Symmetry approach to classification of integrable PDEs

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Part 1. Basic notions of the symmetry approach

The symmetry approach has been developed in 1979-2012 by:

A.Shabat, A.Zhiber, N.Ibragimov, A.Fokas,
V.Sokolov, S.Svinolupov, A.Mikhailov, R.Yamilov, V.Adler,
P.Olver, J.Sanders, J.P.Wang, V.Novikov,
A.Meshkov, D.Demskoy, H.Chen, Y.Lee, C.Liu,
I.Khabibullin, B.Magadeev, R.Heredero, V.Marikhin,
M.Foursov, S.Startcev, M.Balakhnev, ...

Definition. (1+1)-dimensional PDE is integrable if it possesses infinitely many generalized infinitesimal symmetries.

- **Sokolov V.V., Shabat A.B.**, *Classification of Integrable Evolution Equations*, Soviet Scientific Reviews, Section C, **4**, 221-280, 1984
- **Mikhailov A.V., Shabat A.B., Yamilov R.I.**, *The symmetry approach to the classification of non-linear equations.*, Russian Math. Surveys, **42**(4), 1–63, 1987.
- **Mikhailov A.V., Sokolov V.V., Shabat A.B.**, *The symmetry approach to classification of integrable equations*, in "What is Integrability? Springer Series in Nonlinear Dynamics, 115–184, 1991.
- **Adler V. E., Shabat A. B., Yamilov R. I.**, *Symmetry approach to the integrability problem*, Theoret. and Math. Phys., **125**(3), 1–63, 355—424, 2000
- **Mikhailov A.V., Sokolov V.V.**, *Symmetries of differential equations and the problem of integrability*, in „Integrability“. Lecture Notes in Physics. Springer, **767**, 19–88, 2009.

Infinitesimal symmetries.

Suppose we have a dynamical system

$$\frac{d\,u_i}{dt} = F_i(u_1, \ldots, u_n), \qquad i = 1, \ldots, n\,. \tag{1}$$

Definition. The dynamical system

$$\frac{d\,u_i}{d\tau} = G_i(u_1, \ldots, u_n), \qquad i = 1, \ldots, n \tag{2}$$

is called (infinitesimal) symmetry for (1) iff (1) and (2) are compatible.

The compatibility means that

$$XY - YX = 0,$$

where $X = \sum F_i \frac{\partial}{\partial u_i}$, $Y = \sum G_i \frac{\partial}{\partial u_i}$.

Consider evolution equation

$$u_t = F(u, u_x, u_{xx}, \ldots, u_n), \quad u_i = \frac{\partial^i u}{\partial x^i}. \tag{3}$$

The generalized symmetry (the same: higher symmetry or commuting flow) is an evolution equation

$$u_\tau = G(u, u_x, u_{xx}, \ldots, u_m)$$

that is compatible with (3). Compatibility means that

$$F_*(G) - G_*(F) = 0.$$

Here for any function $a(u, ..., u_k)$ the Fréchet derivative a_* is defined as a linear differential operator of the form

$$a_* = \sum_{i=0}^{k} \frac{\partial a}{\partial u_i} D^i, \qquad \text{where} \quad D = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}.$$

Examples:

Example 1. For any m the linear equation $u_\tau = u_m$ is a symmetry for $u_t = u_n$.

Example 2. The Burgers equation

$$u_t = u_{xx} + 2uu_x$$

has the following third order symmetry

$$u_\tau = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x.$$

Example 3. The simplest higher symmetry for the Korteweg-de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

has the following form

$$u_\tau = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x.$$

Why integrable equations have higher symmetries?

"Explanation". Linear equations have infinitely many higher symmetries. Any integrable nonlinear equation is related to a linear equation by some transformation. The same transformation produces higher symmetries for nonlinear equation from symmetries of the corresponding linear one.

