Algebraic ansatz for heat equation and integrable polynomial dynamical systems

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We will discuss the ansatz that reduces the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t) \tag{1}$$

to a homogeneous polynomial dynamical system.
For any such system in the generic case we obtain a nonlinear ordinary differential equation and algorithm for constructing a solution of this system.
As result we have the corresponding solution of the heat equation.

We give the full classification of nonlinear ordinary differential equations that arise from our ansatz.

Symmetry group

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ and

$$\Gamma(M)\psi(z,t) = \frac{1}{\sqrt{ct+d}} \exp\left(\frac{-cz^2}{2(ct+d)}\right) \psi\left(\frac{z}{ct+d}, \frac{at+b}{ct+d}\right).$$

We have

$$\Gamma(M_1)\big(\Gamma(M_2)\psi(z,t)\big) = \Gamma(M_1 \cdot M_2)\psi(z,t).$$

Lemma

If $\psi(z,t)$ is a solution to the heat equation (1), then so is $\Gamma(M)\psi(z,t)$.

Classical solution:

$$\Gamma(M)1 = \frac{1}{\sqrt{ct+d}} \exp\left(\frac{-cz^2}{2(ct+d)}\right).$$

General construction

Any even (odd) regular at $z = 0$ function $\psi(z, t)$ may be presented in the form

$$\psi(z, t) = e^{-\frac{1}{2}h(t)z^2 + r(t)} z^{2s} \phi(z, t), \qquad (2)$$

where s is a non-negative integer and
the function $\phi(z, t)$ in the vicinity of $z = 0$ is given by the series

$$\phi(z, t) = z^\delta + \sum_{k \geqslant 2} \phi_k(t) \frac{z^{2k+\delta}}{(2k + \delta)!} ,$$

where $\delta = 0$ (correspondingly, $\delta = 1$).
Further we assume $\delta = 0$ or $\delta = 1$.

Lemma

The function $\psi(z,t)$ *of the form* (2) *is a solution to the heat equation*

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t)$$

if and only if $s = 0$, $r' = -\left(\frac{1}{2} + \delta\right) h$, *and the function* $\phi(z,t)$ *is a solution to*

$$\frac{\partial}{\partial t}\phi = \mathcal{H}_2\phi - h\mathcal{H}_0\phi,$$

where

$$\mathcal{H}_2\phi = \left(\frac{1}{2}\frac{\partial^2}{\partial z^2} + uz^2\right)\phi,\; u = \frac{1}{2}\left(h' + h^2\right),\; \mathcal{H}_0\phi = \left(z\frac{\partial}{\partial z} - \delta\right)\phi.$$

We will assume all constants have the grading 0.

For given n set $\mathbf{x} = (x_2, \ldots, x_{n+1})$, $\deg x_q = -4q$, and

$$\Phi(z; \mathbf{x}) = z^\delta + \sum_{k \geqslant 2} \Phi_k(\mathbf{x}) \frac{z^{2k+\delta}}{(2k+\delta)!}, \qquad (3)$$

where z, $\deg z = 2$, is a variable independant of $\mathbf{x}$.

Here $\Phi_k(\mathbf{x})$ are homogeneous polynomials, $\deg \Phi_k(\mathbf{x}) = -4k$.

For example $\Phi_2(\mathbf{x}) = c x_2$, where $c = const$.
A series $\Phi(z; \mathbf{x})$ is called generic if for $k = 2, \ldots, n+1$

$$\frac{\partial \Phi_k(\mathbf{x})}{\partial x_k} \neq 0.$$

Algebraical ansatz

We say that a solution of the heat equation is in the n-*ansatz* if it has the form

$$\psi(z,t) = e^{-\frac{1}{2}h(t)z^2 + r(t)}\Phi(z;\mathbf{x}(t)) \tag{4}$$

for series $\Phi(z;\mathbf{x})$ of form (3).

An n-ansatz is called generic if the series $\Phi(z;\mathbf{x})$ is generic.

