Complex Structures and Moduli Problem in Representation Theory

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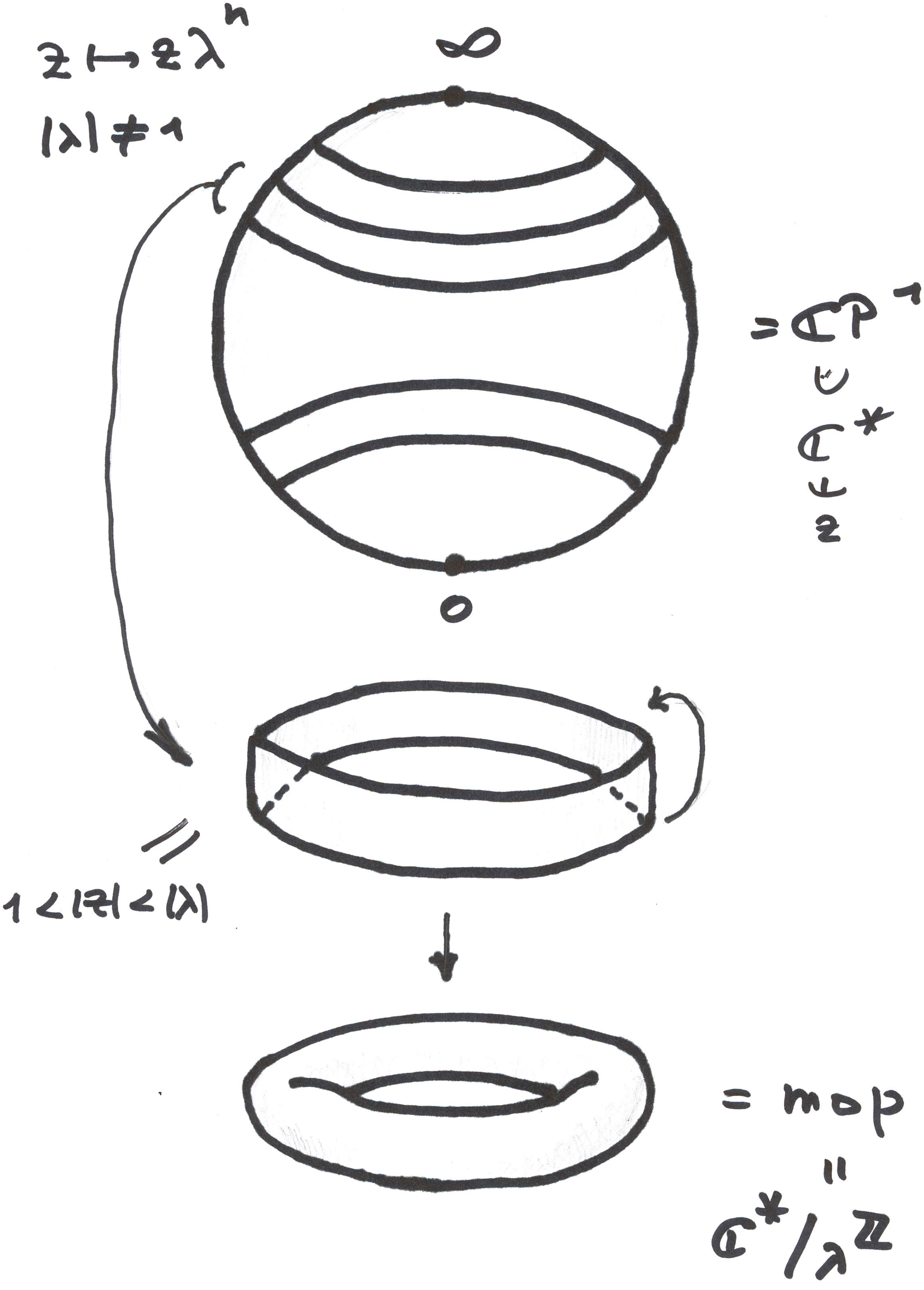
$z \mapsto z\lambda^n$
$|\lambda| \neq 1$
∞
0
$= \mathbb{C}P^1$
$\cup$
$\mathbb{C}^*$
$\circlearrowleft$
$\mathbb{Z}$
$=$
$1 < |z| < |\lambda|$
= тор
$\|$
$\mathbb{C}^*/\lambda^{\mathbb{Z}}$

