,n} = (-1)^{N+n} \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_{n-1}(\gamma_1) & \dots & \beta_{n-1}(\gamma_{N-1}) \\ \beta_{n+1}(\gamma_1) & \dots & \beta_{n+1}(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) \end{vmatrix} \tag{90}$$

The $\tilde{\delta}$'s are particular numbers defined via the zeroes of the zeta-function.

Slide 32:

Normalization

$$\Xi(t) = \sum_{n=1}^{\infty} \beta_n(t) \tag{91}$$

$$\Delta_N(t) = \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) & \beta_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) & \beta_N(t) \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n} \beta_n(t) \tag{92}$$

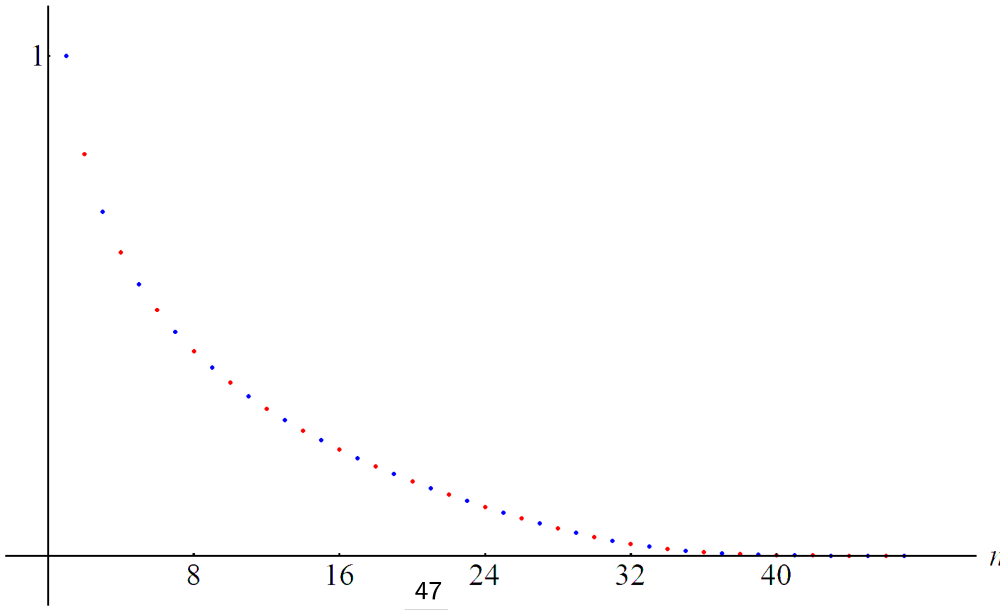
$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}} \qquad \delta_{N,1} = 1 \tag{93}$$

$$\Delta_N(t) = \sum_{n=1}^{N} \tilde{\delta}_{N,1} \delta_{N,n} \beta_n(t) = \tilde{\delta}_{N,1} \sum_{n=1}^{N} \delta_{N,n} \beta_n(t) \tag{94}$$

The normalization doesn't influence zeroes, that is, $\Delta_N(t)$ has the same zeroes as $\sum_{n=1}^{N} \delta_{N,n} \beta_n(t)$.

Slide 33:

Normalized coefficients $\delta_{47,n}$



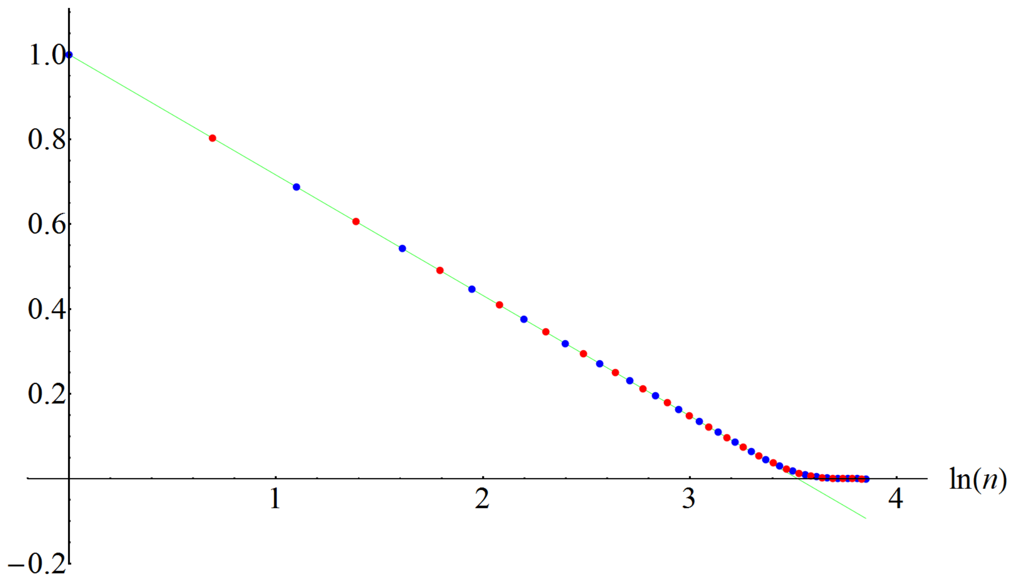
$$\sum_{n=1}^{47} \delta_{47,n}\beta_n(t) \tag{95}$$

We see that the sum (95) is not a *sharp* but a *smooth* truncation of the divergent series (87). It is known that smooth truncation can accelerate convergence of a series and it can even transform a divergent series into a convergent series.

The smoothness of the truncation in (95) might be the first "reason" why the summands of the divergence series (87) are useful for calculation of the zeroes.

Slide 34:

Normalized coefficients $\delta_{47,n}$ with logarithmic scale



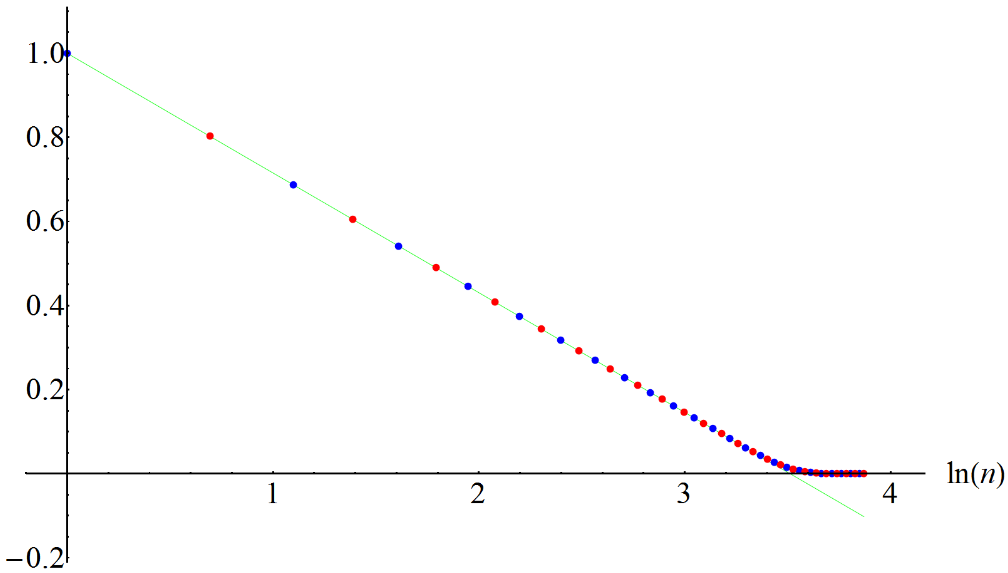
$$\delta_{47,n} \approx 1 + \lambda_{47} \log(n) \tag{96}$$

The curve from Slide 33 on which the coefficients $\delta_{47,n}$ lie looks like a logarithmic curve.

This is better seen on the plot of the same coefficients but with logarithmic scale. Why do the initial coefficients lie approximately on a straight line?

Slide 35:

Normalized coefficients $\delta_{48,n}$ with logarithmic scale

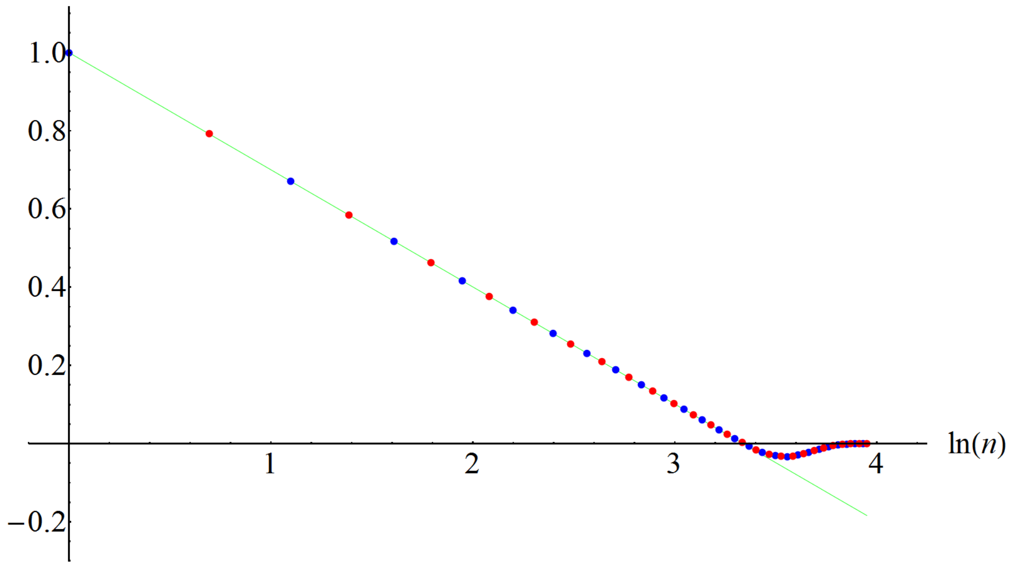


$$\delta_{48,n} \approx 1 + \lambda_{48} \log(n) \quad (97)$$

The graphs for $N = 48, 49, 50, 51$ look very similar to the graph for $N = 47$ from Slide 34.

Slide 36:

Normalized coefficients $\delta_{52,n}$ with logarithmic scale

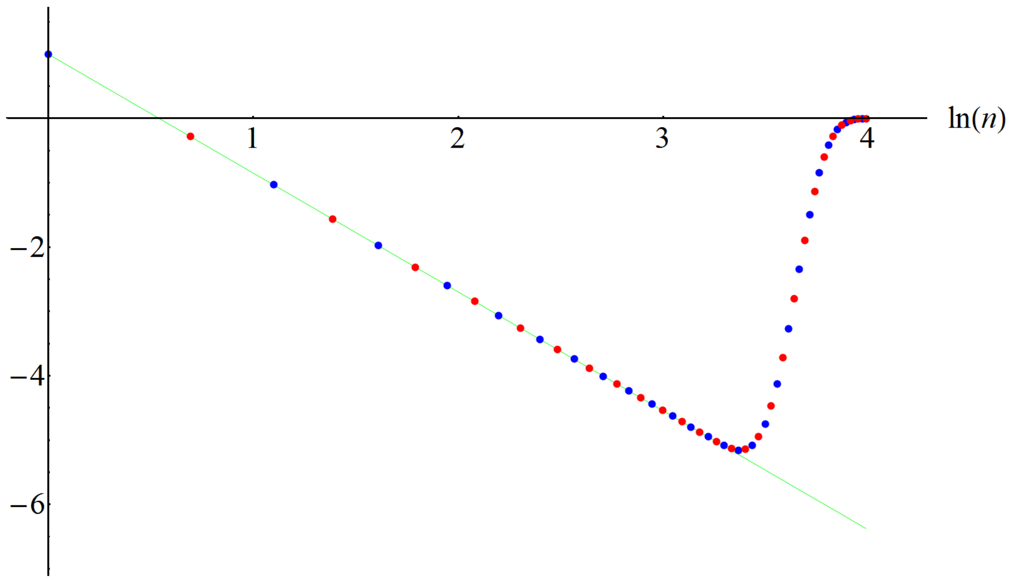


$$\delta_{52,n} \approx 1 + \lambda_{52} \log(n) \tag{101}$$

Normalized coefficients $\delta_{52,n}$ show slightly different behaviour of trailing coefficients.

Slide 37:

Normalized coefficients $\delta_{54,n}$ with logarithmic scale

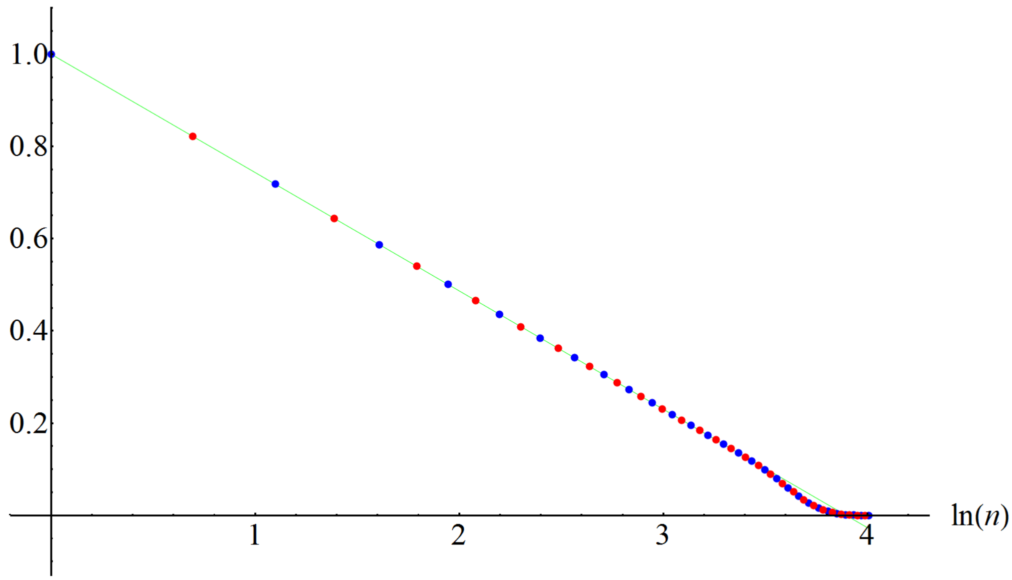


$$\delta_{54,n} \approx 1 + \lambda_{54} \log(n) \qquad (103)$$

What a difference from the preceding graphs! Now most of the coefficients are negative and greater than 1 in absolute value. Nevertheless, the initial coefficients do lie on a straight line and $\Delta_{54}(t)$ gives good predictions for a few of the next zeroes γ_{54}, γ_{55}, ...

Slide 38:

Normalized coefficients $\delta_{55,n}$ with logarithmic scale

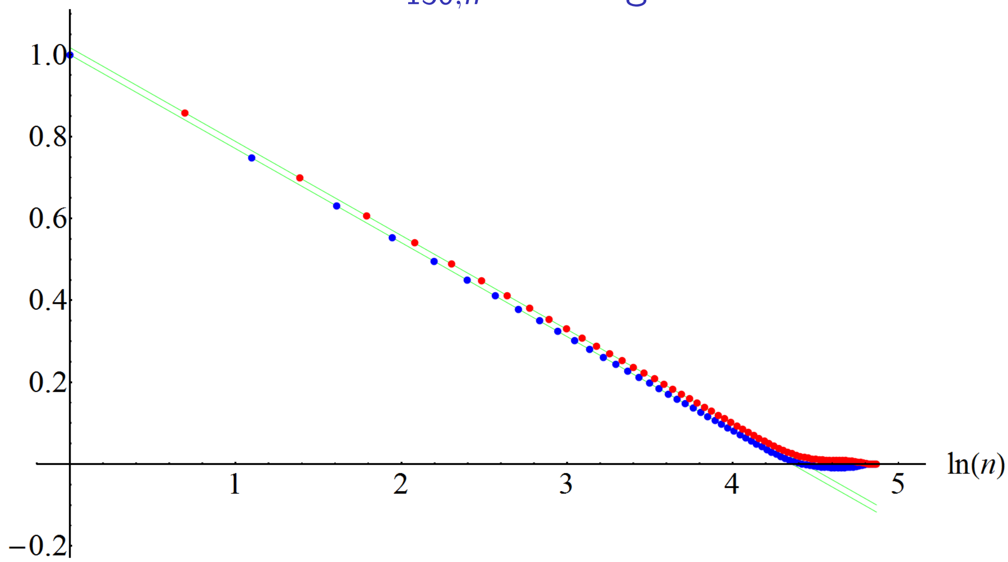


$$\delta_{55,n} \approx 1 + \lambda_{55} \log(n) \qquad (104)$$

PictuGrapfs for $N = 55$, 56, 57 look similar to the graph for $N = 47$. Case $N = 54$ demonstrates sporadic behavour. Why is number 54 special?