Consider a set of homogeneous polynomials $p_q(\mathbf{x})$, $\mathbf{x} = (x_2, \dots, x_{n+1})$, $q = 3, \dots, n+2$, $\deg p_q = -4q$ such that

$$\frac{\partial p_q(\mathbf{x})}{\partial x_q} \neq 0.$$

Introduce the operator

$$L_2 = \sum_{k=2}^{n+1} p_{k+1}(\mathbf{x}) \frac{\partial}{\partial x_k}. \tag{5}$$

Theorem

The heat equation (1) *has a solution in generic n-ansatz if and only if there is an operator* L_2 *of form* (3) *such that*

$$\Phi_3 = 2L_2\Phi_2,$$

$$\Phi_{k+1} = 2L_2\Phi_k + \frac{(2k+\delta)(2k-1+\delta)}{2(1+2\delta)}\Phi_2\Phi_{k-1}, \quad k > 2.$$

Main steps of proof

Let $\Phi(z; \mathbf{x})$ be a series of form (3). By construction $\Phi(z; \mathbf{x})$ is a homogeneous function of degree 2δ.

The homogeneity of $\Phi(z; \mathbf{x})$ implies

$$\mathcal{H}_0 \Phi = L_0 \Phi,$$

where

$$L_0 = \sum_{k=2}^{n+1} 2k x_k \frac{\partial}{\partial x_k}\,. \qquad (6)$$

Lemma

Let L_2 be an operator given by formula (5). Then the function $\Phi(z;\mathbf{x})$ of the form (3) gives a solution to the equation

$$\mathcal{H}_2\Phi = L_2\Phi$$

if and only if

$$\Phi_2 = -4(1+2\delta)u, \qquad \Phi_3 = 2L_2\Phi_2,$$

$$\Phi_{k+1} = 2L_2\Phi_k + \frac{(2k+\delta)(2k-1+\delta)}{2(1+2\delta)}\Phi_2\Phi_{k-1}, \quad k > 2. \qquad (7)$$

Corollary

Under the conditions of this lemma we have $u = -\frac{c}{4(1+2\delta)}x_2$, where $c = const$.

Lemma

Let $\Phi(z;\mathbf{x})$ *be a generic series of the form* (3)*. From the equation*

$$\mathcal{H}_2\Phi = L_2\Phi$$

the coefficients of the operator L_2 *are uniquely defined by the following system*

$$\Phi_2 = -4(1+2\delta)u, \qquad \Phi_3 = 2L_2\Phi_2,$$

and for $k = 3, \ldots, n+1$

$$\Phi_{k+1} = 2L_2\Phi_k + \frac{(2k+\delta)(2k-1+\delta)}{2(1+2\delta)}\,\Phi_2\Phi_{k-1}.$$

Let $p_q(\mathbf{x})$, $q = 3, \ldots, n+2$, be a set of homogeneous polynomials as before. Consider the dynamical system

$$\frac{d}{dt} x_k(t) = p_{k+1}\big(\mathbf{x}(t)\big) - 2kh(t)x_k(t), \quad k = 2, \ldots, n+1. \quad (8)$$

Lemma

For the function $\Phi(z; \mathbf{x})$ of the form (3) with coefficients satisfying (7) the function

$$\phi(z, t) = \Phi\big(z; \mathbf{x}(t)\big),$$

where $x(t)$ is a solution of system (8) gives a solution to the equation

$$\frac{\partial}{\partial t}\phi = \mathcal{H}_2\phi - h\mathcal{H}_0\phi.$$

Heat equation and dynamical systems

Theorem

There exists a one-to-one correspondence between the set of homogeneous polynomial dynamical systems

$$\frac{d}{dt}\,r = -\left(\frac{1}{2}+\delta\right)h, \qquad \frac{d}{dt}\,h = -h^2 - \frac{c}{2(1+2\delta)}\,x_2,$$

$$\frac{d}{dt}\,x_k = p_{k+1}(\mathbf{x}) - 2khx_k, \quad k = 2,\dots,n+1,$$

where $\frac{\partial p_{k+1}(\mathbf{x}(t))}{\partial x_{k+1}} \neq 0$ *for* $k = 2,\dots,n$, *and the set of functions*

$$\psi(z,t) = e^{-\frac{1}{2}h(t)z^2 + r(t)}\Phi(z;\mathbf{x}(t))$$

defining generic n-anzats solutions of the heat equation (1).