Moduli of Elliptic Curves

- $\Delta^*_\pm = \{\lambda \in \mathbb{C}^* \mid |\lambda| \lesseqgtr 1\} =$

  $\{$isomorphism classes of elliptic curves $E_\lambda = \mathbb{C}^*/\{\lambda^n,\ n \in \mathbb{Z}\} \quad +$
  primitive vector, up to a sign, from $\mathrm{H}^1(E_\lambda, \mathbb{Z})\}$
- $M = \Delta^*_+ \sqcup \Delta^*_- \leftarrow \mathcal{E} \leftarrow X$
- $\mathcal{E} = \bigcup_{\lambda \in M} E_\lambda$
- $X = \bigcup_{\lambda \in M} L_\lambda$, where L_λ is the line bundle corresponding to the sheaf $\mathcal{O}_{E_\lambda}(-1)$

особый слой
неособая кривая
= E_λ
$\lambda^N = 1$
0
λ
∞
$|\lambda| < 1$
$|\lambda| = 1$
$|\lambda| > 1$

Moduli of Curves as Moduli of Representations

$\mathcal{E} = \{$equivalence classes of irreducible infinite-dimensional representations of a Heisenberg group $G\}$

$X = \{$equivalence classes of irreducible infinite-dimensional representations of a group $\hat{G} = G \rtimes \mathbb{Z}\}$

What happens with representation when λ tends to λ_0 lying on the circle $|\lambda| = 1$?

Induced representations (definition)

Let G be a group and $H \subset G$ is a subgroup. For a character $\chi : H \to \mathbb{C}^*$ we define the induced representation π_χ of the group G.

Let V_χ be the space of all complex-valued functions f on G which satisfy the following conditions:

1. $f(gh) = \chi(h) f(g)$ for all $h \in G_\chi$.
2. The support $Supp(f)$ is contained in a union of a finite number of left cosets of G_χ.

Left translations define a representation π_χ of the group G on V_χ.

Representations of discrete nilpotent groups

G discrete nilpotent group
$\pi : G \to \mathrm{End}(V)$ irreducible representation
$H \subset G$ a subgroup, $\chi : H \to \mathbb{C}^*$ character

$V(H, \chi) := \{v \in V : \text{for all } h \in H \ \pi(h)v = \chi(h)v\}.$

V is of *finite type* iff there exists a pair H, χ such that $V(H, \chi) \neq (0)$ and $dim V(H, \chi) < \infty$.

Conjecture 1. Any irreducible representation of finite type is monomial (induced by an abelian character $\chi : H \to \mathbb{C}^*$ of a subgroup $H \subset G$).

The monomial property is known for unitary representations on Hilbert spaces (*I. Brown*, 1973).

Discrete Heisenberg groups

H, P, C (finitely generated) abelian groups

$\langle -, - \rangle : H \times P \to \mathrm{C}$ biadditive pairing

The set $H \times P \times \mathrm{C}$ with the composition law

$$(n, p, c)(m, q, a) = (n + m, p + q, c + a + \langle n, q \rangle),$$

where $n, m \in H, p, q \in P$ and $c, a \in \mathrm{C}$,
is called the *discrete Heisenberg group* G.

Theorem (jointly with *S. A. Arnal'*). Every irreducible finite type representation of the discrete Heisenberg group is monomial.

Induced representations (examples)

We introduce the complex tori
$\mathbb{T}_H = \mathrm{Hom}(H, \mathbb{C}^*), \mathbb{T}_P = \mathrm{Hom}(P, \mathbb{C}^*)$ and
$\mathbb{T}_{\mathrm{C}} = \mathrm{Hom}(\mathrm{C}, \mathbb{C}^*) \ni \chi_C$; $\mathbb{T}_G = \mathbb{T}_H \times \mathbb{T}_P \times \mathbb{T}_{\mathrm{C}} \ni \chi$.

There is a pairing $H \times P \to \mathbb{C}^*$ such that :

$$(h, p) \mapsto \chi_{\mathrm{C}}(\langle h, p \rangle)$$

Let H_χ be the left kernel of this pairing. Then $G_\chi = H_\chi PC$ is a subgroup in G and $\chi | G_\chi$ is a character of G_χ .
We set $\pi_\chi = \mathrm{ind}_{G_\chi}^G(\chi)$.