For instance, the Burgers equation $u_t = u_{xx} + 2uu_x$ is linearizable by the Cole-Hopf substitution

$$u = \frac{v_x}{v},$$

which relates the equation to $v_t = v_{xx}$. Moreover, the same substitution maps the third order symmetry of the Burgers equation to

$$v_\tau = v_{xxx},$$

etc.

The first classification result obtained with the symmetry approach was:

Theorem (Zhiber-Shabat 1979). Nonlinear hyperbolic equation of the form

$$u_{xy} = F(u)$$

possesses higher symmetries iff (up to scalings and shifts)

$$F(u) = e^u,\ F(u) = e^u + e^{-u},\ \text{or}\ F(u) = e^u + e^{-2u}.$$

The complete classification of integrable hyperbolic equations of the form

$$u_{xy} = F(u, u_x, u_y)$$

is an open problem till now.

Example:

$$u_{xy} = S(u)\sqrt{u_x^2 - 1}\sqrt{u_y^2 - 1},$$

$$S'^2 = k_1 S^4 + k_2 S^2 + k_3;$$

Let n=ord F, m=ord G and $m > n$. Then the r.h.s. of

$$F_*(G) - G_*(F) = 0.$$

depends on $u, ..., u_{n+m-1}$ and is polynomial in $u_{m+1}, ..., u_{n+m-1}$. By definition, it should be identity w.r.t. $u, ..., u_{n+m-1}$.

Some necessary integrability conditions for F that do not depend on the symmetry order were found by Ibragimov - Shabat - VS. It was proved by Svinolupov-VS that the same conditions fulfilled if equation possesses infinitely many local conservation laws.

To formulate the conditions we will need formal pseudo-differential series of the form

$$A = a_m D^m + a_{m-1} D^{m-1} + \cdots + a_0 + a_{-1} D^{-1} +$$

$$a_{-2} D^{-2} + \cdots \qquad a_k \in \mathcal{K}, \quad m \in \mathbb{Z}.$$

Here by $\mathcal{K}$ we denote the set of all functions depending on $u, u_1, u_2, ...$

The product of two formal series is defined by

$$D^k \circ bD^m = bD^{m+k} + C_k^1 D(b) D^{k+m-1} +$$

$$C_k^2 D^2(b) D^{k+m-2} + \cdots ,$$

where $k, m \in \mathbb{Z}$ and C_n^j is the binomial coefficient

$$C_k^j = \frac{k(k-1)(k-2)\cdots(k-j+1)}{j!} .$$

Definition. The residue of a formal series $A = \sum_{k \leqslant n} a_k D^k$, $a_k \in \mathcal{K}$ is by definition the coefficient at D^{-1}:

$$\operatorname{res}(A) = a_{-1} .$$

The *logarithmic residue* of A is defined as

$$\operatorname{res} \log A = \frac{a_{n-1}}{a_n} .$$

We will use the following important Adler's

Theorem. For any two formal series A, B the residue of the commutator belongs to $\operatorname{Im} D$:

$$\operatorname{res}[A, B] = D(\sigma(A, B)),$$

where

$$\sigma(A, B) = \sum_{p \leqslant \operatorname{ord}(B),\ q \leqslant \operatorname{ord}(A)}^{p+q+1>0} C_q^{p+q+1} \times$$

$$\sum_{s=0}^{p+q} (-1)^s D^s(a_q) D^{p+q-s}(b_q)\,.$$

Formal recursion operator.

Definition. A pseudo-differential series

$$L = l_1 D + l_0 + l_{-1} D^{-1} + \cdots ,$$

where $l_k = l_k(u, \ldots, u_{s_k})$, is called a formal recursion operator (or formal symmetry) for the equation

$$u_t = F(u, \ldots, u_{n-1}, u_n)$$

if

$$L_t = [F_*,\, L], \quad \text{where} \quad F_* = \sum_{k=0}^{n} \frac{\partial F}{\partial u_k} D^k \, .$$

Theorem 1 (Ibragimov-Shabat 1980). If equation $u_t = F$ possesses an infinite hierarchy of higher symmetries

$$u_{\tau_i} = G_i(u, , \ldots, u_{m_i}), \qquad m_i \to \infty,$$

then the equation has a formal recursion operator.