Dynamical systems and ordinary differential equations

A homogeneous polynomial transform

$$X_2 = c_2 x_2, \quad X_k = c_k x_k + q_k(x_2, \ldots, x_{k-1}), \quad k = 3, \ldots, n+1,$$

where $c_k = const \neq 0$ and $\deg q_k = -4k$ brings
the homogeneous polynomial dynamical system

$$\frac{d}{d\tau} x_k = p_{k+1}(\mathbf{x}), \qquad k = 2, \ldots, n+1,$$

to

$$\frac{d}{d\tau} X_k = P_{k+1}(\mathbf{x}), \quad k = 2, \ldots, n+1$$

where

$$P_{k+1}(\mathbf{x}) = c_k p_{k+1}(\mathbf{x}) - \sum_q p_{q+1}(\mathbf{x}) \frac{\partial}{\partial x_q} q_k(\mathbf{x}).$$

A polynomial dynamical system

$$\frac{d}{d\tau} x_k = p_{k+1}(\mathbf{x})$$

is called *reduced* if it is defined by the set of polynomials

$$\begin{aligned} p_k(\mathbf{x}) &= x_k, \quad k = 3, \dots, n+1, \\ p_{n+2}(\mathbf{x}) &= P_n(\mathbf{x}), \end{aligned} \tag{9}$$

where $P_n(\mathbf{x})$ is a homogeneous polynomial of degree $-4(n+2)$.

Lemma

Each generic n-ansatz solution to the heat equation may be obtained by our construction using a reduced system (9) *and a solution to the system*

$$\begin{aligned} \frac{d}{dt}\, r &= -\left(\frac{1}{2} + \delta\right) h, \\ \frac{d}{dt}\, h &= -h^2 + x_2, \\ \frac{d}{dt}\, x_k &= x_{k+1} - 2khx_k, \quad k = 2, \dots, n, \\ \frac{d}{dt}\, x_{n+1} &= P_n(\mathbf{x}) - 2(n+1)hx_{n+1}. \end{aligned} \tag{10}$$

The system (10) is determined by n and $P_n(\mathbf{x})$.
In this system $x_2, \dots, x_{n+1}$ are determined by

$$x_2 = h' + h^2, \quad x_k = x_{k-1}' + 2(k-1)hx_{k-1}, \quad k = 3, \dots, n+1,$$

and r is determined up to a constant r_0 by $r' = -\left(\frac{1}{2} + \delta\right) h$.
Substituting x_k as functions of $h(t)$ into

$$\frac{d}{dt} x_{n+1} = P_n(\mathbf{x}) - 2(n+1)hx_{n+1},$$

we get an ordinary differential equation of order $n+1$

$$\mathcal{D}_{P_n, n+1}(h) = 0.$$

This equation is homogeneous with respect to
the grading $\deg h = -4$, $\deg t = 4$.

Theorem

Each generic n-ansatz solution to the heat equation is defined by the set (n, P_n, h, r_0), *where:*

– n *is a natural number,*

– P_n *is a homogeneous polynomial* $P_n(\mathbf{x})$ *of degree* $-4(n+2)$,

– h *is a solution* $h(t)$ *to the equation* $\mathcal{D}_{P_n,n+1}(h) = 0$,

– r_0 *is a constant.*

Ordinary differential equations, associated with the heat equation

Set

$$\mathcal{D}_1(h) = \left(\frac{d}{dt} + h\right) h \quad \text{and} \quad \mathcal{D}_n(h) = \left(\frac{d}{dt} + 2nh\right) \mathcal{D}_{n-1}(h)$$

for $n \geqslant 2$. We have $\deg \mathcal{D}_n(h) = \deg x_{n+1}$.

Theorem

For the vector-function

$$\mathcal{D}(h) = \big(\mathcal{D}_1(h), \ldots, \mathcal{D}_{n-1}(h)\big)$$

and a homogeneous polynomial $P_n(x_2, \ldots, x_n)$, $\deg P_n = -4(n+2)$, *the formula holds*

$$\mathcal{D}_{P_n, n+1}(h) = \mathcal{D}_{n+1}(h) - P_n(\mathcal{D}(h)).$$

The next result is obtained in collaboration with E. Rees.