Slide 39:

Normalized coefficients $\delta_{130,n}$ with logarithmic scale



$$\delta_{130,n} \approx 1 + \mu_{130,2}\mathrm{dom}_2(n) + \lambda_{130}\log(n) \qquad (107)$$

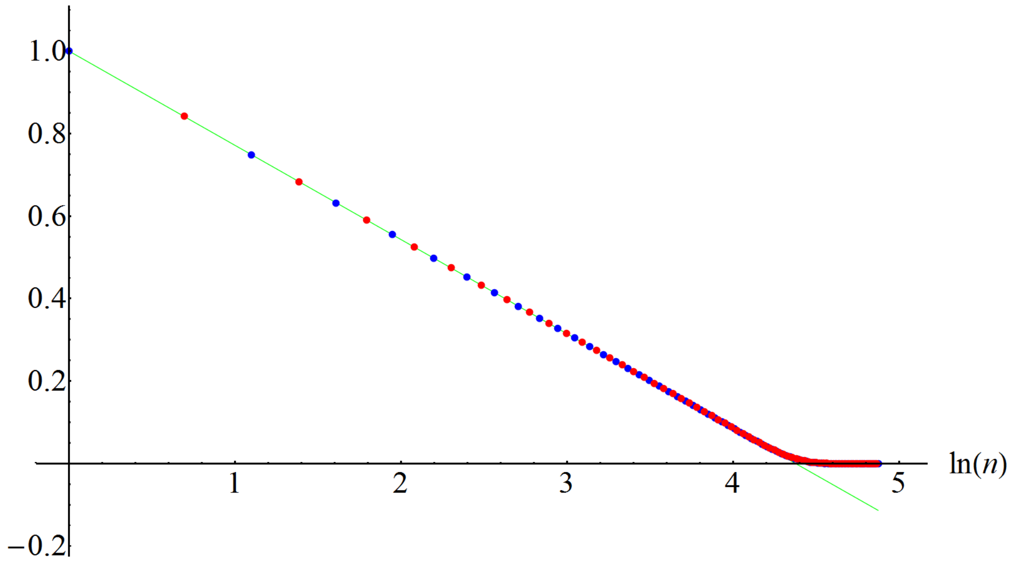
$$\mathrm{dom}_m(k) = \begin{cases} 1, & \text{if } m \mid k \\ 0, & \text{otherwise} \end{cases} \qquad (108)$$

We skip many values of N (those skipped and many other graphs can be found at `http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod`) and examine a new phenomenon in the case $N = 130$.

Now the initial coefficients lie on two parallel lines rather than on a single line, so instead of an analog of (96), (97), (101), (103) and (104), in the case $N = 130$ we should use approximation (107).

Slide 40:

Normalized coefficients $\delta_{131,n}$ with logarithmic scale

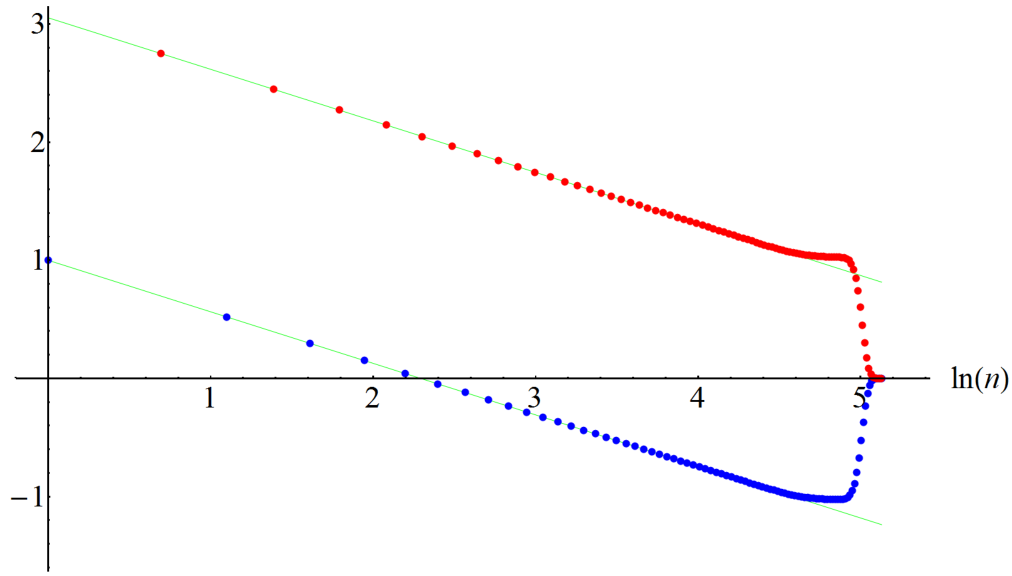


$$\delta_{131,n} \approx 1 + \mu_{131,2}\mathrm{dom}_2(n) + \lambda_{131}\log(n) \qquad (109)$$

Graph for $N = 131$ shows that the case $N = 130$ is sporadic.

Slide 41:

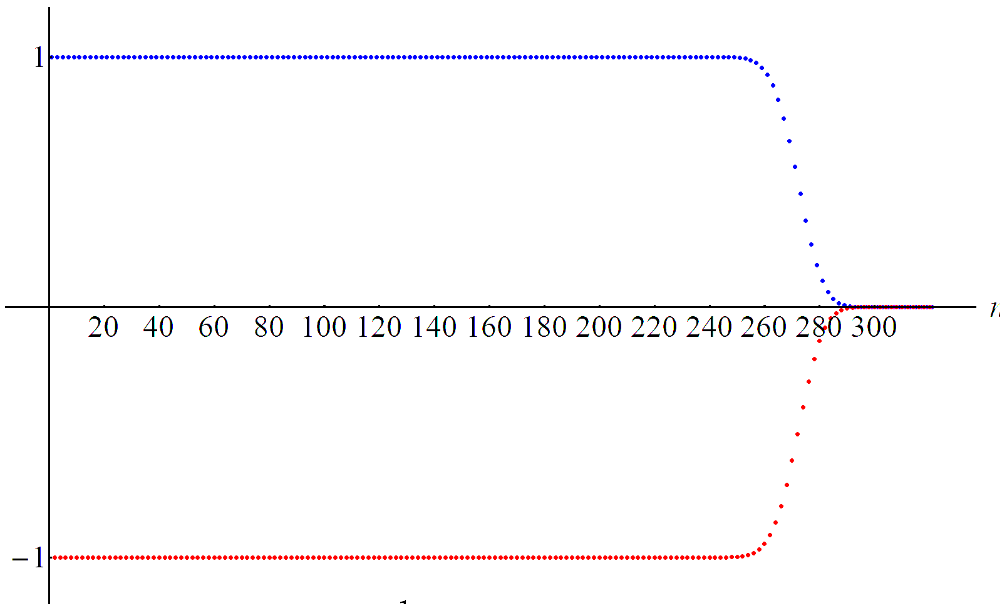
Normalized coefficients $\delta_{169,n}$ with logarithmic scale



$$\delta_{169,n} \approx 1 + \mu_{169,2}\mathrm{dom}_2(n) + \lambda_{169}\log(n) \qquad (111)$$

But with increasing N, splitting into two lines becomes typical.

Normalized coefficients $\delta_{321,n}$



$$\delta_{321,n} \approx (-1)^{n-1} \quad (113)$$

$$\approx 1 + \mu_{321,2}\mathrm{dom}_2(n) \qquad \text{with } \mu_{321,2} \approx -2 \quad (114)$$

Skipping again many pictugraphs, we consider a new pattern exhibited for $N = 321$ – now the initial coefficients alternate between approximately $+1$ and -1.

Partial explanation. II

The function $\Delta_{321}(t)/\tilde{\delta}_{321,n} = \sum_{n=1}^{321} \delta_{321,n}\beta_n(t)$ is a "smooth" truncation, not of the divergent Diriclet series

$$\Xi(t) = \sum_{n=1}^{\infty} \beta_n(t) \tag{115}$$

but of the convergent (for real t) alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\beta_n(t) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\alpha_n(t)+\alpha_n(-t)}{2} \tag{116}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\alpha_n(t) \quad = \quad \sum_{n=1}^{\infty}(-1)^{n-1}g\left(\tfrac{1}{2}+\tfrac{\mathrm{i}t}{2}\right)n^{-\left(\frac{1}{2}+\mathrm{i}t\right)} \tag{117}$$

$$= g\left(\tfrac{1}{2}+\tfrac{\mathrm{i}t}{2}\right)\underbrace{\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\left(\frac{1}{2}+\mathrm{i}t\right)}}_{\eta\left(\frac{1}{2}+\mathrm{i}t\right)} = g\left(\tfrac{1}{2}+\tfrac{\mathrm{i}t}{2}\right)\left(1-2\cdot 2^{-\left(\frac{1}{2}-\mathrm{i}t\right)}\right)\zeta\left(\tfrac{1}{2}+\mathrm{i}t\right)$$

Euler used the function $\eta(s)$ to assign a value to the zeta function outside the area of convergence of the Dirichlet series, and that convergence gives a second "reason" (besides the smoothness of the truncation) why Δ_{321} is so good for predicting the values $\gamma_{321}, \gamma_{322}, \dots$

It is very remarkable that here the function $\eta(s)$ emerges by itself, just from our calculation of the determinants $\Delta_N(t)$ as though they were as clever as Euler.

Still some natural questions remain open. The switching from divergent (115) to convergent (116) partly "explains" why $\Delta_{321}(t)$ is able to "predict" the values of γ_{321} and further zeroes of $\Xi(t)$ but *why does this also happen for, say,* $N = 47$*?* The series (117) converges on the critical line rather slowly, so *why are the zeroes of* $\Delta_{321}(t)$ *so close to those of* $\Xi(t)$*?*

Slide 44:

Functions $\eta_N(s)$

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} \tag{119}$$

$$\eta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s} \tag{120}$$

$$\Delta_N(t) = \frac{1}{2}\tilde{\delta}_{N,1}\Big(g\left(\tfrac{1}{2} - \mathrm{i}t\right)\eta_N\left(\tfrac{1}{2} - \mathrm{i}t\right) + g\left(\tfrac{1}{2} + \mathrm{i}t\right)\eta_N\left(\tfrac{1}{2} + \mathrm{i}t\right)\Big) \tag{121}$$

$$\eta_N(s) \overset{?}{\approx} \eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s) \tag{122}$$

$$\zeta(s) \overset{?}{\approx} \frac{\eta_N(s)}{1 - 2 \cdot 2^{-s}} \tag{123}$$

Based on (113), we introduce functions $\eta_N(s)$ mimicking Euler's function $\eta(s)$. Shouldn't (122) be true? If so, than we'll be able to calculate $\zeta(s)$ via $\eta_N(s)$.

Calculation of $\zeta(s)$

For $s = \frac{1}{2} + 100\mathrm{i}$
$$\left| \frac{\zeta(s)}{\frac{\eta_{321}(s)}{1-2\cdot 2^{-s}}} - 1 \right| = +1.34\ldots \cdot 10^{-12} \tag{124}$$

$$\left| \frac{\zeta(s)}{\frac{\eta_{600}(s)}{1-2\cdot 2^{-s}}} - 1 \right| = -2.95\ldots \cdot 10^{-24} \tag{125}$$

For $s = \frac{1}{4} + 100\mathrm{i}$
$$\left| \frac{\zeta(s)}{\frac{\eta_{600}(s)}{1-2\cdot 2^{-s}}} - 1 \right| = -2.64\ldots \cdot 10^{-24} \tag{126}$$

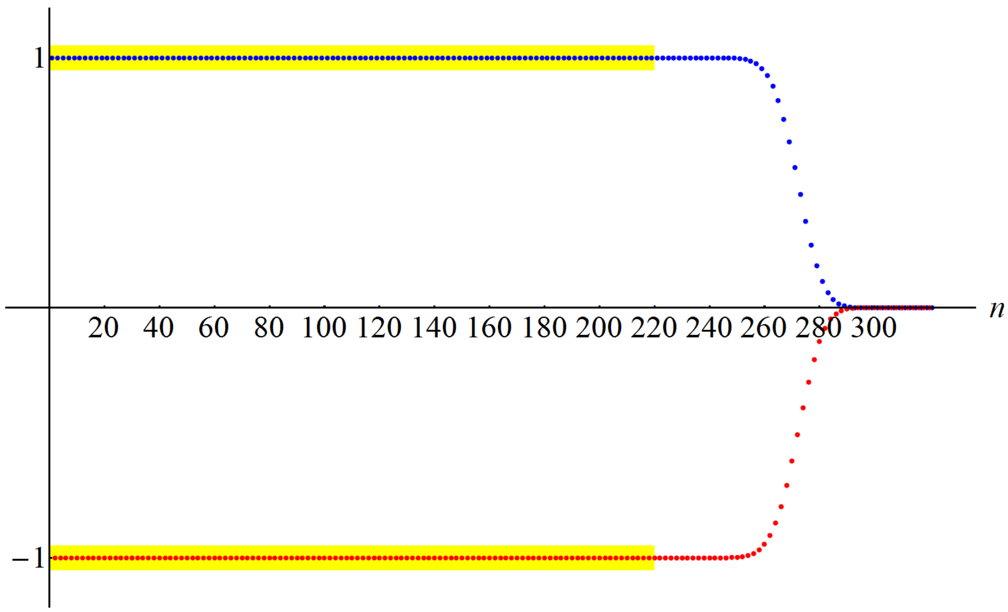
$$\left| \frac{\zeta(s)}{\frac{\eta_{3000}(s)}{1-2\cdot 2^{-s}}} - 1 \right| = +1.19\ldots \cdot 10^{-125} \tag{127}$$

For $s = \frac{1}{4} + 1000\mathrm{i}$
$$\left| \frac{\zeta(s)}{\frac{\eta_{3000}(s)}{1-2\cdot 2^{-s}}} - 1 \right| = +4.07\ldots \cdot 10^{-126} \tag{128}$$

Indeed, such calculations turned out to be possible. For $N = 321$ we get 12 correct decimal digits of $\zeta(s)$, and 24 correct decimal digits in the case $N = 600$. However, earlier (Slides 27 and Slide 29) we got for the same values of N many more correct decimal digits of the γ's. How is this possible?

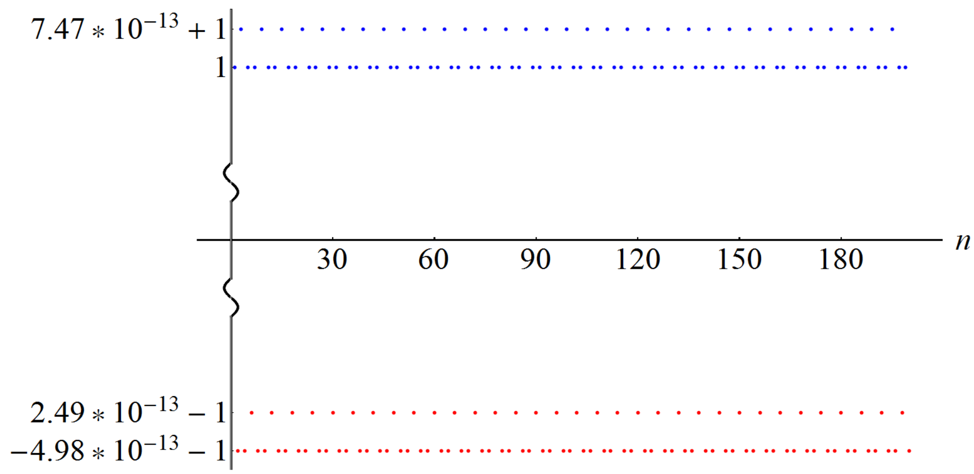
Slide 46:

Normalized coefficients $\delta_{321,n}$



Let us have a closer look at the areas marked in yellow.

Normalized coefficients $\delta_{321,n}$



$$\delta_{321,n} \approx 1 + \mu_{321,2}\mathrm{dom}_2(n) + \mu_{321,3}\mathrm{dom}_3(n) + \lambda_{321}\log(n) \tag{129}$$

$$\mu_{321,2} = -2 - 4.98\ldots \cdot 10^{-13} \qquad \mu_{321,3} = 7.47\ldots \cdot 10^{-13} \tag{130}$$

$$\lambda_{321} = -3.33\ldots \cdot 10^{-18} \tag{131}$$

We observe that actually the plot splits into 4 parallel lines so the initial normalized coefficients are better approximated by (129).

Normalized coefficients: general case

$$\delta_{N,n} \approx \sum_{m=1}^{M} \mu_{N,m} \mathrm{dom}_m(n) + \lambda_N \log(n) \qquad (132)$$

$$\delta_{N,n} = \sum_{m=1}^{n} \mu_{N,m} \mathrm{dom}_m(n) = \sum_{m|n} \mu_{N,m} \qquad n = 1, 2, \ldots \qquad (133)$$

$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta_{N,n} \qquad (134)$$

Möbius function

$$\mu(k) = \begin{cases} (-1)^m, & \text{if } k \text{ is the product of } m \text{ different primes} \\ 0, & \text{otherwise} \end{cases} \qquad (135)$$

$$\frac{1}{\zeta(s)} = \frac{1}{\sum_{n=1}^{\infty} n^{-s}} = \sum_{n=1}^{\infty} \mu(n) n^{-s} \qquad \sum_{n=1}^{\infty} n^{-s} \sum_{n=1}^{\infty} \mu(n) n^{-s} = 1 \qquad (136)$$

In fact, there are similar even smaller splittings depending on the values of n modulo 4, 5, . . . , so we need to use (132).

To give a formal definition of $\mu_{N,n}$ we ignore the logarithmic term in (132) and demand that it to be exact equality when $M = n$. The resulting triangular linear system (133) has solution (134).

The Möbius function also plays an important role in the theory of the zeta-function due to (136).

Slide 49:

Function $\nu_{N,M}(s)$

M	$\mu_{3000,M}$
1	1
2	$-2 - 4.9\ldots \cdot 10^{-126}$
3	$7.40565\ldots \cdot 10^{-126}$
4	$2.85890\ldots \cdot 10^{-284}$
5	$-1.85782\ldots \cdot 10^{-411}$
6	$-2.56167\ldots \cdot 10^{-509}$
7	$-5.02202\ldots \cdot 10^{-585}$
8	$4.39701\ldots \cdot 10^{-641}$
9	$1.08444\ldots \cdot 10^{-681}$
10	$1.90599\ldots \cdot 10^{-716}$

$$\frac{\eta_N(s)}{\zeta(s)} = \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{\infty} n^{-s}} \tag{137}$$

$$= \sum_{n=1}^{\infty} \mu_{N,n} n^{-s} \tag{138}$$

$$\eta_N(s) = \left(\sum_{n=1}^{\infty} \mu_{N,n} n^{-s}\right) \zeta(s) \tag{139}$$

$$\eta_N(s) \approx (1 - 2 \cdot 2^{-s})\zeta(s) \qquad \eta_N(s) \approx \underbrace{\left(\sum_{n=1}^{M} \mu_{N,n} n^{-s}\right)}_{\nu_{N,M}(s)} \zeta(s) \tag{140}$$

Equivalently, numbers $\mu_{N,n}$ can be defined as the coefficients in the ratio (137).