Proof

Linearizing the equations

$$u_t = F(u, u_x, u_{xx}, \ldots, u_n), \qquad u_\tau = G(u, u_x, u_{xx}, \ldots, u_m),$$

we get

$$\psi_t = F_*(\psi), \qquad \psi_\tau = G_*(\psi)$$

The compatibility of these linear equations yields

$$(G_*)_t - [F_*, G_*] = (F_*)_\tau \, .$$

Comparing this with $L_t - [F_*, L] = 0$, we see that if $\mathrm{ord} L = m$ then $m-1$ coefficients of L there exist. Thus we can take first $m-1$ coefficients of $G_*^{\frac{1}{m}}$ for coefficients of the formal recursion operator.

The formal recursion operator allows us to construct local conservation laws for the equation $u_t = F$:

Proposition. The functions

$$\rho_i = res(L^i), \quad i = -1, 1, 2, \dots, \text{ and } \quad \rho_0 = \frac{l_0}{l_1}$$

are conserved densities. We call ρ_i canonical densities.

Proof. Take the trace of both sides of

$$L^i_t = [F_*, \, L^i]$$

and apply Adler's theorem.

Example. For the Korteweg-de Vries equation $u_t = u_3 + 6uu_1$ we can take

$$L = \left(D^2 + 4u + 2u_1 D^{-1}\right)^{1/2}$$

and

$$\rho_1 = 2u, \quad \rho_2 = 2u_1, \quad \rho_3 = 2u_2 + u^2, \dots$$

A simple classification problem.

Consider equations of the form

$$u_t = u_3 + f(u, u_1).$$

Let us find the simplest canonical density ρ_1. Equating the coefficients of $D^3, D^2, \ldots$ in

$$L_t - [F_*,\, L] = 0,$$

where

$$L = l_1 D + l_0 + l_{-1} D^{-1} + \cdots\,,$$

$$F_* = D^3 + \frac{\partial f}{\partial u_1} D + \frac{\partial f}{\partial u},$$

we get:

$$D^3 :\; 3D(l_1) = 0; \quad D^2 :\; 3D^2(l_1) + 3D(l_0) = 0;$$

$$D :\; D^3(l_1) + 3D^2(l_0) + 3D(l_{-1}) + \frac{\partial f}{\partial u_1}\, D(l_1) =$$

$$(l_1)_t + l_1\, D\left(\frac{\partial f}{\partial u_1}\right).$$

If we put $l_1 = 1$ then

$$\rho_1 = l_{-1} = \frac{1}{3}\,\frac{\partial f}{\partial u_1}$$

Thus we discovered a very remarkable fact:

$$\left(\frac{\partial f}{\partial u_1}\right)_t = D_x(\sigma_1)$$

for any integrable equation !

Example. For the mKdV-equation $u_t = u_3 + 3u^2 u_1$ we expect that $\rho_1 = u^2$ is a conserved density. Indeed,

$$(u^2)_t = D(2uu_2 - u_1^2 + \frac{1}{2}u^4).$$

We can eliminate unknown σ_1 applying the Euler operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D \circ \frac{\partial}{\partial u_1} + D^2 \circ \frac{\partial}{\partial u_2} - \cdots$$

As the result we get the first integrability condition

$$0 = \frac{\delta}{\delta u} \left(\frac{\partial f}{\partial u_1} \right)_t = 3u_4 \left(u_2 \frac{\partial^4 f}{\partial u_1^4} + u_1 \frac{\partial^4 f}{\partial u_1^3 \partial u} \right) + \cdots$$

It implies

$$f(u, u_1) = \mu u_1^3 + A(u) u_1^2 + B(u) u_1 + C(u).$$

For such f the first condition is equivalent to

$$\mu A' = 0, \qquad B''' + 8\mu B' = 0,$$
$$(B'C)' = 0, \qquad AB' + 6\mu C' = 0.$$

It is almost enough to complete the classification.