We also have an embedding $H \to \mathbb{T}_P$ and thus action of $h \in H$ by a translation h^* on $\mathbb{T}_P$. The embedding of $\mathbb{T}_{H/H_\chi}$ into $\mathbb{T}_H$ gives a translation t^* on $\mathbb{T}_H$ for any $t \in \mathbb{T}_{H/H_\chi}$.

Induced representations (properties)

The representations $\pi_\chi, \chi = \chi_H \otimes \chi_P \otimes \chi_C$ are irreducible and are of finite type:
there are no nontrivial invariant subspaces, and the Schur lemma holds.

The representations V_χ and $V_{\chi'}$ are equivalent if and only if

1. $\chi_{\mathrm{C}} = \chi'_{\mathrm{C}}$.
2. There exists $h \in H$ such that $\chi'_P = h^*(\chi_P)$.
3. There exists $t \in \mathbb{T}_{H/H_\chi} = \mathrm{Hom}(H/H_\chi, \mathbb{C}^*) \subset \mathbb{T}_H$ such that $\chi'_H = t^*(\chi_H)$.

The equivalence classes of representations $V_\chi \Leftrightarrow$ orbits of the groups $\mathbb{T}_{H/H_\chi} \times H/H_\chi$ in subsets $\mathbb{T}_H \times \mathbb{T}_{H'} \times \{\chi_C\}$ of the torus $\mathbb{T}_G$.

Example: the group $G = \mathrm{Heis}(3, \mathbb{Z})$

$$G = \begin{pmatrix} 1 & n & c \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}$$

where $H = P = C = \mathbb{Z}$ and $\langle n, p \rangle = np$.
Let $\chi_H(n) = w^n,\ \chi_{H'}(p) = z^p,\ \chi_{\mathrm{C}}(c) = \lambda^c$ with $(n, p, c) \in G$. Then

$$\mathbb{T}_H = \{w \in \mathbb{C}^*\}, \quad \mathbb{T}_P = \{z \in \mathbb{C}^*\}, \quad \mathbb{T}_{\mathrm{C}} = \{\lambda \in \mathbb{C}^*\}$$

and if $\chi = (w, z, \lambda)$ and $\chi' = (w', z', \lambda')$ then

$$V_{\chi'} \cong V_\chi \Leftrightarrow \lambda' = \lambda,\ \text{for some}\ \ n \in \mathbb{Z}\ z' = z\lambda^n$$

and if $\lambda^N = 1$ then $(w'w^{-1})^N = 1$.

Type I representations

- First, let $|\lambda| \neq 1$. Then

$$H_\chi = (0), \Rightarrow V_\chi \text{ is infinite-dimensional } = \mathrm{ind}_{PC}^{G}(\chi).$$

- We have the action of the group $\mathbb{C}^* \times \mathbb{Z}$ on the set $\mathbb{T}_G = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ by the formula (where $u \in \mathbb{C}^*,\ n \in \mathbb{Z}$):

$$(u, n)(w, z, \lambda) = (uw, \lambda^n z, \lambda).$$

  We define the space $\mathcal{M}_{GP}$ as the quotient of the domain $\{\chi : |\lambda| \neq 1\} \subset \mathbb{T}_G$ by this action. The quotient-space is a two-dimensional complex manifold $\mathcal{E}$ fibrated over $M = \mathbb{C}^* \setminus \{\lambda| = 1\}$ with the fibers = elliptic curves $E_\lambda = \mathbb{C}^*/\lambda^{\mathbb{Z}}$.
- The space $\mathcal{M}_{GP}$ is the parameter space for this class of the infinite-dimensional representations of G.

Type I representations

The representations from the space $\mathcal{M}_{GP}$ depend on the choice of a subgroup $P \subset G$. This subgroup has the following properties as a subgroup of the quotient $G/C \cong \mathbb{Z} \oplus \mathbb{Z}$:

- P is isomorphic to $\mathbb{Z}$ and is a direct summand in G/C
- P is generated by a vector (x, y) with co-prime integers x, y
- P is defined by a point in projective line $\mathbb{P}^1(\mathbb{Q})$

Theorem. Let V be an irreducible infinite-dimensional finite type representation of the group G. Then class of V belongs to $\mathcal{M}_{GP}$ for some choice of the subgroup P.
If two representations V and V' are equivalent then they belong to the same space $\mathcal{M}_{GP}$.