The small numerical values of $\mu_{3000,n}$ imply that between the initial $\delta_{3000,n}$ there is a number of almost linear relations of the form (134).

To what extent can function $\nu_{N,M}(s)$ play the role of Euler's factor $1 - 2 \cdot 2^{-s}$?

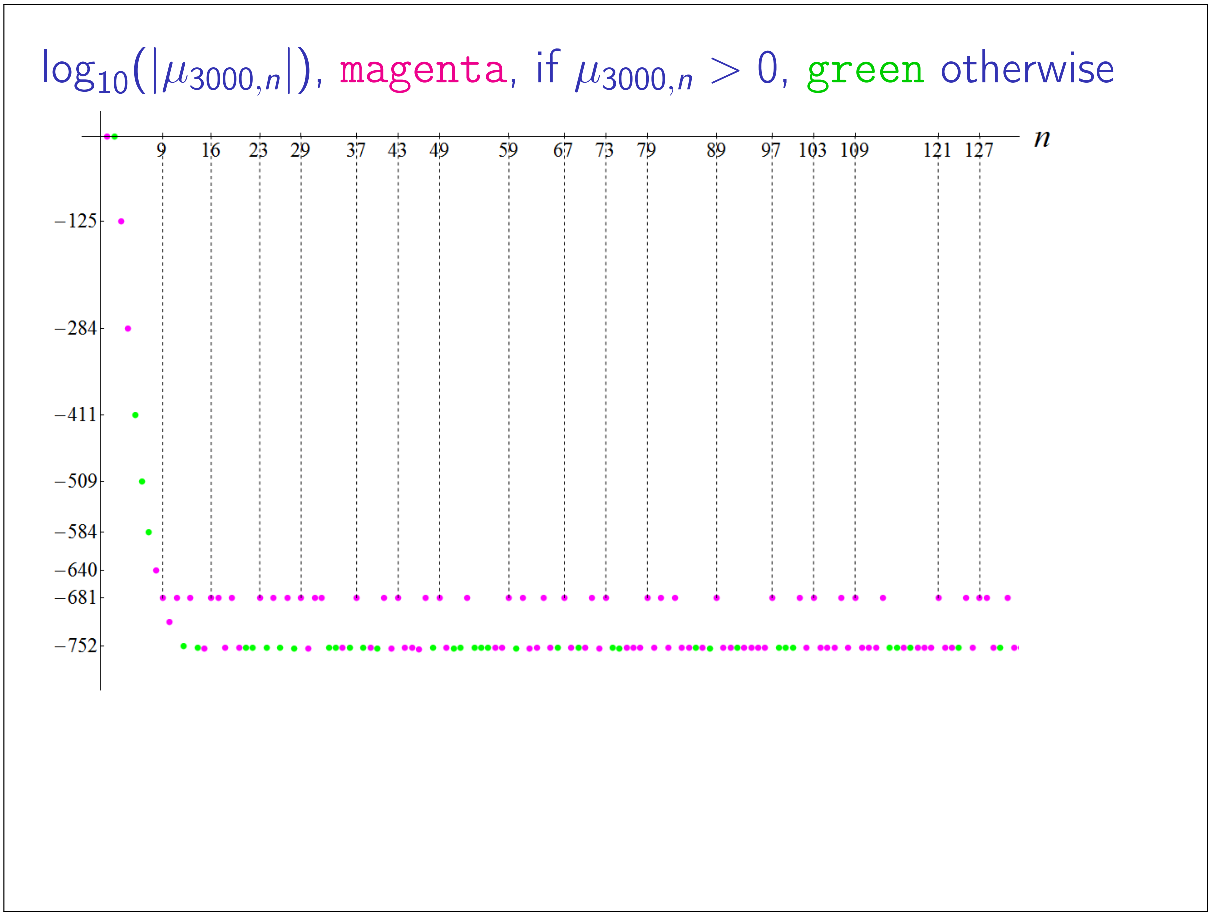
Improved calculation of $\zeta(s)$ for $s = \frac{1}{4} + 1000i$

$$\eta_N(s) \approx \nu_{N,M}(s)\zeta(s) \qquad \nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \tag{141}$$

M	$\mu_{3000,M}$	$\left\lvert \frac{\eta_N(s)}{\nu_{3000,M}(s)\zeta(s)} - 1 \right\rvert$
2	$-2 - 4.9\ldots \cdot 10^{-126}$	$2.46329\ldots \cdot 10^{-126}$
3	$7.40565\ldots \cdot 10^{-126}$	$8.84947\ldots \cdot 10^{-285}$
4	$2.85890\ldots \cdot 10^{-284}$	$5.43870\ldots \cdot 10^{-412}$
5	$-1.85782\ldots \cdot 10^{-411}$	$7.16503\ldots \cdot 10^{-510}$
6	$-2.56167\ldots \cdot 10^{-509}$	$1.35156\ldots \cdot 10^{-585}$
7	$-5.02202\ldots \cdot 10^{-585}$	$1.14450\ldots \cdot 10^{-641}$
8	$4.39701\ldots \cdot 10^{-641}$	$2.13729\ldots \cdot 10^{-681}$
9	$1.08444\ldots \cdot 10^{-681}$	$2.18430\ldots \cdot 10^{-681}$
10	$1.90599\ldots \cdot 10^{-716}$	$2.18430\ldots \cdot 10^{-681}$
11	$2.37291\ldots \cdot 10^{-681}$	$2.56047\ldots \cdot 10^{-681}$

For $M = 2$ we have the almost Euler-like approximation (123).
Increasing M up to 8 greatly improves the accuracy.
But further increasing the value of M doesn't lead to better approximations. The reason is as follows: the (absolute values of) $\mu_{3000,M}$ stop decreasing at $M = 10$.

Slide 51:



For $n = 2 \dots 10$ the absolute values of $\mu_{3000,n}$ decrease, and their logarithms lie (why?) on a smooth curve in spite of the fact that the values themselves have different signs.

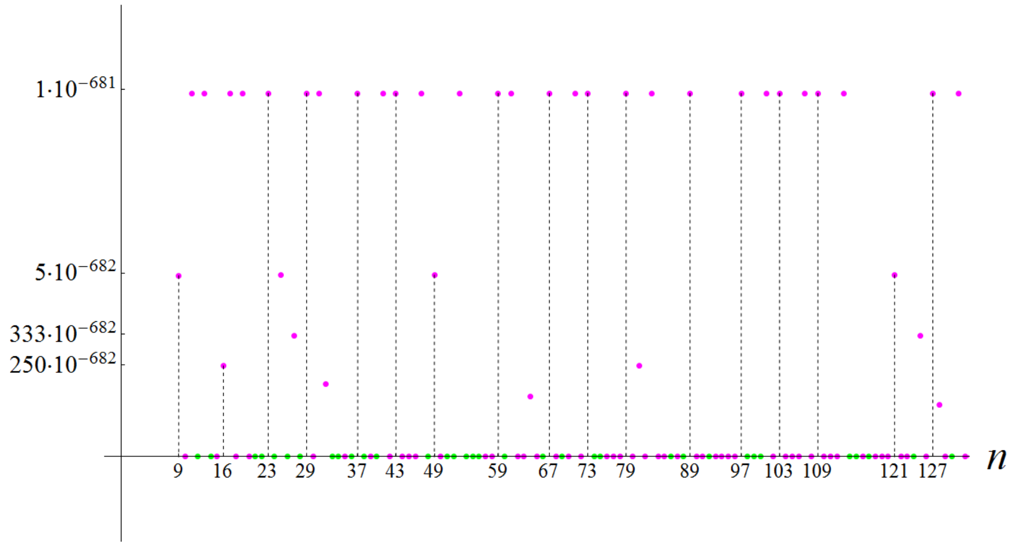
Starting from $n = 11$ the absolute values of $\mu_{3000,n}$ oscillate between approximately 10^{-681} and 10^{-752}. The values of n corresponding to the first of these two cases are all primes or powers of primes, and the corresponding values of $\mu_{3000,n}$ are all positive.

Slide 52:

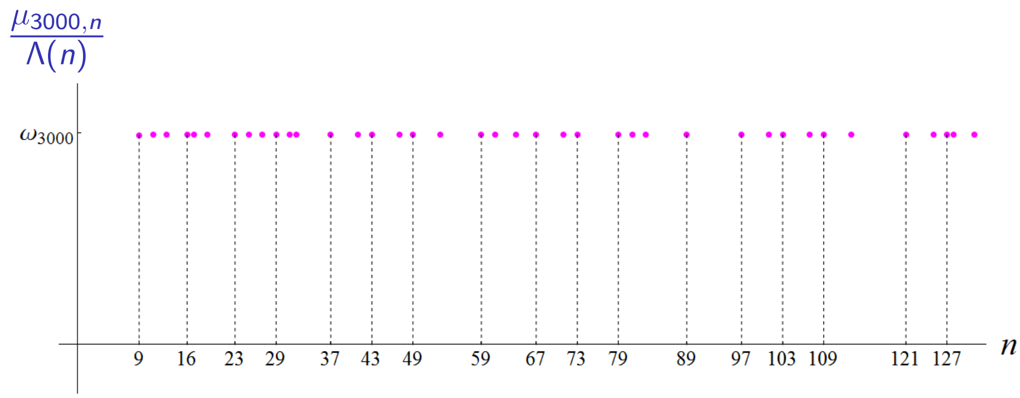
Closer look at the values of $\mu_{3000,n}$ for prime and prime power values of n reveals finer structure.

Slide 53:

$\frac{\mu_{3000,n}}{\ln(n)}$, magenta, if $\mu_{3000,n} > 0$, green otherwise



This structure becomes more transparent when we normalize the μ's by $\ln(n)$.



$$\omega_{3000} = \frac{\mu_{3000,13}}{\ln(13)} = 9.895811\ldots \cdot 10^{-682} \tag{142}$$

$$\left| \frac{\mu_{3000,p^k} / \ln(p)}{\omega_{3000}} - 1 \right| < 3.85\ldots \cdot 10^{-73} \text{ for } 13 \le p^k \le 419, p \text{ is a prime}$$

$$\left| \frac{\mu_{3000,p^k}}{\ln(p)} - \omega_{3000} \right| < 3.81\ldots \cdot 10^{-754} \text{ for } 13 \le p^k \le 419, p \text{ is a prime}$$

It is even better to normalize by von Mangoldt's $\Lambda(n)$ (defined on Slide 17).

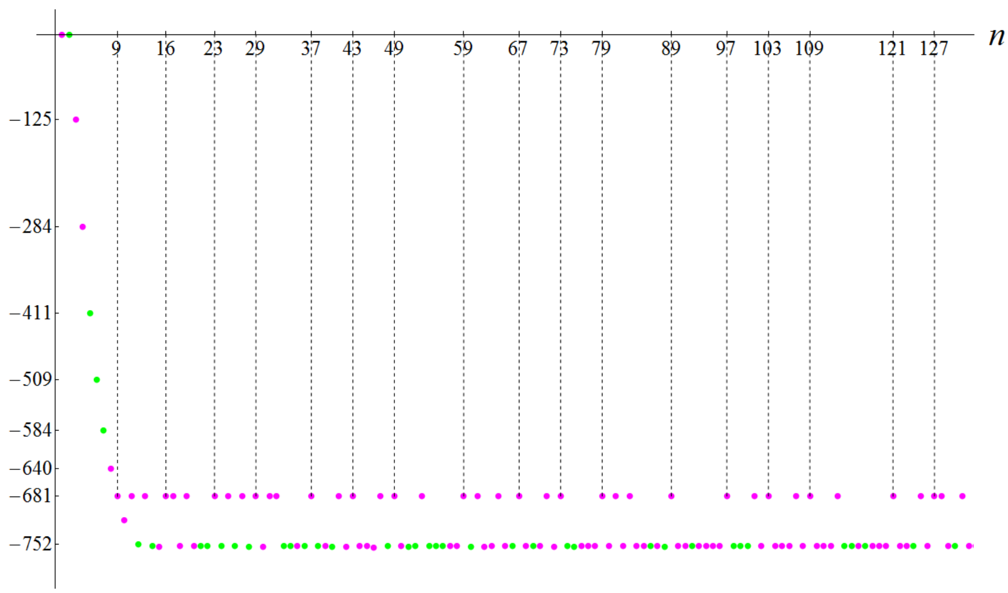
For a long range of prime and prime power values of n the value of the ratio $\frac{\mu_{3000,n}}{\Lambda(n)}$ remains almost constant. For non-prime-power values of n in the same range the values of $|\mu_{3000,n}|$ are more than 70 decimal orders smaller than ω_{3000}, that is, $\mu_{3000,n}$ approximates very well (a multiple of) von Mangoldt's function $\Lambda(n)$.

The connections between the zeta-function and prime numbers is usually attributed to the Euler product (42); however, in the present case it isn't clear how to establish a connection between the Euler product and the values of $\mu_{3000,n}$ in this range.

Another intriguing question is: why are the values of $\mu_{3000,n}$ for a few initial values of n so different? Their structure was analyzed to some extent in

`http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod/talks/leicester2012/leicester_2012_full.pdf`.

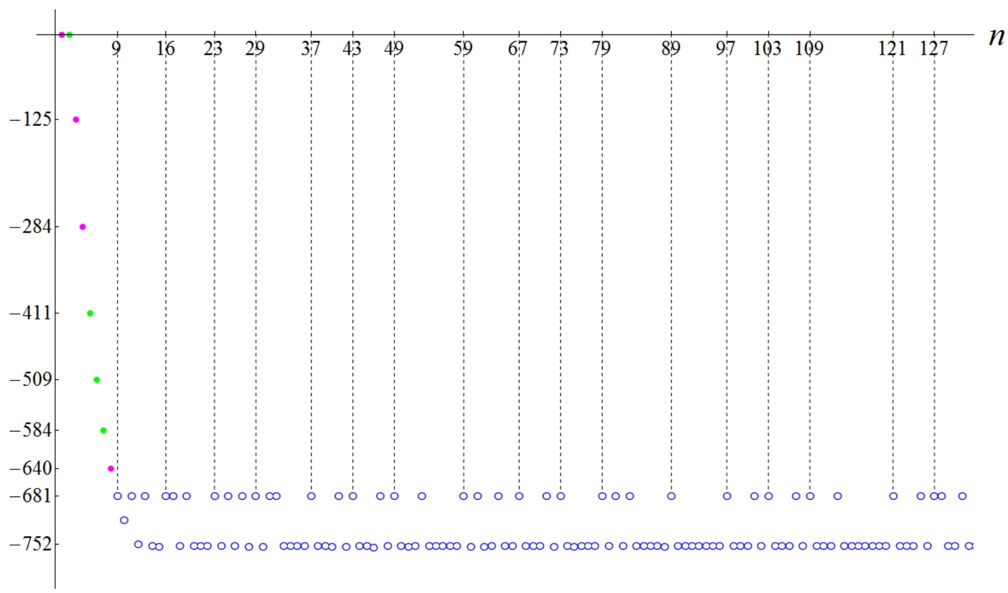
Slide 55:



$$\frac{\eta_{3000}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \mu_{3000,n} n^{-s} \tag{143}$$

This is just a definition of the numbers $\mu_{3000,n}$ (case $N = 3000$ of (138)).

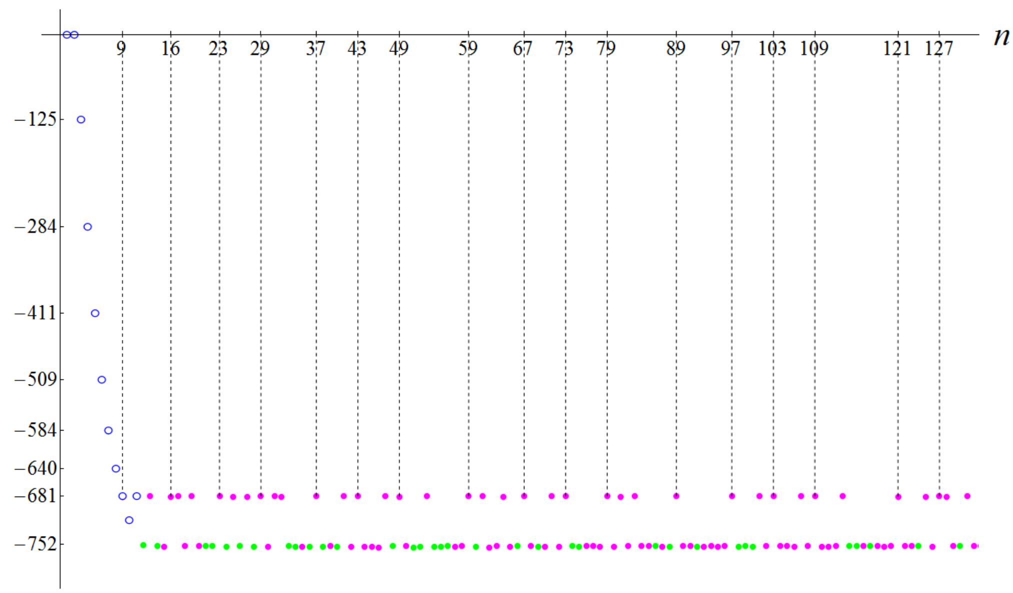
Slide 56:



$$\frac{\eta_{3000}(s)}{\zeta(s)} \approx \sum_{n=1}^{8} \mu_{3000,n} n^{-s}$$

Filling in the table on Slide 50, we dropped all summands in (143) except for a few initial terms, say, the first eight.