The second integrability condition has the form

$$\left(\frac{\partial f}{\partial u}\right)_t = D(\sigma_2)$$

Using this fact we derive several more differential relations between $A(u)$, $B(u)$, $C(u)$. Solving them altogether we obtain the following list of integrable equations

$$\begin{aligned}
u_t &= u_{xxx} + (c_1 u^2 + c_2 u + c_3)u_x \\
u_t &= u_{xxx} - \frac{1}{2}u_x^3 + \left(c_1 e^{2u} + c_2 e^{-2u} + c_3\right) u_x \\
u_t &= u_{xxx} + c_1 u_x^3 + c_2 u_x^2 + c_3 u_x + c_4
\end{aligned}$$

Formal symplectic and Hamiltonian operators

Definition. A pseudo-differential series

$$S = s_1 D + s_0 + s_{-1} D^{-1} + \cdots ,$$

where $s_k = s_k(u, \ldots, u_{s_k})$, is called a formal symplectic operator for the equation

$$u_t = F(u, \ldots, u_{n-1}, u_n)$$

if

$$S_t + F_*^t S + S F_* = 0, \quad S^t = -S.$$

Theorem 2 (Svinolupov-VS 1982). If equation $u_t = F$ possesses an infinite hierarchy of local conservation laws $(\rho_i)_t = (\sigma_i)_x$ then the equation has a formal recursion operator L and a formal symplectic operator S such that

$$L^t = -SLS^{-1}.$$

It follows from the latter identity that $\rho_{2i} \in \operatorname{Im} D$.

Proof. For several first coefficients for S of order $2k$ one can take the coefficients of

$$S_k = \left(\frac{\delta \rho_k}{\delta u}\right)_*,$$

where ρ_k is a conserved density of order k. Then

$$L \approx \left(S_p^{-1} S_q\right)^{\frac{1}{2q-2p}}, \qquad S \approx S_p L^{1-2p}.$$

Similarly we can construct a formal Hamiltonian operator

$$H = h_1 D + h_0 + h_{-1} D^{-1} + \cdots ,$$

such that

$$H_t - H F_*^t - F_* H = 0, \qquad H^t = -H,$$

and

$$L^t = -H^{-1} L H.$$

Theorem (Svinolupov-VS 1982). A complete list (up to „almost invertible“ transformations) of equations of the form

$$u_t = u_{xxx} + f(u, u_x, u_{xx}) \tag{4}$$

with infinite hierarchy of conservation laws can be written as:

$$\begin{aligned}
u_t &= u_{xxx} + u\, u_x, && \text{KdV} \\
u_t &= u_{xxx} + u^2\, u_x, && \text{mKdV} \\
u_t &= u_{xxx} - \frac{1}{2} u_x^3 + (\alpha e^{2u} + \beta e^{-2u}) u_x, && \text{CD1} \\
u_t &= u_{xxx} - \frac{1}{2} Q''\, u_x + \frac{3}{8} \frac{(Q - u_x^2)_x^2}{u_x\,(Q - u_x^2)}, && \text{CD2} \\
u_t &= u_{xxx} - \frac{3}{2} \frac{u_{xx}^2 + Q(u)}{u_x} && \text{KN}\ ,
\end{aligned}$$

where $Q'''''(u) = 0$.

The class of equations (4) is invariant w.r.t. arbitrary point transformations $u \to \phi(v)$.