Let $\mathscr{L}$ be a linear operator satisfying the Leibniz rule such that

$$\mathscr{L} h = 1, \qquad \mathscr{L} h^{(k)} = -(k+1)kh^{(k-1)}, \quad k \geqslant 1.$$

Theorem

A homogeneous differential polynomial

$$\mathscr{D}(h) = h^{(n+1)} + \cdots + \gamma h^{n+2}$$

has the form $\mathcal{D}_{P_n,n+1}(h)$ for some $P_n(\mathcal{D}(h))$
if and only if

$$\mathscr{L}\mathscr{D}(h) \equiv 0.$$

n-ansatz solutions of the heat equation

<u>$n = 0$.</u> We have

$$\mathcal{D}_1(h) = h' + h^2.$$

The general solution to $\mathcal{D}_1(h) = 0$ is

$$h(t) = \frac{a}{at - b}\,.$$

Thus the function

$$\psi(z,t) = \frac{\exp r_0}{(at-b)^{\frac{1}{2}+\delta}} \exp\left(-\frac{az^2}{2(at-b)}\right) z^\delta$$

solves the heat equation. In the case $\delta = 0$, $r_0 = 0$, $b/a = c$ it coincides with the classical solution to the heat equation.

Corollary

For $\delta = 0$ and $b/a < t < \infty$ the 0-ansatz solution is the Gaussian density distribution.

$\underline{n = 1.}$ We have

$$\mathcal{D}_2(h) = \left(\frac{d}{dt} + 4h\right)\mathcal{D}_1(h)$$

and $\deg x_2 = -8$, $\deg P_1 = -12$, thus $P_1(x_2) \equiv 0$.
The system (10) has the form

$$\frac{d}{dt}\,r = -\left(\frac{1}{2} + \delta\right)h, \qquad \frac{d}{dt}\,h = -h^2 + x_2, \qquad \frac{d}{dt}\,x_2 = -4hx_2.$$

Therefore $h(t)$ is a solution to

$$h'' + 6hh' + 4h^3 = 0. \tag{11}$$

The general solution to this equation has the form

$$h(t) = \frac{1}{2}\left(\frac{1}{t-a} + \frac{1}{t-b}\right).$$

Differentiate (11) and put $y(t) = 2h(t)$ to get Chazy-4 equation:

$$y''' = -3yy'' - 3y'^2 - 3y^2y'.$$

Set $\mathcal{G}_1(z,t) = \exp\left(-\frac{1}{2}\,h(t)z^2 + r(t)\right) z^\delta$. Then

$$\mathcal{G}_1(z,t) = \frac{1}{(t-a)^{\frac{1+2\delta}{4}}(t-b)^{\frac{1+2\delta}{4}}} \exp\left(-\frac{1}{4}\left(\frac{1}{t-a} + \frac{1}{t-b}\right)z^2 + r_0\right) z^\delta.$$

The function $\mathcal{G}_1(z,t)$ solves the equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial z^2}\right)\mathcal{G}_1(z,t) = \frac{1}{8}\left(\frac{1}{t-a} - \frac{1}{t-b}\right)^2 z^2 \mathcal{G}_1(z,t)$$

For $a = b$ we get the case $n = 0$:

$$\mathcal{G}_1(z,t) = \frac{1}{(t-a)^{\frac{1}{2}+\delta}} \exp\left(-\frac{1}{2}\frac{1}{(t-a)}z^2 + r_0\right) z^\delta.$$

Now let $a \neq b$. Then $x_2(t) = \frac{1}{2}\left(\frac{(a-b)}{2(t-a)(t-b)}\right)^2$.

According to our construction, we have

$$\Phi(z; x_2) = z^\delta + \sum_{q=1}^{\infty} \Phi_{2q}(x_2) \frac{z^{4q+\delta}}{(4q+\delta)!}, \tag{12}$$

where

$$\Phi_2 = 4(1+2\delta)x_2, \quad \Phi_{2q} = 2(4q+\delta-3)(4q+\delta-2)x_2\Phi_{2q-2}, \quad q = 2, 3 \ldots.$$