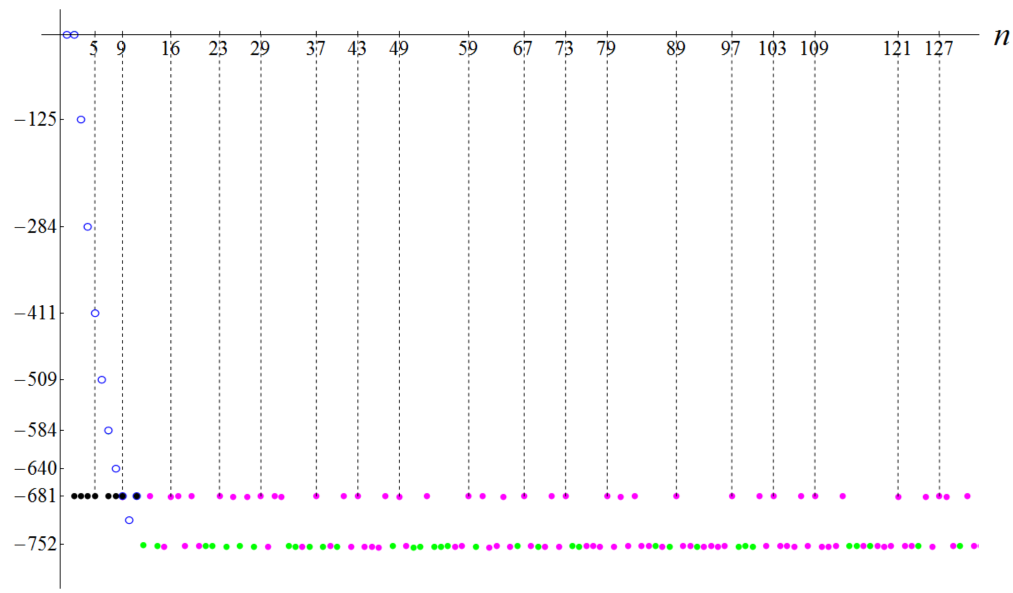
Slide 57:



$$\frac{\eta_{3000}(s)}{\zeta(s)} - \sum_{n=1}^{11} \mu_{3000,n} n^{-s}$$

Let us now do the opposite – drop the first 11 summands from (143)

Slide 58:



$$\frac{\eta_{3000}(s)}{\zeta(s)} - \sum_{n=1}^{11} \big(\underbrace{\mu_{3000,n} - \omega_{3000}\Lambda(n)}_{\tilde{\mu}_{3000,n}} \big) n^{-s} \approx \omega_{3000} \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \qquad (145)$$

$$\stackrel{?}{\approx} -\omega_{3000} \frac{\zeta'(s)}{\zeta(s)}$$

and then add a few "missing" summands for $n = 2, 3, 4, 5, 7, 8, 9, 11$. Taking into account the fact that for non-prime-power values of n the values of $|\mu_{3000,n}|$ are more than 70 decimal orders smaller than ω_{3000}, we see that the initial summands in the left-hand side of (145) (with $\frac{\eta_{3000}(s)}{\zeta(s)}$ replaced by the right-hand side of (143)) are very close to the initial summands in the right-hand side. Shouldn't we expect approximate equality of the numerical values of the left- and right-hand sides of (145)?

Recall that according to (65), the right-hand side of (145) is just a multiple of the logarithmic derivative of the zeta function.

Slide 59:

Calculating zeta derivative at zeros

$$\frac{\eta_{3000}(s)}{\zeta(s)} - \sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s} \stackrel{?}{\approx} -\omega_{3000} \frac{\zeta'(s)}{\zeta(s)} \tag{147}$$

$$\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s} \right) \zeta(s) \stackrel{?}{\approx} -\omega_{3000} \zeta'(s) \tag{148}$$

$$\eta_{3000}\left(\tfrac{1}{2} + \mathrm{i}\gamma_k\right) \stackrel{?}{\approx} -\omega_{3000} \cdot \zeta'\left(\tfrac{1}{2} + \mathrm{i}\gamma_k\right) \tag{149}$$

$$\left| \frac{\eta_{3000}\left(\frac{1}{2} + \mathrm{i}\gamma_{100}\right)}{-\omega_{3000}\zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_{100}\right)} - 1 \right| = 1.024\ldots \cdot 10^{-36} \tag{150}$$

$$\left| \frac{\eta_{3000}\left(\frac{1}{2} + \mathrm{i}\gamma_{500}\right)}{-\omega_{3000}\zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_{500}\right)} - 1 \right| = 2.786\ldots \cdot 10^{-74} \tag{151}$$

To begin with, let us check our guess at a pole of $\zeta(s)$ when (148) simplifies to (149). Indeed, we get a lot of correct digits. This is rather surprising for two reasons. First, we are calculating outside the semiplane $\mathrm{Re}(s) > 1$ of convergence of (65). Second, passing from (147) to (148) we multiplied both sides by $\zeta(s)$; but in our case its value is zero.

Slide 60:

Calculating zeta derivative

$$\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s}\right) \zeta(s) \overset{?}{\approx} -\omega_{3000}\zeta'(s) \tag{152}$$

$$s = \frac{1}{4} + 1000\mathrm{i} \tag{153}$$

$$\left| \frac{\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s}\right) \zeta(s)}{-\omega_{3000}\zeta'(s)} - 1 \right| = 6.44\ldots \cdot 10^{-73} \tag{154}$$

We can get many digits of $\zeta'(s)$ for s different from a pole as well. However, for this we need to know the value of $\zeta(s)$ with much higher accuracy than that from table on Slide 50.

Slide 61:

Calculating both zeta and its derivative. I

$$\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s}\right) \zeta(s) \approx -\omega_{3000}\zeta'(s) \tag{155}$$

$$\eta_{3500}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3500,n} n^{-s}\right) \zeta(s) \approx -\omega_{3500}\zeta'(s) \tag{156}$$

Solving this system for $s = \frac{1}{4} + 1000\mathrm{i}$ produces 908 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

Nevertherles, we can calculate both $\zeta(s)$ and $\zeta'(s)$ via the δ's and μ's. One way is to use two values of N.

Calculating both zeta and its derivative. II

$$\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s}\right) \zeta(s) \approx -\omega_{3000}\zeta'(s) \quad (157)$$

$$\eta_{3000}(1-s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{s-1}\right) \zeta(1-s) \approx -\omega_{3000}\zeta'(1-s) \quad (158)$$

$$g(s)\zeta(s) = g(1-s)\zeta(1-s) \quad (159)$$

$$g'(s)\zeta(s) + g(s)\zeta'(s) = -g'(1-s)\zeta(1-s) - g(1-s)\zeta'(1-s) \quad (160)$$

Solving this system for $s = \frac{1}{4} + 1000\mathrm{i}$ produces 752 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

Another way is to use the functional equation relating the values of $\zeta(s)$ and $\zeta'(s)$ with values of $\zeta(1-s)$ and $\zeta'(1-s)$.

Near the pole

$$\eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s) \tag{161}$$

$$\eta_N(s) \approx \nu_{N,M}(s)\zeta(s) \qquad \nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \tag{162}$$

M	$\mu_{3000,M}$	$\nu_{3000,M}(1)$
2	$-2 - 4.9\ldots \cdot 10^{-126}$	$-2.46855\ldots \cdot 10^{-126}$
3	$-7.40566\ldots \cdot 10^{-126}$	$-7.14726\ldots \cdot 10^{-285}$
4	$2.85891\ldots \cdot 10^{-284}$	$3.71565\ldots \cdot 10^{-412}$
5	$-1.85782\ldots \cdot 10^{-411}$	$4.26945\ldots \cdot 10^{-510}$
6	$-2.56167\ldots \cdot 10^{-509}$	$7.17431\ldots \cdot 10^{-586}$
7	$-5.02202\ldots \cdot 10^{-585}$	$-5.49626\ldots \cdot 10^{-642}$
8	$4.39701\ldots \cdot 10^{-641}$	$-5.73467\ldots \cdot 10^{-681}$
9	$1.08444\ldots \cdot 10^{-681}$	$-5.61417\ldots \cdot 10^{-681}$
10	$1.90599\ldots \cdot 10^{-716}$	$-5.61417\ldots \cdot 10^{-681}$
11	$2.37291\ldots \cdot 10^{-681}$	$-5.39845\ldots \cdot 10^{-681}$

The role played by Euler's factor $1 - 2 \cdot 2^{-s}$ in (161) was to cancel the pole of the zeta function. To what extent does our factor $\nu_{N,M}(s)$ play this role?

The small values of the ν's imply an almost linear relation between the δ's, the coefficients being rational numbers with small denominators.

Slide 64:

Near the pole

$$\eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s) \tag{161}$$

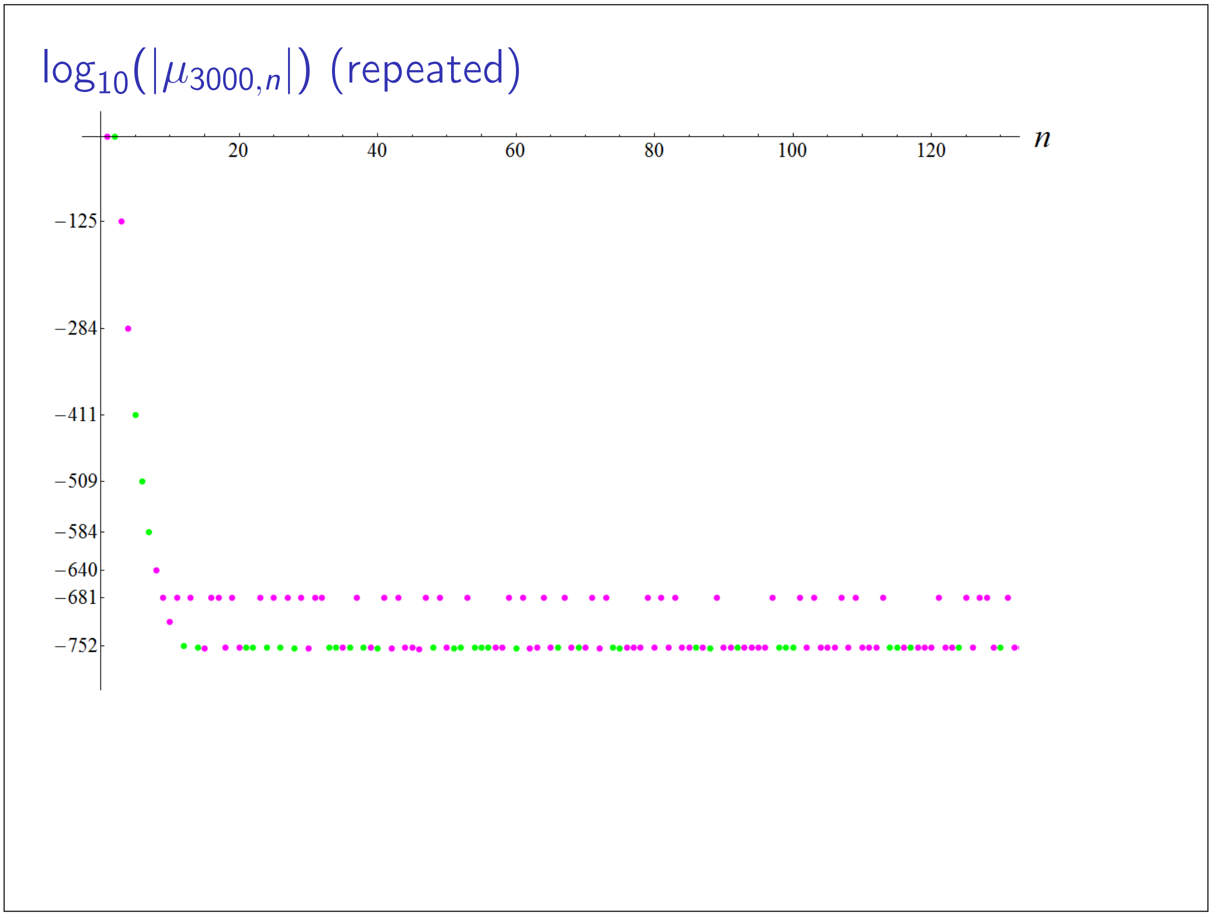
$$\eta_N(s) \approx \nu_{N,M}(s)\zeta(s) \qquad \nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \tag{162}$$

$$\begin{aligned}
\eta(1) &= \lim_{s\to 1}(1 - 2 \cdot 2^{-s})\zeta(s) && (163)\\
&= \lim_{s\to 1}(1 - 2 \cdot 2^{-s})\left(\frac{1}{s-1} + \dots\right) && (164)\\
&= (1 - 2 \cdot 2^{-s})'\Big|_{s=1} && (165)\\
\eta_N(1) &\approx \nu'_{N,M}(1) && (166)
\end{aligned}$$

$$\frac{\eta_{3000}(1)}{\nu'_{3000,9}(1)} = 1 - 4.1515... \cdot 10^{-680} \tag{167}$$

In spite of the fact that the function $\nu_{3000,M}(s)$ doesn't vanish as $s = 1$, its derivative approximates $\eta_{3000}(1)$ very well.

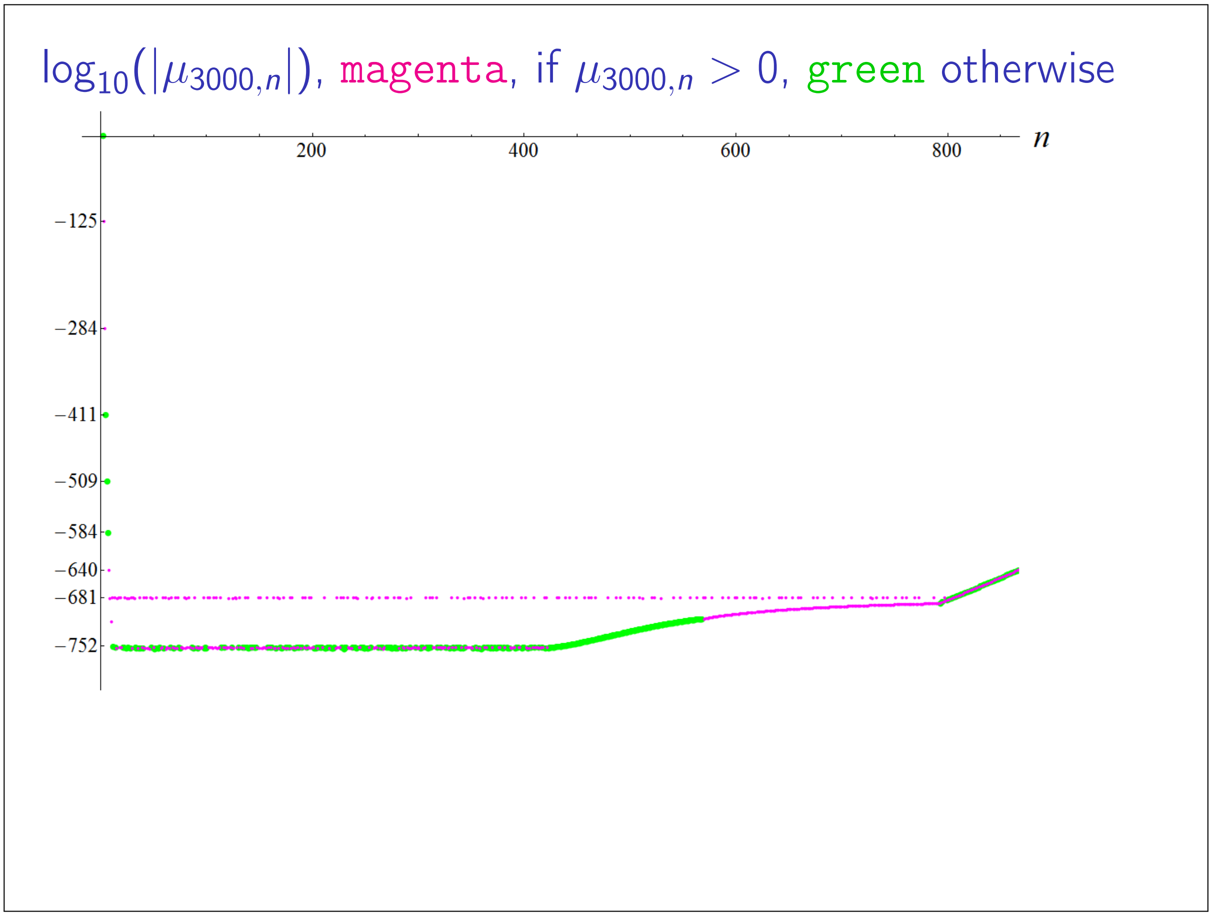
Slide 65:



We have analyzed the fine structure of numbers $\mu_{N,n}$ for prime and prime power values of n. Is there any interesting pattern for other values of n? For example, for such n the value of $\mu_{N,n}$ can be either positive or negative, and the sign depends on both n and N. Is there any simple rule for determining the signs?