In particular, if $Q' \neq 0$, then we can perform the transformation $u = f(v)$, where $(f')^2 = Q(f)$ in the equations CD2 and KN. As the result we get

$$v_t = v_{xxx} - \frac{3 v_x v_{xx}^2}{2(v_x^2+1)} - \frac{3}{2}\wp(v) v_x (v_x^2+1),$$

and

$$v_t = v_{xxx} - \frac{3 v_{xx}^2}{2 v_x} + \frac{1}{v_x} - \frac{3}{2}\wp(v) v_x^3,$$

where $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$,

$$g_2 = \frac{4}{3}c_2^2 - 4c_1c_3 + 16c_0c_4,$$

$$g_3 = \frac{8}{27}c_2^3 - \frac{4}{3}c_1c_2c_3 - \frac{32}{3}c_0c_2c_4 + 4c_0c_3^2 + 4c_1^2c_4.$$

A complete list of equations (4) having generalized symmetries has been obtained by S.Svinolupov and VS in 1982. It consists of 13 equations up to the point transformation. A proof of this statement can be found in

Meshkov A.G., Sokolov V.V., *Integrable evolution equations with constant separant*, Ufa Mathematical Journal, **4**(3), 104–154, 2012, ISSN 2074-1863.

In particular, the proof contains an algorithm for a bringing of any integrable equation to one of the 13 canonical forms.

In this survey we present also the following recursion formula for the canonical densities.

All canonical densities $(\rho_n)_t = (\theta_n)_x, \quad n = 0, 1, ...,$ are given by

$$\rho_{n+2} = \frac{1}{3}\Bigg[\theta_n - \delta_{n,0} F_0 - F_1 \rho_n -$$

$$F_2\Big(D(\rho_n) + 2\rho_{n+1} + \sum_{s=0}^{n} \rho_s\, \rho_{n-s}\Big)\Bigg]$$

$$-\sum_{s=0}^{n+1} \rho_s\, \rho_{n+1-s} - \frac{1}{3} \sum_{0 \leqslant s+k \leqslant n} \rho_s\, \rho_k\, \rho_{n-s-k}$$

$$-D\Bigg[\rho_{n+1} + \frac{1}{2}\sum_{s=0}^{n} \rho_s\, \rho_{n-s} + \frac{1}{3}D(\rho_n)\Bigg], \quad n \geqslant 0,$$

where $F_i = \frac{\partial F}{\partial u_i}$, and

$$\rho_0 = -\frac{1}{3}F_2, \qquad \rho_1 = \frac{1}{9}F_2^2 - \frac{1}{3}F_1 + \frac{1}{3}D(F_2).$$

All equations of the form

$$u_t = u_5 + F(u, u_1, u_2, u_3, u_4),$$

possessing higher conservation laws were found by Drinfeld-VS-Svinolupov 1985.

Examples: Well-known equations

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1,$$

$$u_t = u_5 + 5uu_3 + \tfrac{25}{2}u_1u_2 + 5u^2u_1$$

$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2$$
$$-5u_1^3 + 5u^4u_1$$

A new equation

$$\begin{aligned} u_t \;=\; & u_5 + 5(u_2 - u_1^2 + \lambda_1 e^{2u} - \lambda_2^2 e^{-4u})\,u_3 \\ & -5u_1u_2^2 + 15(\lambda_1 e^{2u} + 4\lambda_2^2 e^{-4u})\,u_1u_2 + u_1^5 \\ & -90\lambda_2^2 e^{-4u}\,u_1^3 + 5(\lambda_1 e^{2u} - \lambda_2^2 e^{-4u})^2\,u_1 \end{aligned}$$

Part 2. Integrable vector systems

Main concepts of the symmetry approach can be generalized to systems of evolution equations (A. Mikhailov, A. Shabat, R. Yamilov). However, component-wise computations in this case are very tedious. The only one serious classification problem has been solved: all systems of the form

$$u_t = u_2 + F(u,\, v,\, u_1,\, v_1), \qquad u_t = -v_2 + G(u,\, v,\, u_1,\, v_1)$$

possessing higher conservation laws were listed.