Note the function $\Phi(z;x_2)$ (see (12)) solves

$$\frac{d^2}{dz^2}\Phi(z;x_2) - 2x_2 z^2 \Phi(z;x_2) = 0. \tag{13}$$

Lemma

$\Phi(z;x_2) = z^\delta \gamma(z^4; x_2, \delta)$,
where $\gamma(v;x_2,\delta)$ is a solution to the differential equation

$$\frac{4v}{3+2\delta}\gamma''(v) + \gamma'(v) = \lambda\gamma(v), \qquad \textit{with} \quad \lambda = \frac{1}{2(3+2\delta)}x_2$$

with the initial condition $\gamma(0) = 1$.
Thus the function $\gamma(v)$ is the eigenfunction of the generalized shift operator, defined by the generator

$$\frac{4}{3+2\delta}v\frac{d^2}{dv^2} + \frac{d}{dv}.$$

$\underline{n=2.}$ We have $P_2(x_2, x_3) = c_4 x_2^2$, where $c_4 = const$ and

$$\mathcal{D}_3(h) = \left(\frac{d}{dt} + 6h\right)\mathcal{D}_2(h), \qquad \mathcal{D}_{P_2,3}(h) = \mathcal{D}_3(h) - c_4\mathcal{D}_1(h)^2.$$

The system (10) has the form

$$\frac{d}{dt}\,r = -\left(\frac{1}{2} + \delta\right)h, \qquad \frac{d}{dt}\,h = -h^2 + x_2,$$
$$\frac{d}{dt}\,x_2 = -4hx_2 + x_3, \qquad \frac{d}{dt}\,x_3 = c_4 x_2^2 - 6hx_3. \tag{14}$$

Therefore $h(t)$ is a solution to

$$h''' + 12hh'' - 18(h')^2 + (24 - c_4)(h' + h^2)^2 = 0. \tag{15}$$

Equation (15) is brought by the substitution $y(t) = -6h(t)$ to

$$y''' = 2yy'' - 3(y')^2 + \frac{24 - c_4}{216}(6y' - y^2)^2. \tag{16}$$

For $\frac{24-c_4}{216} = -\frac{4}{k^2-36}$ the equation (16) is Chazy-12
if k is integer and $k > 1$, $k \neq 6$.

For $c_4 = 24$ this equation is Chazy-3.

The function $-2\left(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{1}{t-a_3}\right)$ is the general solution to the equation Chazy-12 with $c_4 = -3$ (or $k = 2$). In this case

$$h(t) = \frac{1}{3}\left(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{1}{t-a_3}\right).$$

Classical solutions related with Chazy-3 equation.

I. Consider an elliptic curve in standard Weierstrass form

$$V = \{(\lambda, \mu) \in \mathbb{C}^2 : \mu^2 = 4\lambda^3 - g_2\lambda - g_3\}.$$

Weierstrass sigma function $\sigma(z; g_2, g_3)$ is an entire function. At $z = 0$ it is given by a series of z with *homogeneous polynomial* in g_2, g_3 coefficients, $\deg g_k = -4k$, $k = 2, 3$.

We have

$$\sigma(z; g_2, g_3) = z - \frac{g_2}{2}\,\frac{z^5}{5!} - 6g_3\frac{z^7}{7!} - 9\frac{g_2^2}{4}\,\frac{z^9}{9!} + \ldots .$$

Set $\deg z = 2$, then $\sigma(z; g_2, g_3)$ is a homogeneous series in z, g_2, g_3 and $\deg \sigma = 2$.

Consider the fields on $\mathbb{C}^2$

$$l_0 = 4g_2 \frac{\partial}{\partial g_2} + 6g_3 \frac{\partial}{\partial g_3}, \qquad l_2 = 6g_3 \frac{\partial}{\partial g_2} + \frac{1}{3} g_2^2 \frac{\partial}{\partial g_3}.$$

We have $[l_0, l_2] = 2l_2, \quad l_0\Delta = 12\Delta, \quad l_2\Delta = 0$, where $\Delta = g_2^3 - 27g_3^2$ is the *discriminant* of the curve.