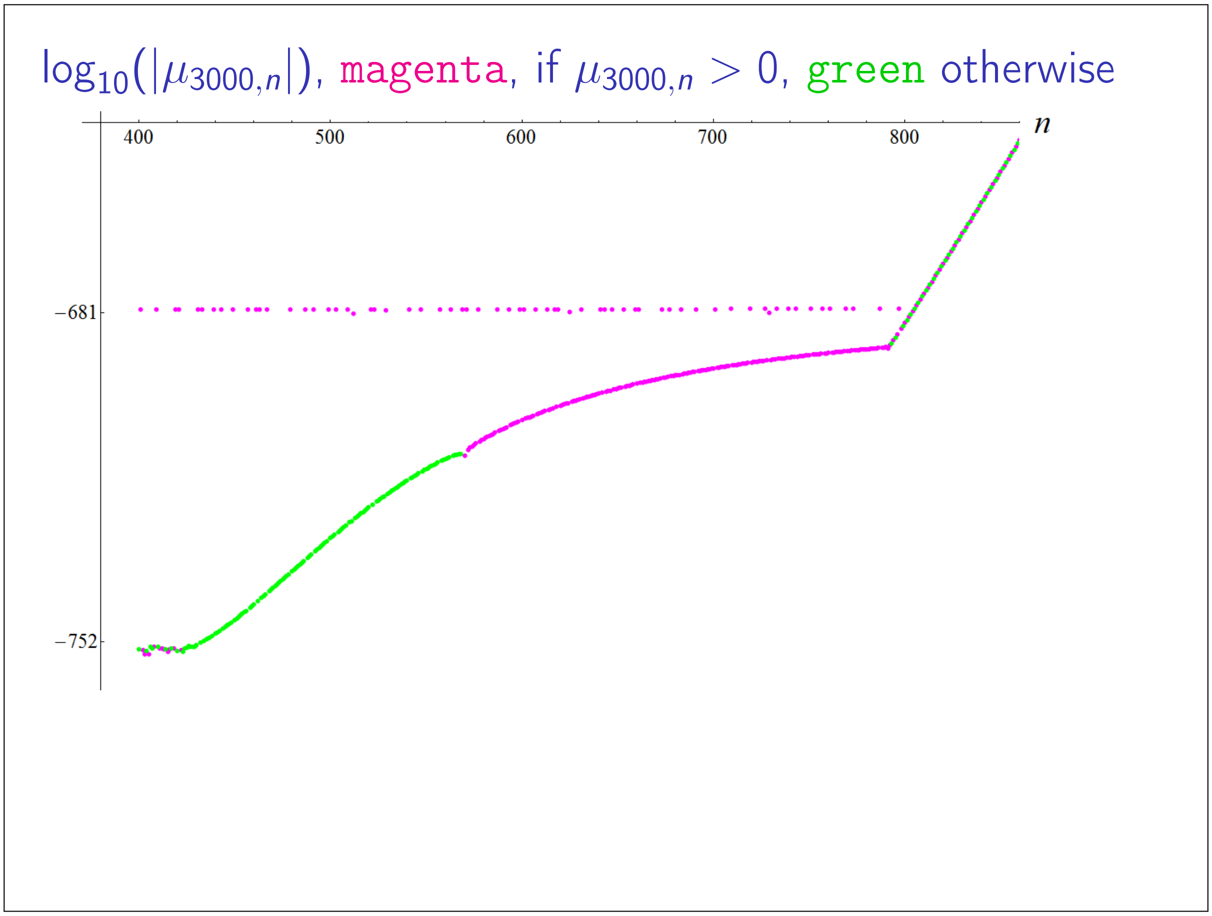
Clearly, $\mu_{3000,n}$ cannot distinguish prime powers from the other integers for all n,

Slide 66:



so let us look at larger values of n.

Slide 67:



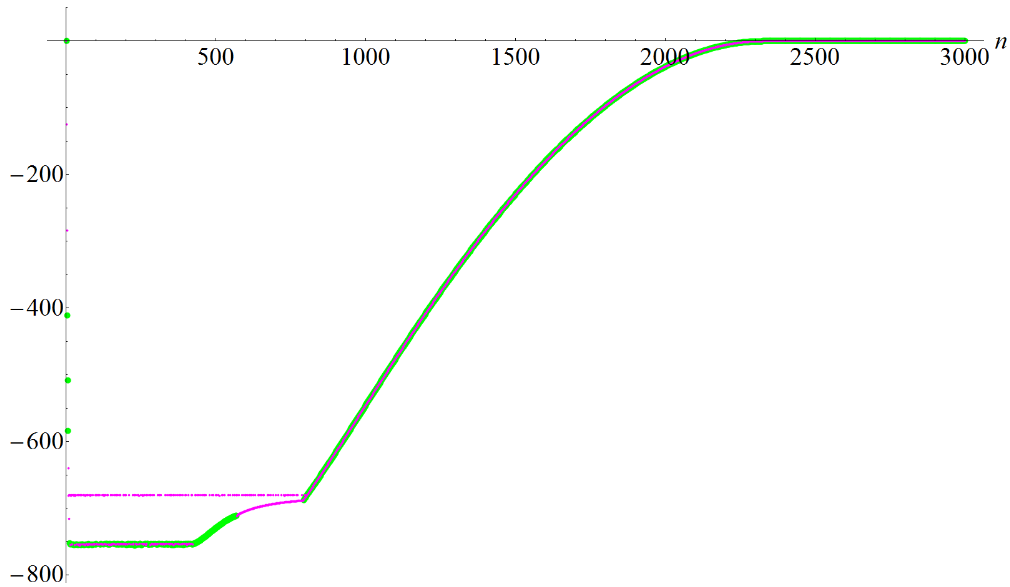
While after $n = 420$ the absolute values of $\mu_{3000,n}$ for non-prime-power values of n begin to increase, for prime power values of n the values of $\mu_{3000,n}$ continue to stay close to 10^{-681}. We can clearly see points for $n = 512,\ 625,\ 729$.

Why does the relatively slow growth of $|\mu_{3000,n}|$ suddenly change to much faster growth after $n = 791$?

Why is there no distinction between prime power and non-powers after $n = 797$?

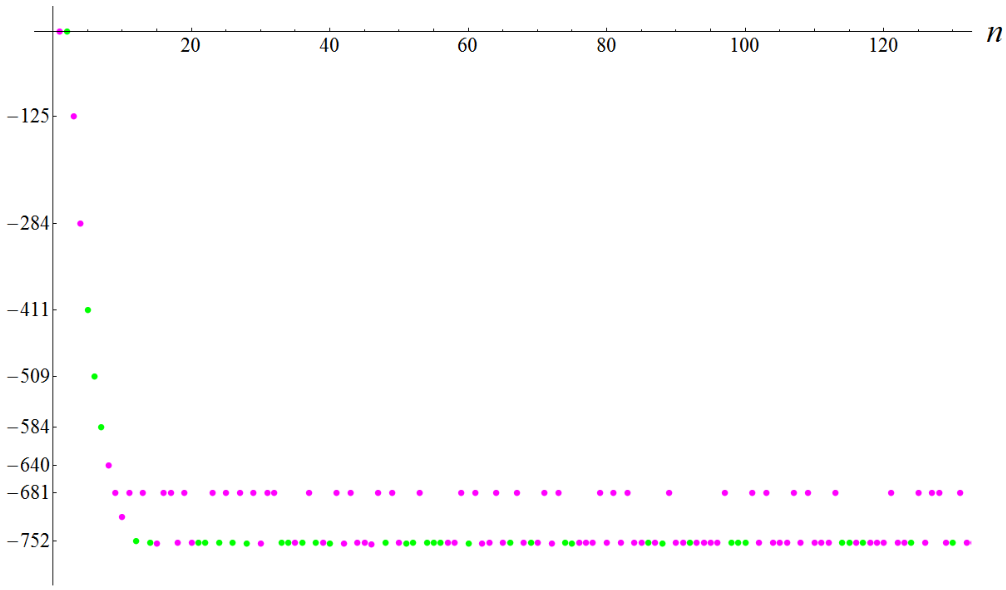
Slide 68:

$\log_{10}(|\mu_{3000,n}|)$, magenta, if $\mu_{3000,n} > 0$, green otherwise



Slide 69:

$\log_{10}(|\mu_{3000,n}|)$ (repeated)



$$\mu_{3000,n} \approx \omega(3000)\Lambda(n) \qquad \psi(x) = \sum_{n=1}^{x} \Lambda(n) \tag{168}$$

Could we use the μ's for studying the growth of $\psi(x)$? To this end we would need to estimate certain sums with $\mu_{N,n}$.

Slide 70:

$\nu_{N,M}(s)$

$$\nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \qquad \mu_{N,n} = \sum_{m|n} \mu\left(\tfrac{n}{m}\right) \delta_{N,n} \tag{169}$$

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}} \qquad \tilde{\delta}_{N,n} = (-1)^{N+n} \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_{n-1}(\gamma_1) & \dots & \beta_{n-1}(\gamma_{N-1}) \\ \beta_{n+1}(\gamma_1) & \dots & \beta_{n+1}(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) \end{vmatrix} \tag{170}$$

$$\beta_n(t) = g\left(\tfrac{1}{2} - \mathrm{i}t\right) n^{-\left(\frac{1}{2} - \mathrm{i}t\right)} + g\left(\tfrac{1}{2} + \mathrm{i}t\right) n^{-\left(\frac{1}{2} + \mathrm{i}t\right)} \tag{171}$$

$$g(s) = \pi^{-\frac{s}{2}} (s-1) \Gamma\left(\tfrac{s}{2} + 1\right) 0 < \gamma_1 < \gamma_2 < \dots \tag{172}$$

Can we expect interesting properties of the individual values of numbers $\nu_{N,M}(s)$ defined in this complicated way?

Slide 71:

$\nu_{N,N}(0)$

$$\nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \tag{173}$$

$$\nu_{N,M}(0) = \sum_{n=1}^{M} \mu_{N,n} \tag{174}$$

$$\begin{aligned} \nu_{3000,3000}(0) &= -\tfrac{1}{2} + 1.23\ldots \cdot 10^{-126} \\ \nu_{3001,3001}(0) &= -\tfrac{3}{2} + 6.67\ldots \cdot 10^{-126} \end{aligned} \tag{175}$$

$$\begin{aligned} \nu_{6000,6000}(0) &= -\tfrac{1}{2} - 5.48\ldots \cdot 10^{-166} \\ \nu_{6001,6001}(0) &= -\tfrac{3}{2} - 3.43\ldots \cdot 10^{-166} \end{aligned} \tag{176}$$

The numbers $\nu_{N,N}(0)$ seem to be close to $\frac{1}{2} + (-1)^N$.

Slide 72:

$\nu_{N,N}(1)$

$$\nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \qquad \nu_{N,M}(1) = \sum_{n=1}^{M} \frac{\mu_{N,n}}{n} \tag{177}$$

$$N = 2520 = \mathrm{LCM}(1,2,3,4,5,6,7,8,9,10) \tag{178}$$

$$2N \cdot \nu_{N,N}(1) = 0.9998015873172093... \tag{179}$$

$$= \cfrac{1}{1+\cfrac{1}{5039+\cfrac{1}{2520+\cfrac{1}{1680+\cfrac{1}{1260+\cfrac{1}{1008+\cfrac{1}{840+\cfrac{1}{720+\cfrac{1}{630+\cfrac{1}{560+\cfrac{1}{504+\cfrac{1}{\ddots}}}}}}}}}}}} \tag{180}$$

$5039 = 2N-1,\ 2520 = \frac{2N}{2},\ 1680 = \frac{2N}{3},\ 1260 = \frac{2N}{4},\ 1008 = \frac{2N}{5},$
$840 = \frac{2N}{6},\ 720 = \frac{2N}{7},\ 630 = \frac{2N}{8},\ 560 = \frac{2N}{9},\ 504 = \frac{2N}{10}$

In all other examples the choice of N wasn't crucial, neighbouring values would suit equally well. Here neither $N = 2519$ nor $N = 2521$ would do, because it is important that $2 \cdot 2520$ is divisible by 2,3,...,10. Similar, the pattern doesn't continue because $2 \cdot 2520$ isn't divisible by 11.

Also, this is the only place where the normalization (93) is crucial.

Slide 73:

$\nu_{N,N}(1)$

$$\phi(N) = \frac{1}{2N} \cdot \cfrac{1}{1 + \cfrac{1}{2N - 1 + \cfrac{1}{\frac{2N}{2} + \cfrac{1}{\frac{2N}{3} + \cfrac{1}{\frac{2N}{4} + \cfrac{1}{\frac{2N}{5} + \cfrac{1}{\frac{2N}{6} + \ddots}}}}}}} \tag{181}$$

$$= \frac{1}{2}\psi\left(\frac{N}{2} + 1\right) - \frac{1}{2}\psi\left(\frac{N+1}{2}\right) \tag{182}$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{183}$$

The equivalence of the two definitions of $\phi(N)$ was established by comparing a few initial terms in the asymptotic expansions, I don't have a rigorous proof.

$\nu_{N,N}(1)$

$$\nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \qquad \nu_{N,M}(1) = \sum_{n=1}^{M} \frac{\mu_{N,n}}{n} \tag{184}$$

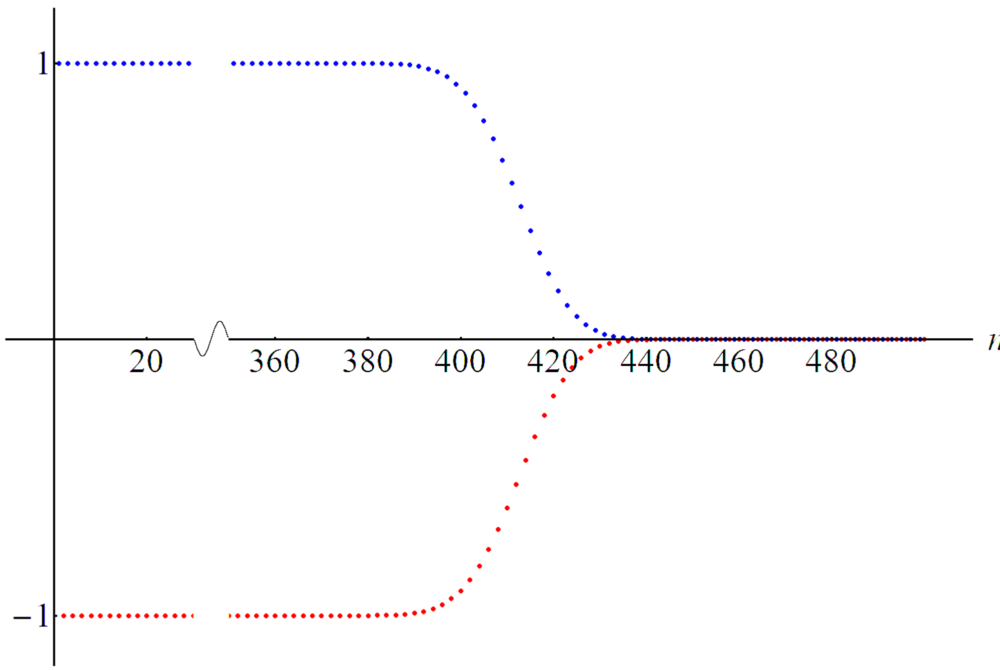
$$\phi(N) = \frac{1}{2}\psi\left(\frac{N}{2}+1\right) - \frac{1}{2}\psi\left(\frac{N+1}{2}\right) \qquad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{185}$$

$$\frac{\nu_{3000,3000}(1)}{\phi(3000)} = 1 - 2.46827839\ldots \cdot 10^{-126} \tag{186}$$

$$\frac{\nu_{6000,6000}(1)}{\phi(6000)} = 1 + 1.09736771\ldots \cdot 10^{-165} \tag{187}$$

Slide 75:

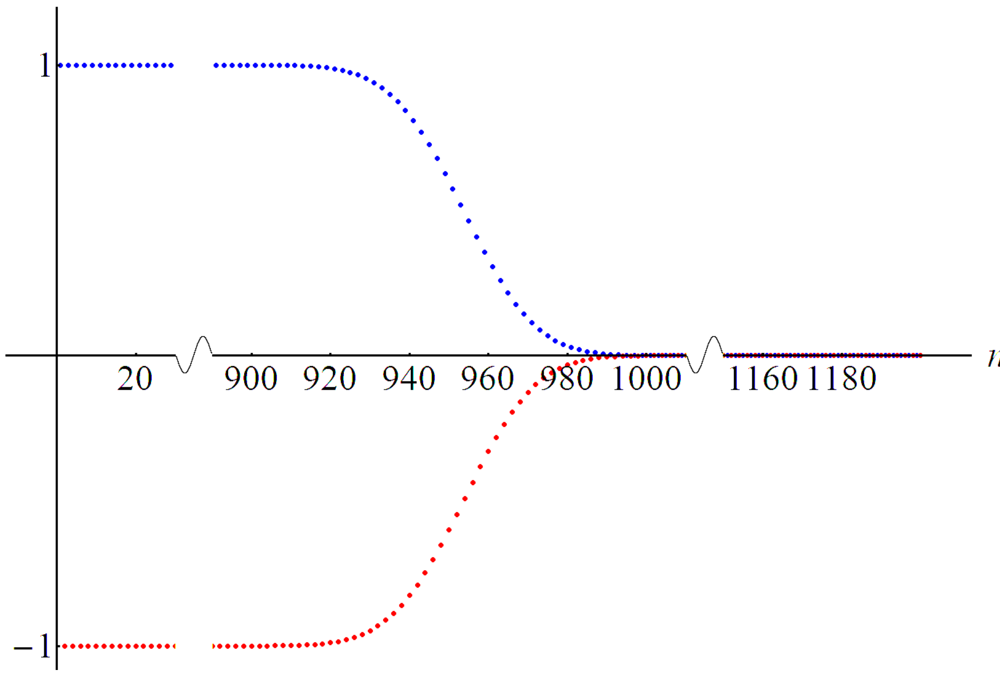
Normalized coefficients $\delta_{500,n}$



This graph looks similar to the case $N = 321$ on Slide 42.