Examples: Well-known NLS-equation

$$u_t = u_2 + u^2 v, \qquad v_t = -v_2 - v^2 u;$$

one of the versions of the Boussinesq equation

$$u_t = u_2 + (u+v)^2, \qquad v_t = -v_2 + (u+v)^2;$$

and Landau-Lifshitz equation

$$u_t = u_2 - \frac{2u_1^2}{u+v} - \frac{4\,(p(u,v)\,u_1 + r(u)\,v_1)}{(u+v)^2}$$
$$v_t = -v_2 + \frac{2v_1^2}{u+v} - \frac{4\,(p(u,v)\,v_1 + r(-v)\,u_1)}{(u+v)^2},$$

where $r(y) = c_4y^4 + c_3y^3 + c_2y^2 + c_1y + c_0$ and

$$p(u,v) = 2c_4u^2v^2 + c_3(uv^2 - vu^2) - 2c_2uv + c_1(u-v) + 2c_0,$$

are basic models in a very long list of integrable systems.

But there exist several classes of systems where we can avoid the component-wise computations.

Integrable isotropic and anisotropic vector evolution equations of the form

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2\mathbf{u}_{xx} + f_1\mathbf{u}_x + f_0\mathbf{u} \qquad (5)$$

were studied by A.Meshkov and VS. Here $\mathbf{u}$ is N-component vector, the coefficients f_i depend on some scalar products between $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}$.

Equation (5) is called isotropic if the coefficients f_i are scalar functions in the following six variables:

$$(\mathbf{u},\, \mathbf{u}),\ (\mathbf{u}, \mathbf{u}_x),\ (\mathbf{u}_x, \mathbf{u}_x),\ (\mathbf{u}, \mathbf{u}_{xx}),\ (\mathbf{u}_x, \mathbf{u}_{xx}),\ (\mathbf{u}_{xx}, \mathbf{u}_{xx}). \qquad (6)$$

It is clear that isotropic models are invariant with respect to the group $SO(N)$.

Examples. The following vector mKdV-systems:

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\,\mathbf{u}_x.$$

and

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\,\mathbf{u}_x + (\mathbf{u}, \mathbf{u}_x)\,\mathbf{u}$$

are integrable for any N.

We consider equations (5) that are integrable for arbitrary dimension N. In virtue of the arbitrariness of N, variables (6) can be regarded as independent.

By integrability of such equations we mean the existence of infinite series of generalized symmetries

$$\mathbf{u}_{\tau_k} = g_k\,\mathbf{u}_k + g_{k-1}\,\mathbf{u}_{k-1} + \cdots + g_1\,\mathbf{u}_x + g_0\,\mathbf{u}, \qquad \mathbf{u}_i = \frac{\partial^i \mathbf{u}}{\partial x^i},$$

whose coefficients g_i depend on all possible scalar products between $\mathbf{u}, \ldots, \mathbf{u}_k$.

Theorem (A.Meshkov, VS 2002).

- i). If equation

$$\mathbf{u}_t = f_n\,\mathbf{u}_n + f_{n-1}\,\mathbf{u}_{n-1} + \cdots + f_1\,\mathbf{u}_1 + f_0\,\mathbf{u}, \qquad (7)$$

possesses an infinite series of generalized symmetries of the form

$$\mathbf{u}_\tau = g_m\,\mathbf{u}_m + g_{m-1}\,\mathbf{u}_{m-1} + \cdots + g_1\,\mathbf{u}_1 + g_0\,\mathbf{u}, \qquad (8)$$

then there exists a formal series

$$L = a_1\,D + a_0 + a_{-1}\,D^{-1} + a_{-2}\,D^{-2} + \cdots\,, \qquad (9)$$

satisfying the operator relation

$$L_t = [A,\,L], \qquad A = \sum_0^n f_i\,D^i. \qquad (10)$$

Here f_i are the coefficients of equation (7).