Type II representations

Second, what happens when $|\lambda| = 1$ and $\lambda^N = 1$ for some $N \in \mathbb{Z}$.

Theorem (jointly with *S. A. Arnal'*). Let V be an irreducible monomial representation of the group G, $V = ind_{H(V)}^G(\chi)$, where $\chi : H \to \mathbb{C}^*$ is a character and χ_C be it's restriction on the center of G. The following properties are equivalent:

- V is finite-dimensional and $dimV = N$
- $\chi_C(c) = \lambda^c$ with λ of finite order N in $\mathbb{C}^*$
- $[G : H(V)] < \infty$
- $H(V) = N\mathbb{Z} \cdot \mathbb{Z} \cdot \mathbb{Z} = NH \cdot P \cdot C$

We see that inducing subgroup is the same for all representations of given dimension.

Classification of irreducible representations. Viewpoint of functional analysis

G a locally compact group, H Hilbert space

- Unitary representation
  $\pi : G \to U(H)$ continuous homomorphism into the unitary group
- $\check{G} = \{$equivalence classes of irreducible unitary representations for the group $G\}$ with a natural topology (where the matrix elements of representations are continuous).
- **Classification problem**. Describe the topological space $\check{G}$

Classification of irreducible representations. Viewpoint of algebraic geometry

G a group

- Family of (irreducible) representations$/k$ where k is an algebraically closed field:
  $f : E \to S$ vector bundle over a variety S/k and
  $\pi : G \to Aut_S(E)$ homomorphism such that
  for any point $s \in S(k)$ the representation π_s on the fiber E_s is irreducible.
- $\mathcal{M}_G(S) = \{$equivalence classes of families of representations over S for the group $G\}$
- **Moduli problem**. Find a universal family $F : \mathcal{E} \to \mathcal{M}$ such that for all S there is a natural bijection

$$Hom(S, \mathcal{M}) \to \mathcal{M}_G(S),$$

where the $\mathcal{M}$ is a variety or scheme or a stack.

Solutions of the moduli problem

- *Parameter space* is a variety M_G together with a family of representations E_G such that
  - for every point $P \in M_G$ the representation $(E_G)_P$ is irreducible,
  - for every irreducible representation $\pi : G \to Aut_{\mathbb{C}}(V)$ of the group G there exists a unique point $P \in M_G$ and an isomorphism $V \cong (E_G)_P$.
- a parameter space M_G is the coarse moduli scheme for the group G or the moduli problem can be solved by a stack M_G^{st}.The natural map $M_G \to M_G^{st}$ is bijection for the $\mathbb{C}$-points.

The moduli space

- Let $\mathcal{M}_G^{fin}$ be the parameter space of all finite-dimensional representations of the group G and let μ_N be the group of N-th roots of unity. Then

$$\mathcal{M}_G^{fin} = \bigsqcup_{N,\mu_N} \mathbb{C}^*/\mu_N \times \mathbb{C}^*/\mu_N$$

Define

$$\mathcal{M}_G = \bigsqcup_{P \in \mathbb{P}^1(\mathbb{Q})} \mathcal{M}_{GP} \bigsqcup \mathcal{M}_G^{fin} = \mathcal{M}_G^{infin} \bigsqcup \mathcal{M}_G^{fin}$$

Theorem*. $\mathcal{M}_G^{fin}$ is the moduli stack of the finite-dimensional representations of the group G. $\mathcal{M}_G^{infin}$ is the moduli stack of the infinite-dimensional representations of the group G.

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Theorem*. $\mathcal{M}_G^{fin}$ is the moduli stack of the finite-dimensional representations of the group G. $\mathcal{M}_G^{infin}$ is the moduli stack of the infinite-dimensional representations of the group G.
- In this theory we have no room for the representations V_χ with another λ's. They are unitary representations for the C^*-algebras with irrational rotations.

Trace problem

- The representations V_χ have no characters (the traces of representation $\pi_\chi(g), \ g \in G$ operators will diverge). This problem can be solved by an extension of the group G to a larger one. We use a well-known construction from the theory of loop groups. We have to add some "loop rotations " to G .