Slide 76:

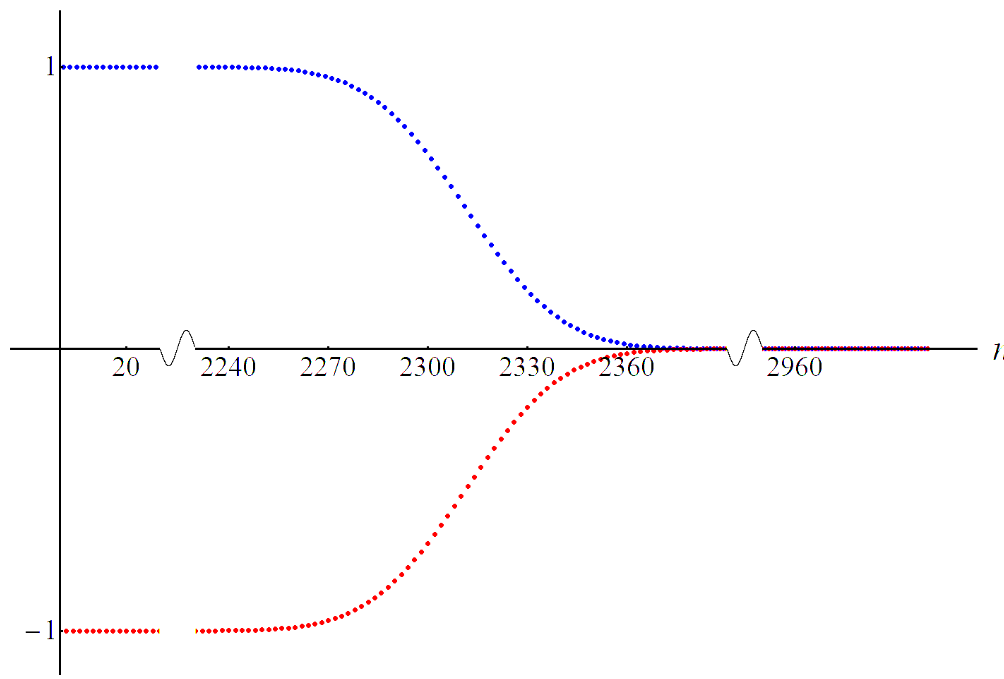
Normalized coefficients $\delta_{1200,n}$



And this pictugraph looks similar to the case $N = 321$ on Slide 42.

Slide 77:

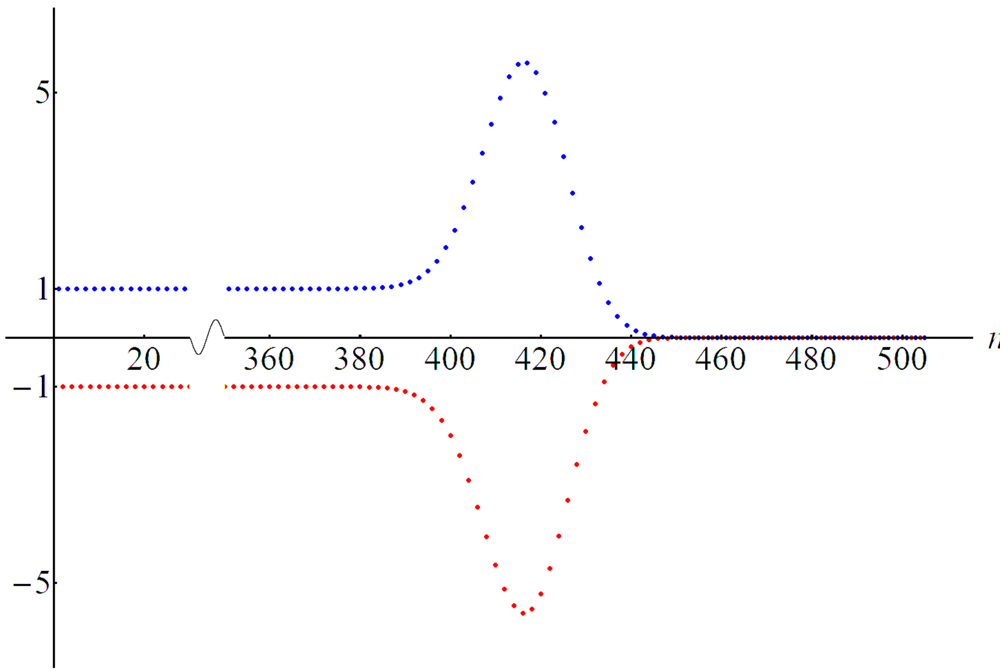
Normalized coefficients $\delta_{3000,n}$



Again similar graph.

Slide 78:

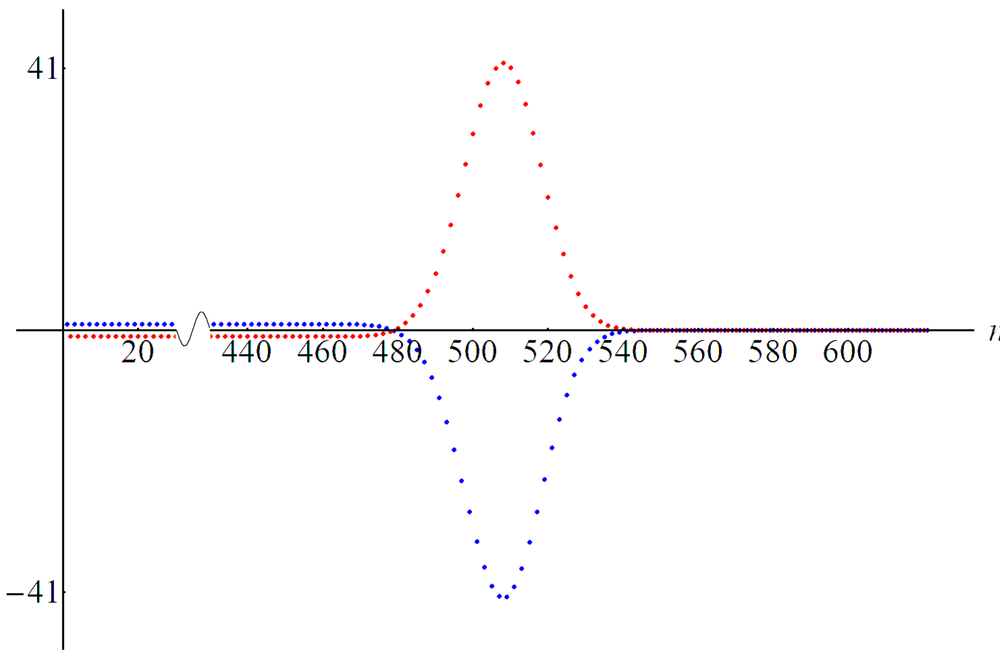
Normalized coefficients $\delta_{505,n}$



This is a sporadic graph.

Slide 79:

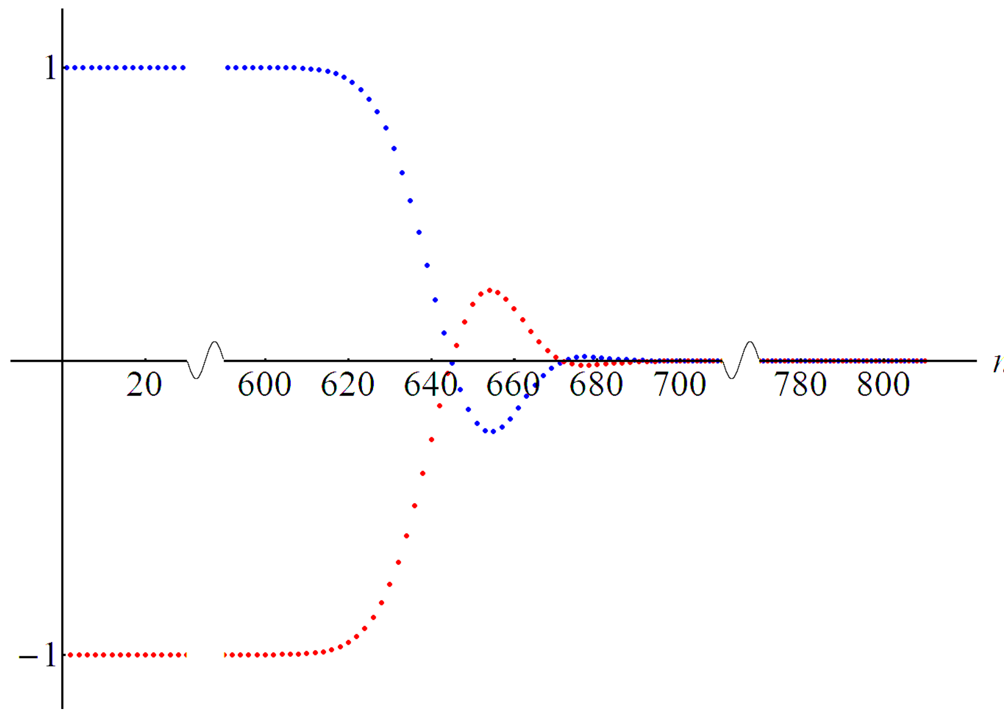
Normalized coefficients $\delta_{621,n}$



And this graph is sporadic.

Slide 80:

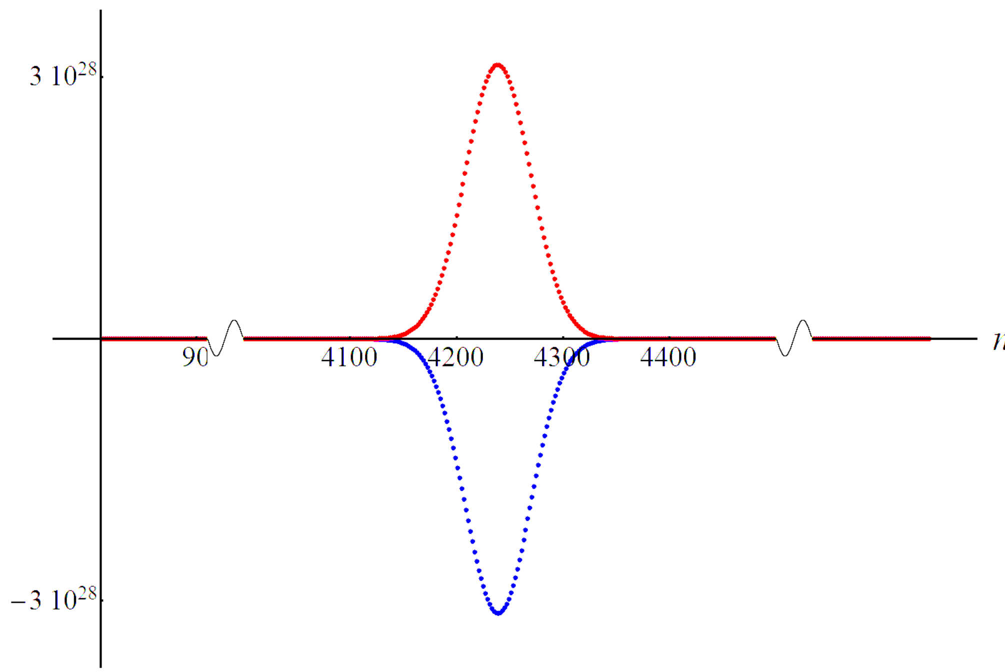
Normalized coefficients $\delta_{810,n}$



Notice multiple changes of sign.

Slide 81:

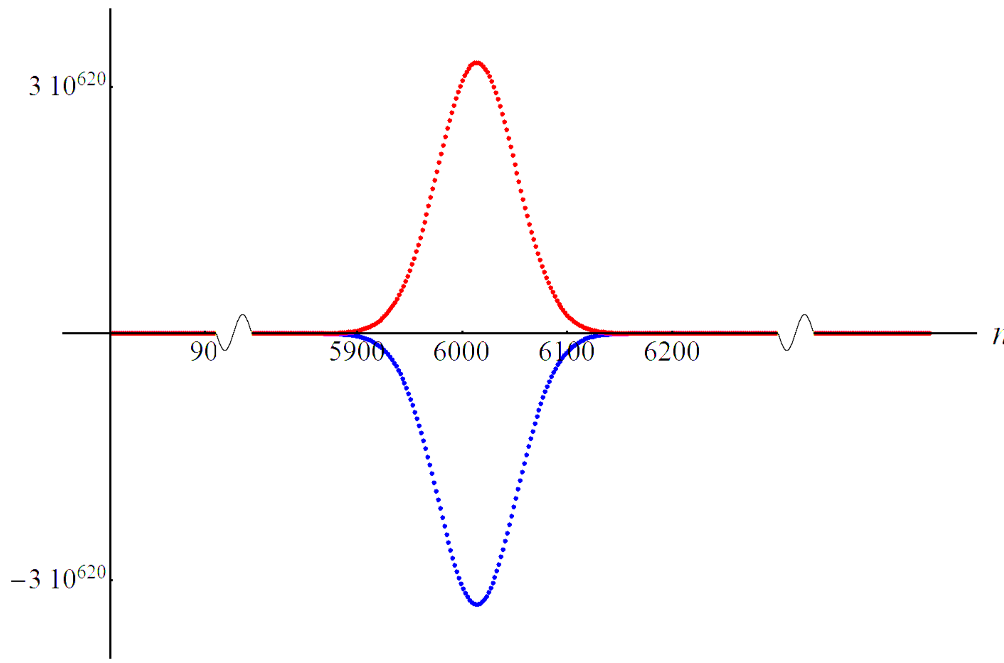
Normalized coefficients $\delta_{5600,n}$



But for large values of N such a graph become typical.

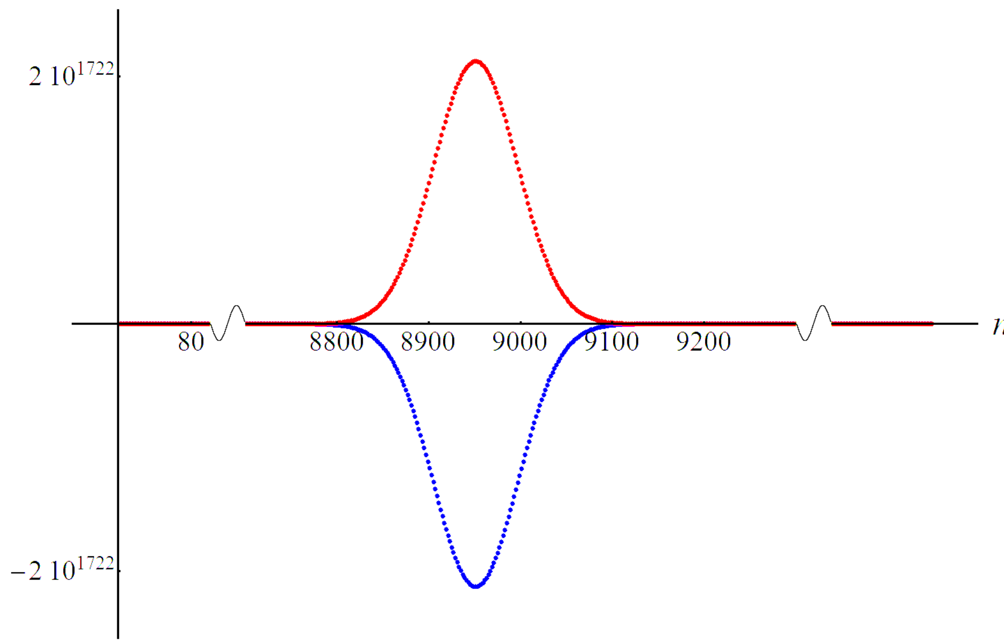
Slide 82:

Normalized coefficients $\delta_{8000,n}$



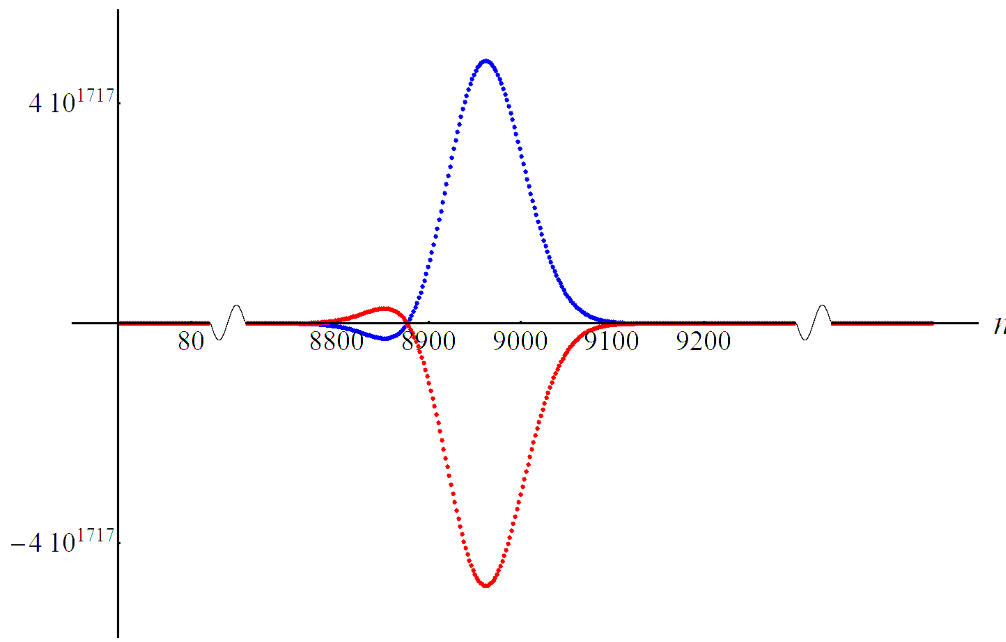
Slide 83:

Normalized coefficients $\delta_{12000,n}$



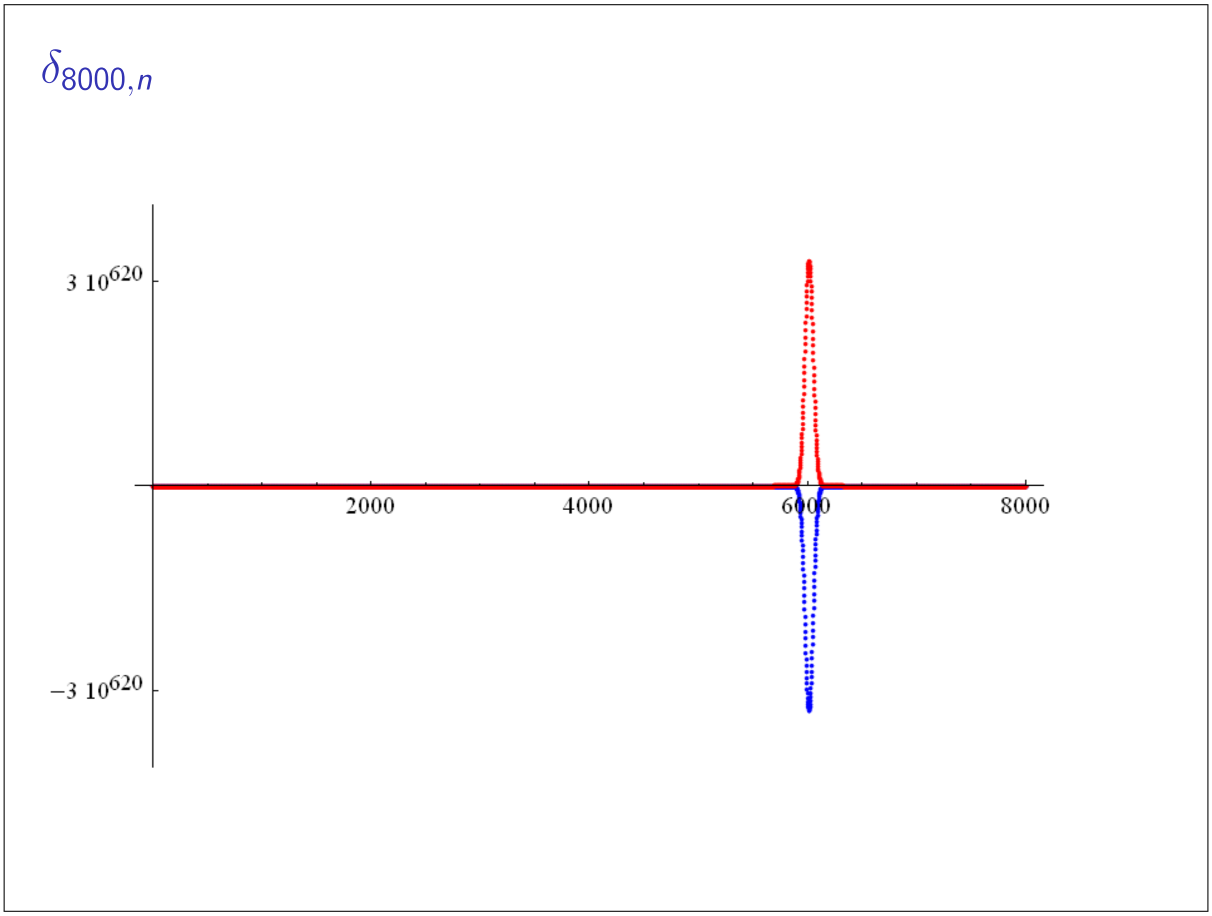
Slide 84:

Normalized coefficients $\delta_{11981,n}$



Again, there are sporadic cases.

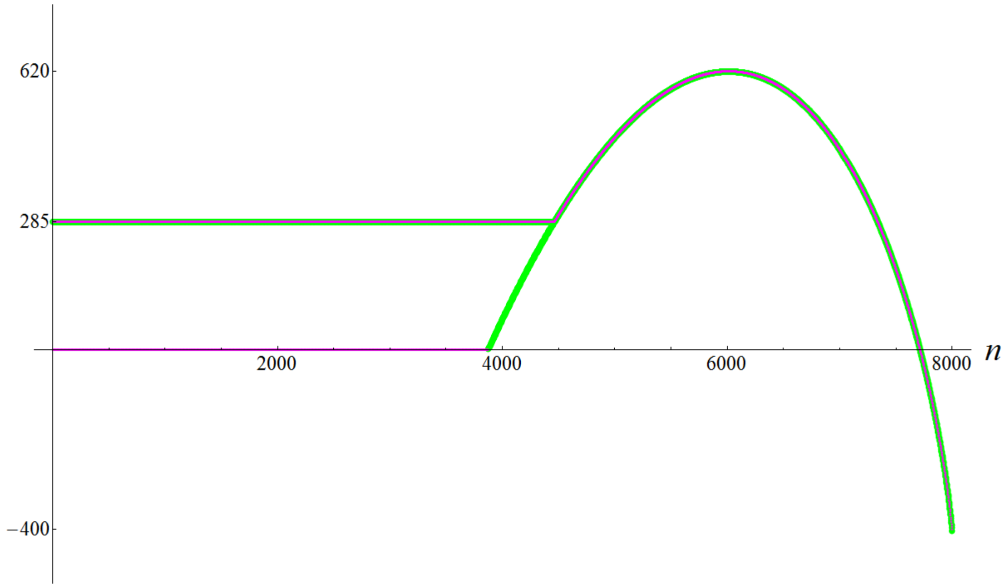
Slide 85:



This graph exploits neither logarithmic scaling nor cuts in the axes; as a result no details can be seen.

Slide 86:

$\log_{10}(|\delta_{8000,n}|)$, magenta, if $\delta_{8000,n} > 0$, green otherwise

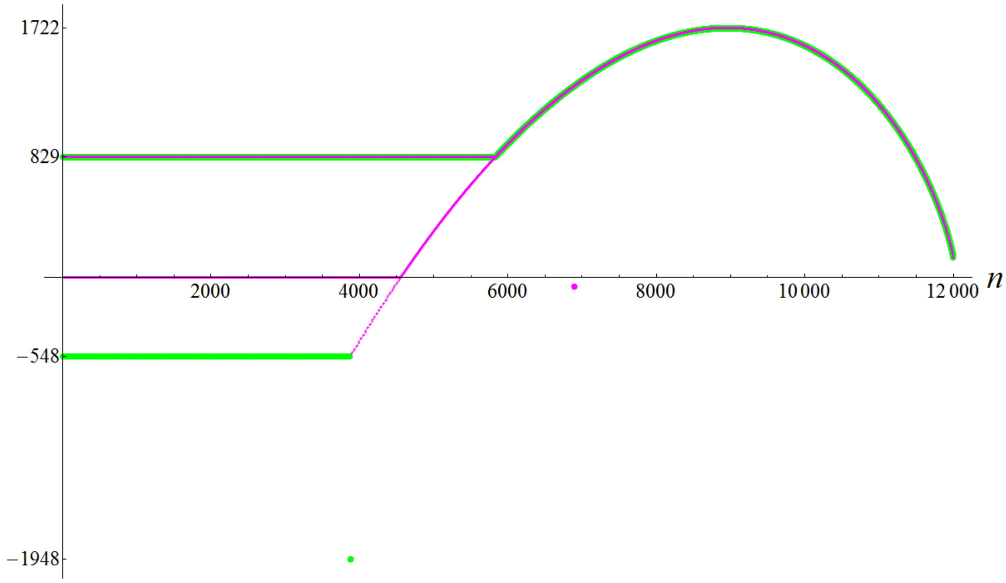


$$\mu_{8000,2} = -3.9784578642060744693593899018\ldots \cdot 10^{285}$$
$$\mu_{8000,3} = 5.9676867963091117040390848527\ldots \cdot 10^{285}$$

The two horizontal lines are due to the large values of $\mu_{8000,2}$ and $\mu_{8000,3}$. Of what form is the arc?

Slide 87:

$\log_{10}(|\delta_{12000,n}|)$, magenta, if $\delta_{12000,n} > 0$, green otherwise



$$\mu_{12000,2} = 6.218564\ldots \cdot 10^{828} \qquad \mu_{12000,4} = 6.624976\ldots \cdot 10^{552}$$

$$\mu_{12000,3} = -9.327846\ldots \cdot 10^{828} \qquad \mu_{12000,5} = 1 - 2.380857\ldots \cdot 10^{-548}$$

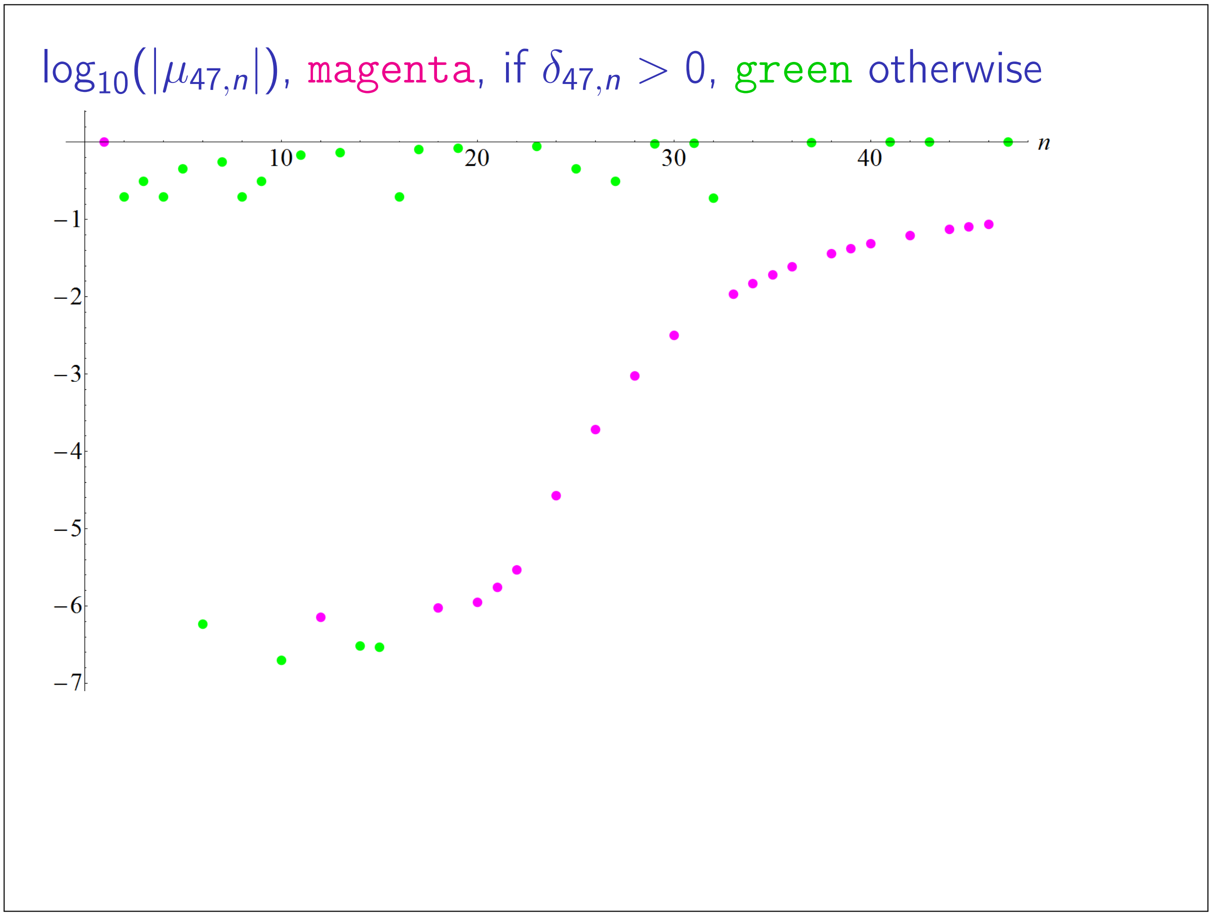
The three horizontal lines are due to the large values of $\mu_{12000,2}$ and $\mu_{12000,3}$ and the values of $\mu_{12000,3}$ being very close to 1.

The two dots at $n = 3875$ and $n = 6892$ are due to the changes of sign of values of $\delta_{12000,2m}$ and $\delta_{12000,2m+1}$.

So the δ's show three typical behaviour:

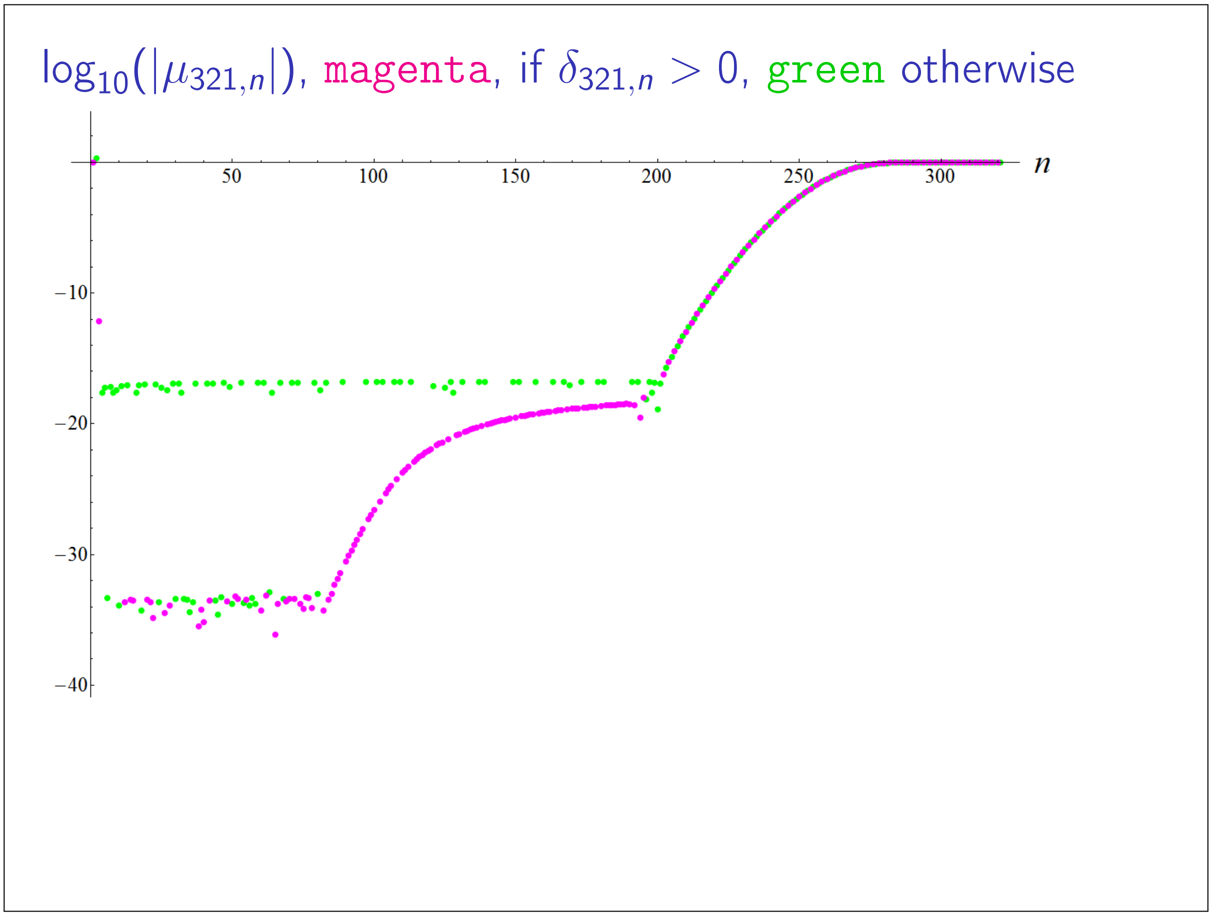
- "logarithmic" for "small" N like on Slide 33,
- "parallel" for "medium" N like on Slide 42,
- "normally distributed" for larger N like on Slide 85.