- ii). The following functions

$$\rho_{-1} = \frac{1}{a_1}, \qquad \rho_0 = \frac{a_0}{a_1}, \qquad \rho_i = \operatorname{res} L^i, \qquad i \in \mathbb{N} \qquad (11)$$

  are conserved densities for equation (7).
- iii). If equation (7) possesses an infinite series of conserved densities, then there exist a series L satisfying (10), and a series S of the form

$$S = s_1\, D + s_0 \, + s_{-1}\, D^{-1} + s_{-2}\, D^{-2} + \cdots\,,$$

  such that

$$S_t + A^t\, S + S\, A = 0, \quad S^t = -S, \quad L^t = -S^{-1} L S,$$

  where the upper index t stands for a formal conjugation.
- iiii). Under the conditions of item iii) densities (11) with $i = 2k$ are of the form $\rho_{2k} = D(\sigma_k)$ for some functions σ_k.

Isotropic equations on the sphere

Let us assume that $\mathbf{u}^2 = 1$. Then $(\mathbf{u}, \mathbf{u}_x) = 0$ and $(\mathbf{u}, \mathbf{u}_{xx}) = -(\mathbf{u}_x, \mathbf{u}_x)$. Therefore we have only three independent scalar products

$$(\mathbf{u}_x, \mathbf{u}_x), \quad (\mathbf{u}_x, \mathbf{u}_{xx}), \quad (\mathbf{u}_{xx}, \mathbf{u}_{xx})$$

in the coefficients of the equation.

Theorem (Meshkov-VS 2002). Suppose an equation of the form (5) on the sphere $\mathbf{u}^2 = 1$ has an infinite series of generalized symmetries or conserved densities; then this equation belongs to the following list:

List of integrable isotropic equations on the sphere

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_{xx} + \frac{3}{2}\frac{u_{[2,2]}}{u_{[1,1]}}\,\mathbf{u}_x,$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_{xx} + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2\,(1 + a\,u_{[1,1]})}\right)\mathbf{u}_x,$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} + \frac{3}{2}\left(\frac{a^2\,u_{[1,2]}^2}{1 + a\,u_{[1,1]}} - a\,(u_{[2,2]} - u_{[1,1]}^2) + u_{[1,1]}\right)\mathbf{u}_x + \\ + 3\,u_{[1,2]}\,\mathbf{u},$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{(q+1)\;u_{[1,2]}}{2\,q\,u_{[1,1]}}\,\mathbf{u}_{xx} + 3\,\frac{(q-1)\;u_{[1,2]}}{2\,q}\,\mathbf{u}$$

$$+\frac{3}{2}\left(\frac{(q+1)\,u_{[2,2]}}{u_{[1,1]}} - \frac{(q+1)\,a\,{u_{[1,2]}}^2}{q^2 u_{[1,1]}} + u_{[1,1]}\,(1-q)\right)\mathbf{u}_x,$$

where $u_{[i,j]} := (\mathbf{u}_i, \mathbf{u}_j)$ and $q = \sqrt{1 + a\,u_{[1,1]}}$.

Notice that if $a = 0$ and therefore $q = \pm 1$ then the latter equation yields

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_{xx} + 3\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\mathbf{u}_x,$$

and

$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3\,u_{[1,1]}\,\mathbf{u}_x + 3\,u_{[1,2]}\,\mathbf{u},$$

Anisotropic equations

Consider the following equation (I. Golubchik-VS 2000):

$$\mathbf{u}_t = \Big(\mathbf{u}_{xx} + \frac{3}{2}(\mathbf{u}_x,\ \mathbf{u}_x)\mathbf{u}\Big)_x + \frac{3}{2}\langle \mathbf{u}, \mathbf{u}\rangle \mathbf{u}_x, \qquad \mathbf{u}^2 = 1. \qquad (12)$$

Here $\langle \mathbf{a}\,, \mathbf{b}\rangle = (\mathbf{a},\ R\mathbf{b})$, where R is an arbitrary constant symmetric matrix R. One can assume that $R = \mathrm{diag}(r_1, \ldots, r_N)$. Equation (12) has a Lax pair whose spectral parameter lies on an algebraic curve of genus $1 + (N-3)2^{N-2}$. If $N = 3$, then (12) is a symmetry for the famous Landau-Lifshitz equation.