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- $G \quad \Rightarrow \quad \hat{G} = G \rtimes A,$
  where $A \subset \mathrm{Hom}(H, P), A \neq (0),$

$$A \sim \begin{pmatrix} \mathrm{Id} & 0 \\ A & \mathrm{Id} \end{pmatrix} \quad \text{acts on } H \oplus P.$$

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  $$A \sim \begin{pmatrix} \mathrm{Id} & 0 \\ A & \mathrm{Id} \end{pmatrix} \quad \text{acts on } H \oplus P.$$
- One needs to extend the automorphisms of the abelian group $H \oplus P$ to automorphisms of the entire Heisenberg group.

Trace properties

Under simple and natural conditions, the representation of G on V_χ can be extended to a representation $\hat{\pi}_\chi$ of the extended group $\hat{G}$ on the same space. Let $\mathbb{T}_A = \mathrm{Hom}(A, \mathbb{C}^*)$ and $\mathbb{T}_{\hat{G}} = \mathbb{T}_G \times \mathbb{T}_A$. If $\hat{\chi} = (\chi, \chi_A) \in \mathbb{T}_{\hat{G}}$, then we set

$$\hat{\pi}_{\hat{\chi}} = \hat{\pi}_\chi \otimes \chi_A.$$

We have $\mathrm{Tr}\ \hat{\pi}_{\hat{\chi}} = \mathrm{Tr}\ \hat{\pi}_\chi \cdot \chi_A$ if the trace is defined.

- The trace $\mathrm{Tr}\ \hat{\pi}_{\hat{\chi}}(g)$ is defined on a sufficiently big "domain" $\hat{G}(\chi)$ in the group $\hat{G}$.
- Let $\hat{\chi}, \hat{\chi}^{'} \in \mathbb{T}_{\hat{G}}$. The representations $\hat{\pi}_{\hat{\chi}}$ and $\hat{\pi}_{\hat{\chi}'}$ are equivalent if and only if $\hat{G}(\chi) = \hat{G}(\chi')$ and $\mathrm{Tr}\ \hat{\pi}_{\hat{\chi}}(g) = \mathrm{Tr}\ \hat{\pi}_{\hat{\chi}'}(g)$ for all $g \in \hat{G}(\chi)$.

Example: $\hat{G} = \mathrm{Heis}(3, \mathbb{Z}) \rtimes \mathbb{Z}$

Let $\hat{G} = \begin{pmatrix} 1 & n & \frac{1}{2}n(n-1) & c \\ 0 & 1 & n & p \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

$\chi_H(n) = w^n,\ \chi_P(p) = z^p,\ \chi_C(c) = \lambda^c$ and $\chi_A(k) = t^k$.

Then

$$\begin{array}{lcll} \hat{G}(\hat{\chi}) & = & \{g = (n, p, c, k) | k > 0\} & \text{if } |\lambda| < 1, \\ \hat{G}(\hat{\chi}) & = & G & \text{if } |\lambda| = 1,\ \lambda^N = 1,\ N \in \mathbb{N} \\ \hat{G}(\hat{\chi}) & = & \{g = (n, p, c, k) | k < 0\} & \text{if } |\lambda| > 1, \end{array}$$

and

$$\mathrm{Tr}\ \hat{\pi}_{\hat{\chi}}(0, p, c, k) = \lambda^c t^k z^p \sum_{n \in \mathbb{Z}} z^{kn} \lambda^{np + 1/2kn(n-1)}$$

converges for all $z \in \mathbb{C}^*$ and $|\lambda| \lessgtr 1,\ k \gtrless 0$.

$$\mathrm{Tr}\ \hat{\pi}_{\hat{\chi}}(n, p, c, k) = 0 \text{ for } n \neq 0.$$

Characters and theta-functions

We have

$$\mathrm{Pic}(E_\lambda) = H^1(E_\lambda, \mathcal{O}^*) = H^1(H, \mathcal{O}^*(\mathbb{T}_P)) \to A = \mathbb{Z} \ni k,$$

$$\mathrm{Pic}(E_\lambda) = \{\varphi(n,z) = a^{-n} z^{-kn} \lambda^{-1/2kn(n-1)} : a \in \mathbb{C}^*,\ k \in \mathbb{Z}\}.$$

Let L be the line bundle corresponding to a cocycle φ. Then

$$H^0(E_\lambda, L) = \{f(z), z \in \mathbb{T}_P : f(\lambda^n z) = \varphi(n,z) f(z)\}$$

and it suffices to impose the condition $f(\lambda z) = a^{-1} z^{-k} f(z)$. The theta-series

$$\vartheta_{p,k,a}(z,\lambda) := z^p \sum_{n\in\mathbb{Z}} a^n z^{kn} \lambda^{np+1/2kn(n-1)}$$