The Weierstrass theorem

The operators

$$Q_0 = z\frac{\partial}{\partial z} - 1 - l_0, \quad Q_2 = \frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{24} g_2 z^2 - l_2$$

annihilate the sigma-function, that is

$$Q_0 \sigma(z; g_2, g_3) = 0, \qquad Q_2 \sigma(z; g_2, g_3) = 0.$$

As a corollary of the Weierstrass theorem we obtain

Theorem

The function

$$\psi(z,t) = e^{-\frac{1}{2}h(t)z^2 + r(t)}\sigma\big(z, g_2(t), g_3(t)\big)$$

is a solution to the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t)$$

if and only if the function $y(t) = -6h(t)$ satisfies the Chazy-3 equation $y''' = 2y''y - 3y'^2$ and the formulas hold

$$r' = -\frac{3}{2}h, \quad g_2 = 12h' + 12h^2, \quad g_3 = \frac{1}{6}g_2' + \frac{2}{3}hg_2.$$

II. The *z-periodic odd* function

$$\psi(z,t) = \exp\left(-\frac{1}{2}a^2 t\right)\frac{\sin az}{a}, \quad a = const,$$

with $\psi(0,t) = 0$, $\psi'(0,t) = \exp\left(-\frac{1}{2}a^2 t\right)$, is a solution to the heat equation.

We have

$$r = -\frac{1}{2}\,a^2 t, \quad h = \frac{1}{3}\,a^2, \quad g_2 = \frac{4}{3}\,\gamma^4, \quad g_3 = \frac{8}{27}\,\gamma^6.$$

III. The *decreasing* when $z \to \pm\infty$ solution is

$$\psi(z,t) = \psi_*(z,t) - \psi_*(-z,t),$$

where

$$\psi_*(z,t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{(z-a)^2}{2t}\right)$$

with $\psi(0,t) = 0$ and $\psi'(0,t) = \frac{2a}{t\sqrt{t}} \exp\left(-\frac{a^2}{2t}\right)$.

We have

$$r = \ln\left(\frac{2a}{\sqrt{t^3}}\right) - \frac{a^2}{2t}, \quad h = -\frac{a^2-3t}{3t^2}, \quad g_2 = \frac{4}{3}\frac{a^4}{t^4}, \quad g_3 = -\frac{8}{27}\frac{a^6}{t^6}.$$

$\underline{n = 3.}$

$$\mathcal{D}_4(h) = \left(\frac{d}{dt} + 8h\right)\mathcal{D}_3(h).$$

We have $P_3(\mathbf{x}) = c_5 x_2 x_3$, where $c_5 = const$ and

$$\mathcal{D}_{P_3,4}(h) = \mathcal{D}_4(h) - c_5\mathcal{D}_1(h)\mathcal{D}_2(h).$$

The system (10) has the form

$$\frac{d}{dt} r = -\left(\frac{1}{2} + \delta\right) h, \qquad \frac{d}{dt} h = -h^2 + x_2, \qquad \frac{d}{dt} x_2 = x_3 - 4hx_2,$$

$$\frac{d}{dt} x_3 = x_4 - 6hx_3, \qquad \frac{d}{dt} x_4 = c_5 x_2 x_3 - 8hx_4.$$

Therefore $h(t)$ is a solution to

$$h'''' + 20hh''' - 24h'h'' + 96h^2h'' - 144h(h')^2 + \\ + (48 - c_5)(h' + h^2)(h'' + 6hh' + 4h^3) = 0. \quad (17)$$

In the case $c_5 = -16$ with the substitution $y(t) = 4h(t)$ equation (17) takes the form

$$y'''' + 5yy''' + 10y'y'' + 10y^2y'' + 15y(y')^2 + 10y'y^3 + y^5 = 0. \quad (18)$$

This equation appeared in the work of N.A.Kudryashov (2001).

Theorem

The general solution to (18) *is*

$$h(t) = \frac{1}{4}\left(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{1}{t-a_3} + \frac{1}{t-a_4}\right).$$

The question of finding all c_5 for which equation (17) has Painlevé property is still open.