In this case the coefficients in

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2\mathbf{u}_{xx} + f_1\mathbf{u}_x + f_0\mathbf{u}$$

depends on two different independent scalar products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$.

Theorem (Meshkov-VS 2002). Suppose equation (5) on the sphere $(\mathbf{u},\mathbf{u})=1$ with

$$f_i = f_i(u_{[1,1]},\, u_{[1,2]},\, u_{[2,2]},\, v_{[0,0]},\, v_{[0,1]},\, v_{[1,1]}),$$

where $v_{[i,j]} := \langle \mathbf{u}_i, \mathbf{u}_j\rangle$. has an infinite series of symmetries or conserved densities; then this equation is one of the above or belongs to the following list:

$$\mathbf{u}_t = \mathbf{u}_3 - \frac{3\,u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_2 + \left(\frac{3\,u_{[2,2]}}{2\,u_{[1,1]}} + \frac{3\,u_{[1,2]}^2}{2\,u_{[1,1]}^2} + \frac{c\,v_{[1,1]}}{u_{[1,1]}}\right)\mathbf{u}_1,$$

$$\mathbf{u}_t = \mathbf{u}_3 + \left(c\,v_{[0,0]} + \frac{3}{2}\,u_{[1,1]}\right)\mathbf{u}_1 + 3\,u_{[1,2]}\,\mathbf{u}_0,$$

$$\mathbf{u}_t = \mathbf{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{(v_{[0,0]}+a)\,u_{[1,2]}^2}{q\,u_{[1,1]}^2} - 2\frac{v_{[0,1]}u_{[1,2]}}{q\,u_{[1,1]}} + \frac{v_{[1,1]}}{u_{[1,1]}} - \frac{v_{[0,1]}^2}{q\,u_{[1,1]}}\right)\mathbf{u}_1,$$

where $q = u_{[1,1]} + v_{[0,0]} + a$.

The classification of anisotropic equations on the sphere with

$$f_i = f_i(u_{[1,1]},\, u_{[1,2]},\, u_{[2,2]},\, v_{[0,0]},\, v_{[0,1]},\, v_{[1,1]},\, v_{[0,2]},\, v_{[1,2]},\, v_{[2,2]})$$

was completed by M. Balakhnev and A. Meshkov in 2005.

Example (Balakhnev-Meshkov 2005)

$$\mathbf{u}_t = \mathbf{u}_3 - 3\frac{v_{[0,1]}}{v_{[0,0]}}\mathbf{u}_2 - 3\left(\frac{v_{[0,2]}}{v_{[0,0]}} - 2\frac{v_{[0,1]}^2}{v_{[0,0]}^2}\right)\mathbf{u}_1 + 3\left(u_{[1,2]} - \frac{v_{[0,1]}}{v_{[0,0]}}u_{[1,1]}\right)\mathbf{u},$$

Integrable hyperbolic vector equations the the sphere were studied by A. Meshkov and VS.

Example (Meshkov-VS 2012)

$$\mathbf{u}_{xy} = \frac{\mathbf{u}_x}{\langle \mathbf{u}, \mathbf{u}\rangle}\left(\langle \mathbf{u}, \mathbf{u}_y\rangle + \sqrt{1 + \langle \mathbf{u}, \mathbf{u}\rangle |\mathbf{u}_x|^{-2}}\ \phi\right) - (\mathbf{u}_x, \mathbf{u}_y)\mathbf{u},$$

$$\text{where} \qquad \phi = \sqrt{\langle \mathbf{u}, \mathbf{u}_y\rangle^2 + \langle \mathbf{u}, \mathbf{u}\rangle(1 - \langle \mathbf{u}_y, \mathbf{u}_y\rangle)}$$