The Lüroth problem and the Cremona group

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Lüroth (1875): yes for curves.

(Quite easy with Riemann surface theory; but Lüroth's proof is algebraic.)

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Around 1971 three "modern" counter-examples appeared:

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(*Fano threefolds of the first species* : modern classification due to Iskovskikh).

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- The proof is very simple – maybe the simplest non-rationality proof available.
- Real motivation: it completes the work of Prokhorov on the finite simple subgroups of Cr_3.

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(because on H^3, we have v^* and v_* with $v_* v^* = \mathrm{Id}$).

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Usually by studying the geometry of the theta divisor (singular locus, Gauss map, ...).

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In particular, $\mathfrak{A}_7 \subset \mathrm{Aut}(JV)$. Note: $\dim JV = 20$.

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Torelli: $$\mathrm{Aut}(JC) = \begin{cases} \mathrm{Aut}(C) & \text{if } C \text{ hyperelliptic} \\ \mathrm{Aut}(C) \times \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

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(more subtle: e.g. $\mathrm{Aut}(E^{20}) \supset \mathfrak{S}_{20}$).

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Thus $\# O = 7$ or 15; contradiction!

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Up to conjugacy, $SL(2, \mathbb{F}_8)$ admits only one embedding in Cr_3, and $PSp(4, \mathbb{F}_3)$ two.

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Since V is not rational, only one embedding $\mathfrak{A}_7 \subset Cr_3$. ∎

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Question : Is it true that $\mathrm{crdim}(\mathfrak{S}_n) = n-3$?

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Happy birthday, Alberto!