(they are the Poincare series for φ) are convergent for all $z \in \mathbb{C}^*, 0 < |\lambda| < 1, k > 0$ and form a basis in the space $H^0(E_\lambda, L)$ for $0 \le p < k$. Finally,

$$\mathrm{Tr}\hat{\pi}_{\hat{\chi}}(0,p,c,k) = \lambda^c t^k \vartheta_{p,k,1}(z,\lambda).$$

Let L_λ be the line bundle corresponding to the sheaf $\mathcal{O}_{E_\lambda}(-1)$. Denote by $L_\lambda^* \subset L_\lambda$ the corresponding G_m-bundle. Then we have

- The parameter space for representations $\pi_{\hat{\chi}}$ $\hat{\chi} \sim (\lambda, w, z, t| \in \mathbb{C}^*)$ with $|\lambda| \neq 1$ is

$$X = \bigcup_{\lambda} L_\lambda^*,$$

  which is quotient of $(\lambda, z, t) \in \mathbb{C}^*$ by the relation

$$\lambda' = \lambda,\ z' = z\lambda^n,\ t' = tz^n\lambda^{\frac{1}{2}n(n-1)} \quad \text{for some } n \in \mathbb{Z}$$

- The characters of representations $\pi_{\hat{\chi}}$ are the functions on L_λ^* coming from the sections of the line bundles L_λ^{-k}.

Boundary behavior

We assume:

- the extension Q of the pairing $\langle n, k(n)\rangle,\ n \in H$ on $V = H \otimes \mathbb{R}$ is positive-definite
- $C = \mathbb{Z},\ \lambda \in \mathbb{T}_C = \mathbb{C}^*$ and $\chi(c) = \lambda^c$
- $\chi_H \equiv 1$ and $\chi_P \equiv 1$

The classical limit formulas for theta-functions imply the following behavior of the trace near the $\chi_0 = \lambda_0^c$, where λ_0 is a root of unity:

$$\mathrm{Tr}\ \hat{\pi}_{\hat{\chi}}(g) \sim \mathrm{Tr}\ \hat{\pi}_{\hat{\chi}_0}(g) \cdot [H : H_{\chi_0}]^{-1} (\mathrm{Det}_V Q)^{-1} A \log|\lambda|^{-\frac{1}{2}\mathrm{rk} H},$$

when $\chi \to \chi_0$. Here $A = (\sqrt{\pi}/2)^{\mathrm{rk} H}$. The trace of the (finite-dimensional) representation $\hat{\pi}_{\hat{\chi}_0}$ can be computed in terms of a Gauss sum.

Examples

- 1) finite-dimensional representations/k of semi-simple Lie algebras with $char(k) > 0$
  ($sl(2, k)$, *I. R. Shafarevich, A. N. Rudakov*, 1967;
  arbitrary algebra, *R. Veldkamp* ,1972)
- 2) smooth representations/$\mathbb{C}$ of reductive algebraic groups over a p-adic field
  (*J. N. Bernstein* , *A. V. Zelevinsky*, 1976)
- 3) finite-dimensional representations/$\mathbb{C}$ of discrete finitely presented groups (*A. Lubotzky, A. Magid*, 1985)
- **Answers**: 1) affine variety (actually a stack).
  2) countable disjoint unions of complex tori as parameter spaces ($=$ moduli stacks,remarked by *S. O. Gorchinskiy*)
  3) affine varieties as parameter spaces ($=$ moduli stacks, *D. Mumford*, GIT, 1965, *C. S. Seshadri* 1967, *C. T. Simpson* 1994)

Classification problem for representations of discrete nilpotent groups

G discrete nilpotent group
V irreducible representation of finite type, according to Conjecture 1 a monomial one induced from some subgroup H

Conjecture 2. There exists a moduli stack $\mathcal{M}_G$ of these V's glued from complex tori

$$\mathcal{M}_G = \bigcup_{H \subset G} \mathbb{T}_{H/[H,H]} / \sim$$

In particular, how to glue the two stacks $\mathcal{M}_G^{infin}$ and $\mathcal{M}_G^{fin}$ from the previous example of the Heisenberg group $Heis(3, \mathbb{Z})$?