For some function $h = h(t)$ and a constant b consider the $(n+2) \times (n+2)$-matrix

$$S_n(h) = \begin{pmatrix} bh & -1 & \dots & 0 & 0 \\ bh' & bh & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{b}{n!} h^{(n)} & \frac{b}{(n-1)!} h^{(n-1)} & \dots & bh & -(n+1) \\ \frac{b}{(n+1)!} h^{(n+1)} & \frac{b}{n!} h^{(n)} & \dots & bh' & bh \end{pmatrix}$$

We have

$$\frac{1}{b} \det S_n(h) = h^{(n+1)} + (n+2) b h h^{(n)} + \dots + b^{n+1} h^{n+2}.$$

The equation

$$\frac{1}{b}\det S_n(h) = 0 \tag{19}$$

is an ordinary differential equation homogeneous with respect to the grading $\deg h = -4$, $\deg t = 4$.

Theorem

For any integer $n \geqslant 0$ the function

$$h(t) = h_n(t) = \frac{1}{b}\sum_{k=1}^{n+1}\frac{1}{t-a_k}$$

is the general solution to the equation (19).

Lemma

For the operators

$$\frac{1}{b}\det S_n(h) \quad \textit{and} \quad \mathcal{D}_{P_n,n+1}(h)$$

to coincide for some P_n it is necessary that

$$b = (n+1).$$

It has been proved that this condition is sufficient for $n \leqslant 8$.
For $n > 8$ this problem is still open.

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ADDENDUM.
Fuchs–Poincaré problem.

In 1884 L. Fuchs and H. Poincaré have considered the general problem of integrating differential equations and came to the conclusion that it is closely connected to the *problem* of defining *new functions* by means of nonlinear ordinary differential equations.

Movable singularities.

A point is called a *singularity* of a function if this function is not analytic (possibly not defined) in that point.

A singularity of a function is called *a critical singularity* if going around this singularity changes the value of the function.

L. Fuchs remarked that differential equation solutions can have *movable singularities*, that is singularities whose location depends on the initial data.

Consider the first-order explicit differential equation

$$y' = F(t, y)$$

with F being a rational function in y and a locally-analytical function in t.

L. Fuchs proved that among such equations only the Riccati equation

$$y' = P_0(t) + P_1(t)y + P_2(t)y^2$$

does not have movable critical singularities.

All first-order *algebraic* differential equations without movable critical singularities can be transformed into the Riccati equation or the Weierstrass equation

$$(y')^2 = 4y^3 - g_2 y - g_3.$$

Both this equations are integrable in terms of previously known special functions.

In 1888 S. Kovalevskaya solved the classical precession of a top under the influence of gravity problem.

S. Kovalevskaya's approach to the problem is based on finding solutions with no movable critical singularities.

She proved that there exists only three cases with such solutions. Two of them are the famous Euler and Lagrange tops.

In the third case (now named Kovalevskaya top) she found new solutions and thus was first to discover the advantages of solving differential equations whose solutions have no movable critical singularities.

Painlevé property.

The property of a differential equation that its solutions have no movable critical singularities is well known now as the *Painlevé property*.

The general solution to equation with Painlevé property leads to the *single-valued* function.

All *linear* ordinary differential equations have the Painlevé property, but it turns out that this property is rare for *nonlinear* differential equations.

Around 1900, P. Painlevé studied second order explicit nonlinear differential equations

$$y'' = F(t, y, y')$$

with F being a rational function in y, y' and a locally-analytical function in t.

It turned out that among such equations up to certain transformations only *fifty* equations have the Painlevé property, and among them *six* are not integrable in terms of previously known functions.

P. Painlevé and B. Gambier have introduced new special functions, now known as *Painlevé transcendents*, as the general solutions to this equations.

Chazy equations.

In 1910 J. Chazy extended Painlevé's work to higher order equations.

He considered the problem of classification of all third-order explicit differential equations of the form

$$y''' = F(t, y, y', y''),$$

where F is a polynomial in y, y', y'' and locally analytic in t, having the Painlevé property.

The most known *autonomic* Chazy equations are

Chazy-3 equation: $y''' = 2yy'' - 3(y')^2,$

Chazy-12 equation: $y''' = 2yy'' - 3(y')^2 - \dfrac{4}{k^2 - 36}(6y' - y^2)^2,$

where $k \in \mathbb{N}$, $k > 1$, $k \neq 6$.

It was recently shown by M. J. Ablowitz, S. Chakravarty, R. Halburd that this Chazy equations are reductions of the self-dual Yang-Mills equations with an infinite-dimensional gauge algebra.