Slide 88:



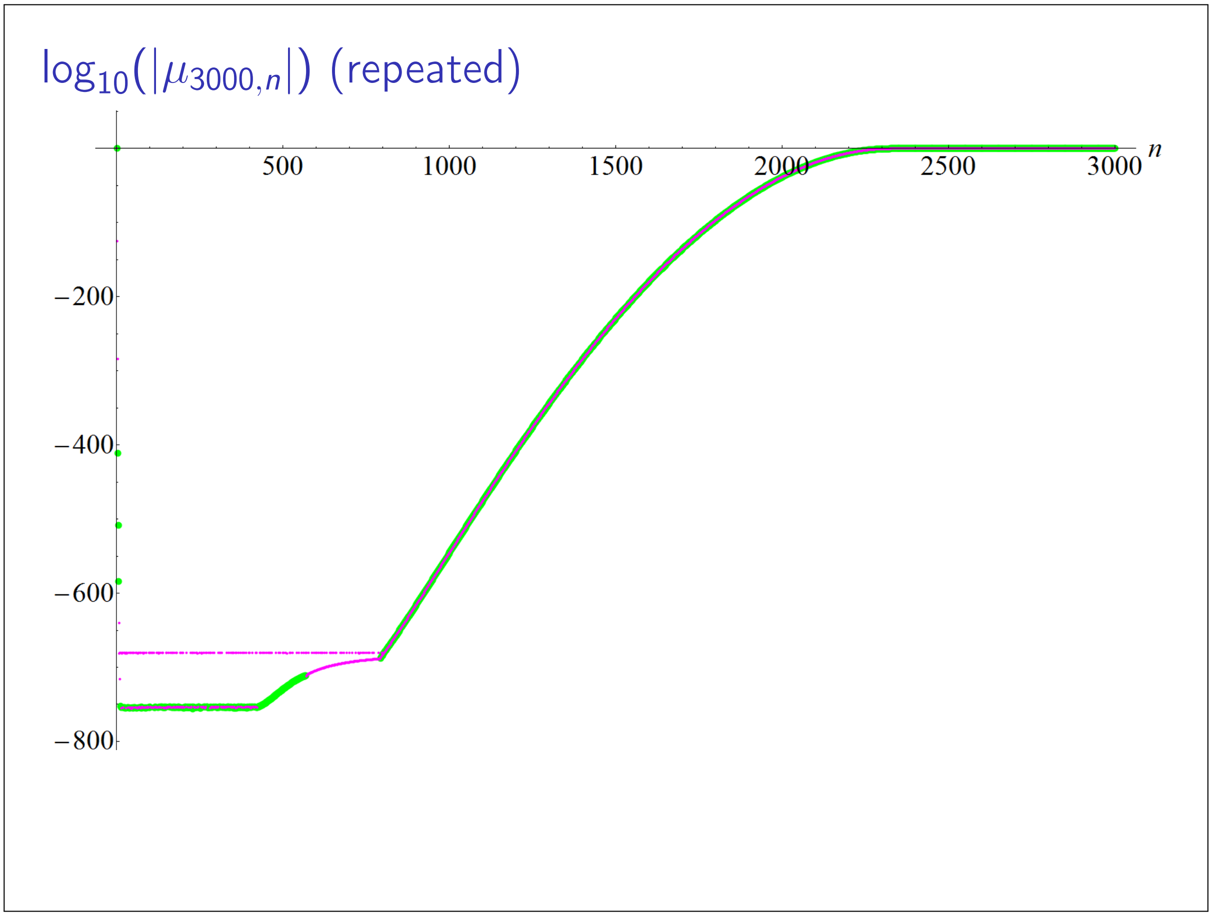
The μ's behave more uniformly. Already for N as small as 47 we can distinguish primes and their powers.

Slide 89:



Here we can see $n = 25,\ 27,\ 32,\ 49,\ 64,\ 81,\ 121,\ 125,\ 128.$

Slide 90:

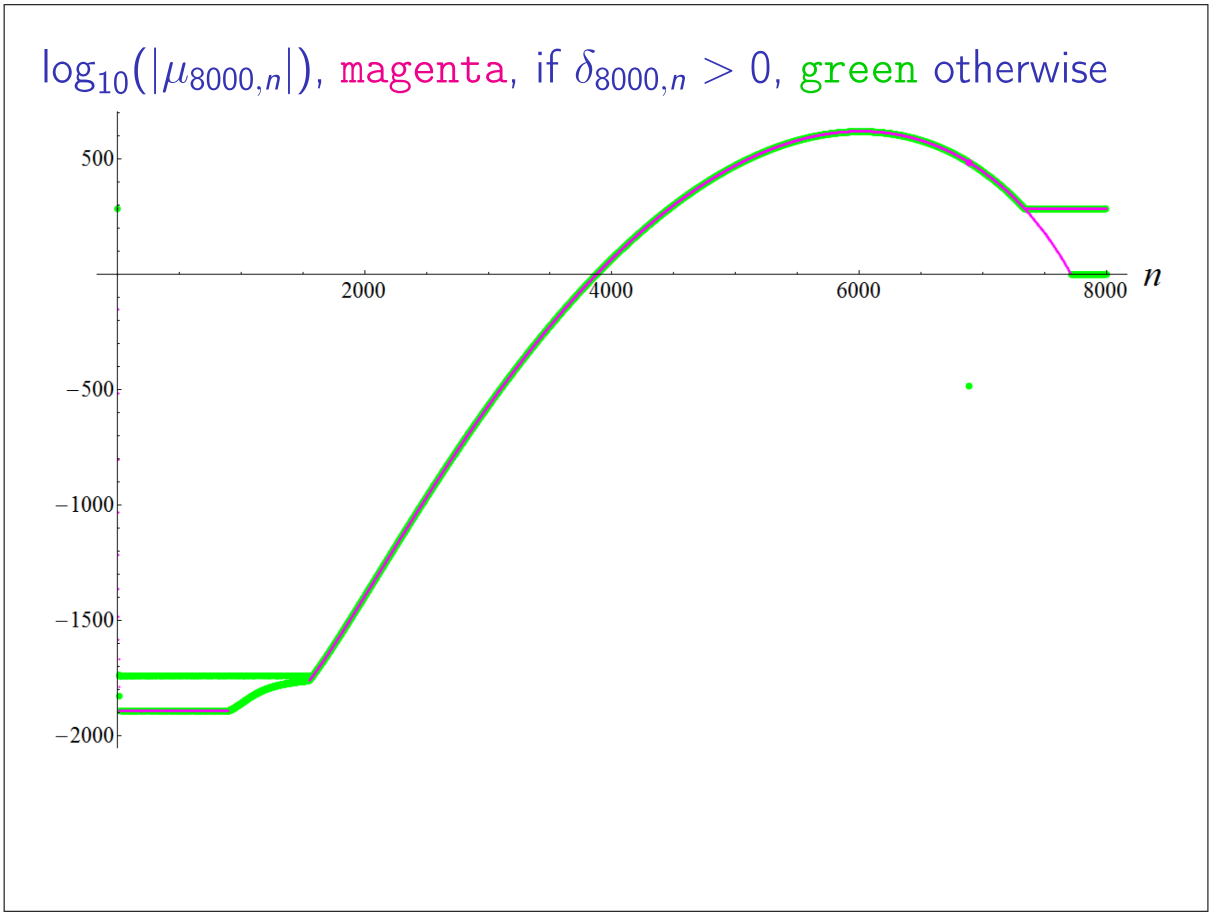


$\log_{10}(|\mu_{3000,n}|)$ (repeated)
500
1000
1500
2000
2500
3000
n
−200
−400
−600
−800

Slide 91:

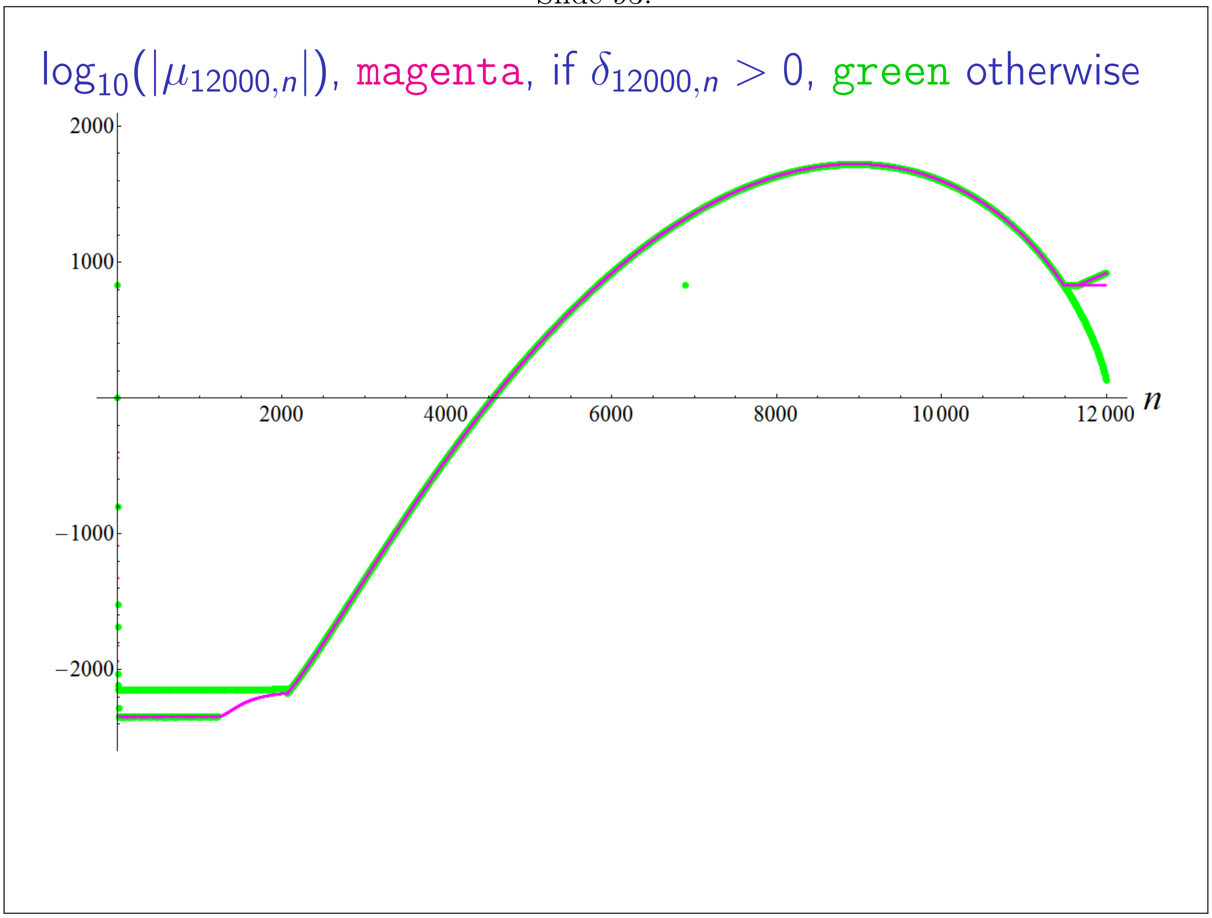
For $N = 6000$ the trailing values of $\log_{10}(|\mu_{N,n}|)$ are almost zero; again the switch to such behaviour is very sudden.

Slide 92:



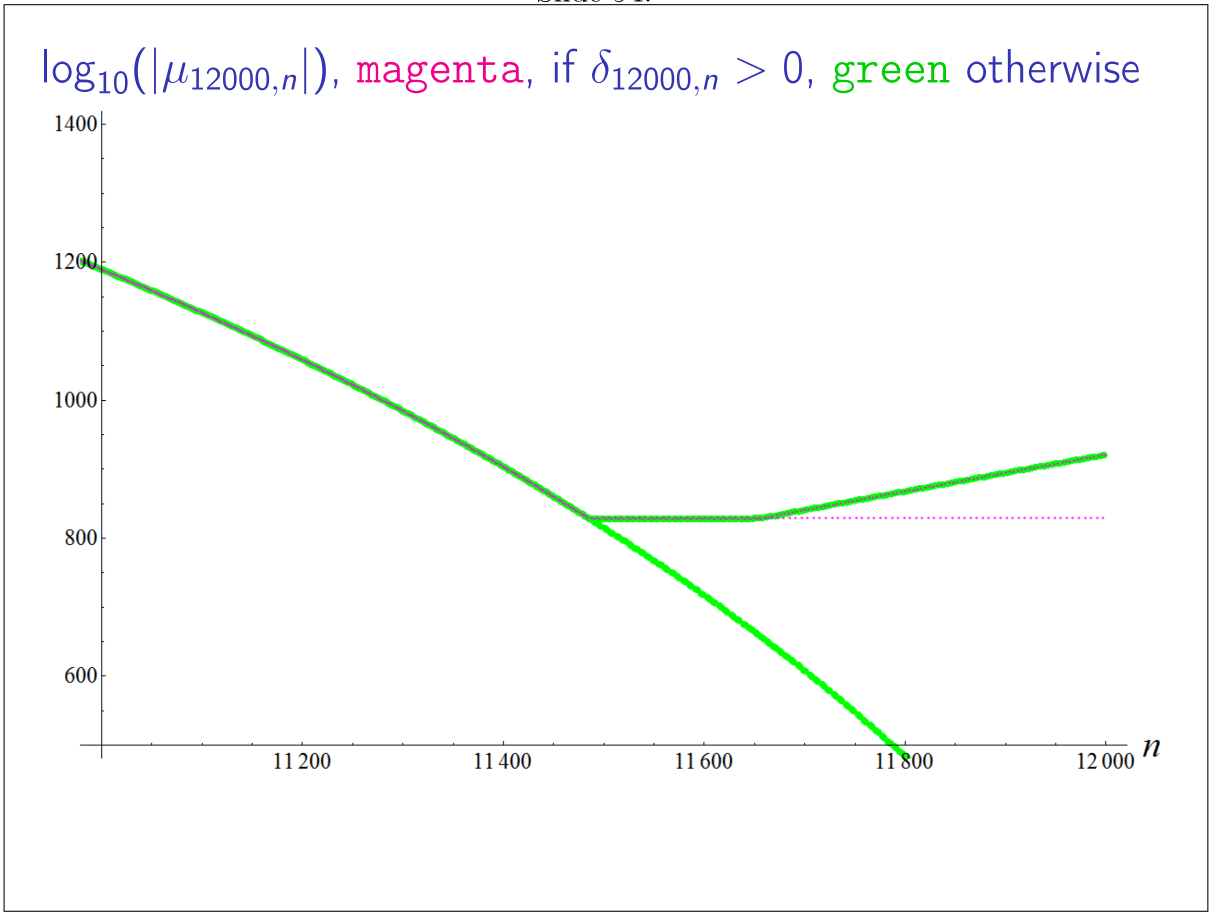
The trailing splitting occurs according to the parity of n.

Slide 93:



$\log_{10}(|\mu_{12000,n}|)$, magenta, if $\delta_{12000,n} > 0$, green otherwise
2000
1000
−1000
−2000
2000
4000
6000
8000
10 000
12 000
n

Slide 94:



Now the trailing splitting is determined by the divisibility of n by both 2 and 3.

What next?

N	$\mu_{N,2}$	$\mu_{N,3}$	$\mu_{N,5}$
800	$-2-6.04\dots\cdot 10^{-33}$	$+9.06\dots\cdot 10^{-33}$	$+7.17\dots\cdot 10^{-100}$
2000	$-2-5.98\dots\cdot 10^{-84}$	$+8.98\dots\cdot 10^{-84}$	$+2.42\dots\cdot 10^{-268}$
2800	$-2+1.15\dots\cdot 10^{-117}$	$-1.73\dots\cdot 10^{-117}$	$+4.92\dots\cdot 10^{-383}$
3600	$-2+1.55\dots\cdot 10^{-151}$	$-2.32\dots\cdot 10^{-151}$	$-1.31\dots\cdot 10^{-498}$
4400	$-2+9.65\dots\cdot 10^{-185}$	$-1.44\dots\cdot 10^{-184}$	$+1.96\dots\cdot 10^{-614}$
5200	$-2-5.63\dots\cdot 10^{-215}$	$+8.45\dots\cdot 10^{-215}$	$+1.00\dots\cdot 10^{-726}$
6000	$-2+2.19\dots\cdot 10^{-165}$	$-3.29\dots\cdot 10^{-165}$	$-3.55\dots\cdot 10^{-759}$
6800	$-2+4.84\dots\cdot 10^{-26}$	$-7.26\dots\cdot 10^{-26}$	$-8.73\dots\cdot 10^{-703}$
7600	$-6.31\dots\cdot 10^{170}$	$+9.47\dots\cdot 10^{170}$	$-2.25\dots\cdot 10^{-588}$
8000	$-3.97\dots\cdot 10^{285}$	$+5.96\dots\cdot 10^{285}$	$+4.81\dots\cdot 10^{-516}$
8800	$+5.57\dots\cdot 10^{538}$	$-8.36\dots\cdot 10^{538}$	$-6.24\dots\cdot 10^{-346}$
9600	$-3.52\dots\cdot 10^{818}$	$+5.29\dots\cdot 10^{818}$	$-1.81\dots\cdot 10^{-149}$
10400	$+6.40\dots\cdot 10^{1049}$	$-9.60\dots\cdot 10^{1049}$	$-1-2.36\dots\cdot 10^{-69}$
11200	$+2.29\dots\cdot 10^{1030}$	$-3.44\dots\cdot 10^{1030}$	$-1+3.39\dots\cdot 10^{-302}$
12000	$+6.21\dots\cdot 10^{828}$	$-9.32\dots\cdot 10^{828}$	$-1-2.38\dots\cdot 10^{-548}$

Comments to Slide 87 list three different typical behaviours of δ's but the above table suggests that with further increasing N the picture may change once (or many times?) again. For this reason I believe that it would be very interesting to continue calculating for $N > 12000$, but this exceeds the computational resources I have at my disposal at present; international cooperation seems to be necessary.

Also it would be very interesting to see if there are similar phenomena for other L-functions. We saw that for the Riemann zeta function many things are connected with its pole, so, probably, we should deal not with individual Dirichlet L-functions but with the product of all of them corresponding to fixed prime p.

The determinants $\Delta_N(t)$ were defined under the assumptions that all zeroes satisfy Riemann's Hypothesis and are simple. What would happen if Riemann's Hypothesis is valid but there are multiple zeroes? It might completely spoil the methods considered above for calculating the values of the zeta function, and its derivative and zeroes in the case of large N. More likely, it will only be necessary to modify the definitions (84) and (90) by replacing columns with repeated values of zeroes by columns with derivatives of β_n. But maybe there is no need in such modification and proving it would give a proof (under RH) of the simplicity of all zeroes.

A possible non-real zero γ^* of $\Xi(t)$ wouldn't spoil definitions (84) and (90) but would require making a decision as to which of the two zeroes, γ^* or $\overline{\gamma^*}$, should be used first. When N is such that an odd number of non-real zeroes of $\Xi(t)$ are used in (90), the values of $\delta_{N,n}$ and $\mu_{N,n}$ will become in general non-real, making $\Lambda(n)$-like behaviour of $\mu_{N,n}$ doubtful. Does this observation open a way to proving RH?

Slide 96:

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http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod

More graphs, tables, texts can be found on this site devoted to the ongoing research.