The relation of considered equations to mathematical physics arises from an observation by M. J. Ablowitz and H. Segur that reductions of a partial differential equation of the soliton type give rise to ordinary differential equations that posess the Painlevé property.

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ADDENDUM. Solutions in terms of Weierstrass functions

It is non-degenerate for $g_2^3 \neq 27 g_3^2$. Set

$$2\omega_k = \oint_{a_k} \frac{d\lambda}{\mu}, \qquad 2\eta_k = -\oint_{a_k} \frac{\lambda d\lambda}{\mu}, \qquad k = 1, 2,$$

where $\frac{d\lambda}{\mu}$ and $\frac{\lambda d\lambda}{\mu}$ are basis holomorphic differentials and a_k are basis cycles on the curve.

We have

$$\eta_1\omega_2 - \omega_1\eta_2 = \frac{\pi i}{2}.$$

A non-degenerate elliptic curve V defines
the rank 2 lattice $\Gamma \subset \mathbb{C}$
generated by $2\omega_1$ and $2\omega_2$ with $\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$.

The *Jacobian* of the curve V is the complex torus $\mathbb{T} = \mathbb{C}/\Gamma$.

An *elliptic function* is a meromorphic function on $\mathbb{C}$ such that

$$f(z + 2\omega_1) = f(z), \quad f(z + 2\omega_2) = f(z).$$

The *Weierstrass function*

$$\wp(z) = \wp(z; g_2, g_3)$$

is the unique elliptic function with periods $2\omega_1$, $2\omega_2$ and poles only in lattice points such that

$$\lim_{z\to 0}\left(\wp(z) - \frac{1}{z^2}\right) = 0.$$

It defines the uniformization of the elliptic curve in the standard Weierstrass form

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

The *Weierstrass function*

$$\sigma(z) = \sigma(z; g_2, g_3)$$

is the entire *odd* function such that

$$\big(\ln \sigma(z)\big)'' = -\wp(z) \qquad \text{and} \qquad \lim_{z\to 0}\left(\frac{\sigma(z)}{z}\right) = 1.$$

Periodic properties:

$$\sigma(z + 2\omega_k) = -\sigma(z) \exp\big(2\eta_k(z + \omega_k)\big), \quad k = 1, 2.$$

Degenerate case:

$$\sigma\left(z; \frac{4}{3}a^4, \frac{8}{27}a^6\right) = \frac{1}{a}\exp\left(\frac{1}{6}a^2 z^2\right)\sin az.$$

Elementary solutions

The function $\psi(z,t)$ of the form

$$\psi(z,t) = e^{-\frac{1}{2}h(t)z^2}\phi(z,t),$$

where $\phi(z,t)$ is a polynomial of z,
is a solution to the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t)$$

if and only if it is a linear combination with constant coefficients of the function

$$\frac{1}{\sqrt{t-c}}\exp\left(\frac{-z^2}{2(t-c)}\right)$$

and its derivatives with respect to z. Here $c = const$.

ADDENDUM.
Ramanujan system.

Set $a_n(q) = \frac{q^n}{1-q^n}$. Consider the classical Eisenstein series for $|q| < 1$

$$P(q) = 1 - 24\sum_{n=1}^{\infty} na_n(q); \qquad Q(q) = 1 + 240\sum_{n=1}^{\infty} n^3 a_n(q);$$

$$R(q) = 1 - 504\sum_{n=1}^{\infty} n^5 a_n(q).$$

In 1916 S.Ramanujan employed the trigonometric series identities to proof that $P(q)$, $Q(q)$ and $R(q)$ are the solutions to system

$$12q\frac{d}{dq}P = P^2 - Q; \qquad 3q\frac{d}{dq}Q = PQ - R; \qquad 2q\frac{d}{dq}R = PR - Q^2.$$

Ramanujan system is brought by substitution $q = \exp(-12t)$ to

$$\frac{d}{dt} P = -P^2 + Q;$$

$$\frac{d}{dt} Q = -4PQ + 4R;$$

$$\frac{d}{dt} R = 6Q^2 - 6PR.$$

It is the system (14) for $h = P$, $x_2 = Q$, $x_3 = 4R$ and $c_4 = 24$. Hence P is a solution to Chazy-3 equation.