Kinetic equation for a soliton gas: a new integrable system?

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Outline

- **Motivation & Background**
- **Zakharov's kinetic equation for a rarefied soliton gas**
- **Kinetic equation for solitons as the thermodynamic limit of the Whitham equations**
- **Hydrodynamic reductions and integrability**
- **Conclusions**

Motivation & Background

- **Main premise**: nonlinear wave systems integrable by the IST can demonstrate complex behaviour demanding a statistical description;
- **Recent experimental/observational evidence** of the presence of turbulent regimes in physical systems well described by integrable equations: nonlinear optics, BECs, shallow-water waves;
- **Two problems**:
  - Wave (weak) turbulence
  - Soliton turbulence.
- **Three important references:**
  - V.E. Zakharov, Kinetic equation for solitons, Sov. Phys. JETP, 1971;
  - P.D Lax, The zero-dispersion limit, a deterministic analogue of turbulence, Comm. Pure Appl. Math., 1991
  - V.E. Zakharov, Turbulence in integrable systems, Stud. Appl. Math., 2009

Zakharov's kinetic equation for a rarefied soliton gas.

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0\,.$$

Consider a rarefied soliton gas – an infinite sequence of the KdV solitons randomly distributed on $\mathbb{R}$ with small density $\rho \ll 1$.

More precisely: it is assumed that at each moment t the solution almost everywhere can be represented in the form $u \approx \sum_{n=1}^{\infty} 2\eta_n^2 \cosh^{-2}(\eta_n(x - x_n))$, where x_n is a random discrete variable distributed by Poisson with small density.

Main assumption: 'local' validity of the N-soliton solution.

Introduce the continuous spectral distribution function $f(\eta)$ for η_n such that the number of solitons with $\eta_n \in [\eta, \eta + d\eta]$ in the interval $[x, x + dx]$ is $f(\eta)d\eta dx$.

- Free "η-soliton" velocity: $S = 4\eta^2$ ('trial' soliton)
- Each collision with a "μ-soliton" ($\mu \neq \eta$) leads to a shift in its position: $\delta(\eta, \mu) = \pm\frac{1}{\eta} \ln \left|\frac{\eta+\mu}{\eta-\mu}\right|$ ("$+$" if $\eta > \mu$ and "$-$" if $\eta < \mu$).
- Owing to collisions, the path covered by the trial η-soliton over large time interval t will differ from $4\eta^2 t$

Zakharov's kinetic equation for a rarefied soliton gas.

Average (over a large distance) number of collisions of a "trial" η-soliton with all other "μ-solitons" ($\eta > \mu$), $\mu \in [\mu, \mu + d\mu]$, per second
$=$ relative velocity $\times$ density of μ-solitons $\approx (4\eta^2 - 4\mu^2) f(\mu) d\mu$

Then the mean (averaged over a large distance) speed of the η-soliton, $s(\eta; x, t)$ is determined by the balance relation:

$$s(\eta)dt = 4\eta^2 dt + \text{'Total phase shift over } [t, t+dt]\text{'} \implies$$

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_0^\infty \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[4\eta^2 - 4\mu^2]d\mu + O(\rho^2). \qquad (1)$$

Here $\rho = \int_0^\infty f(\eta)d\eta \ll 1$ is the spatial density of solitons.
Now, consider (1) as a local relationship in a spatially nonuniform soliton gas and introduce

$$f(\eta) \equiv f(\eta; x, t), \qquad s(\eta) \equiv s(\eta; x, t); \qquad \Delta x, \Delta t \gg 1.$$

Then isospectrality of the KdV dynamics implies:

$$f_t + (sf)_x = 0, \qquad (2)$$

Eqs. (1), (2): approximate **kinetic description** of a rarefied soliton gas.

Soliton gas: two approaches

***N*-solitons:**

- IST: reflectionless potentials (N-soliton solutions);
- Finite-gap theory: closing all spectral bands in the N-gap potential leads to the N-soliton.

Soliton gas: $N \to \infty$; a generalised reflectionless potential (Marchenko) with shift invariant probability measure on it.

- IST:
  - *Gurevich, Mazur and Zybin (2000); Mazur, Geogjaev, Gurevich and Zybin (2002)*: statistical version of the Lax-Levermore approach (KdV and defocusing NLS)
  - *Kotani (2008)*: KdV flow on generalized reflectionless potentials (Marchenko's approach to non-decaying reflectionless potentials)
- Finite-gap theory: *GE, Krylov, Molchanov and Venakides (2001)*: soliton turbulence as the thermodynamic limit of finite-gap potentials.

Soliton gas as the thermodynamic limit of finite-gap potentials

GE & Krylov (1999); GE, Krylov, Molchanov & Venakides (2001).

Starting point: quasi-periodic analogs of N-soliton solutions – nonlinear multiphase solutions parameterized by $2N+1$ constants λ_j.

$$u_N(x) = u_N(\theta_1, \dots, \theta_N | \lambda_1, \dots, \lambda_{2N+1});$$

$$u_N(\theta_1, \dots, \theta_j + 2\pi, \dots \theta_N) = u_N(\theta_1, \dots, \theta_j, \dots, \theta_N)$$

$$\theta_j = k_j x + \theta_j^0 \, , \qquad k_j = k_j(\lambda_1, \dots, \lambda_{2N+1}), \quad j = 1, \dots, N$$

$\theta \in \mathrm{Tor}^N$, and k_j are the torus 'frequencies';

Let θ_j^0 be random values uniformly distributed on Tor^N then u_N becomes a random process.

Soliton gas as the thermodynamic limit of finite-gap potentials

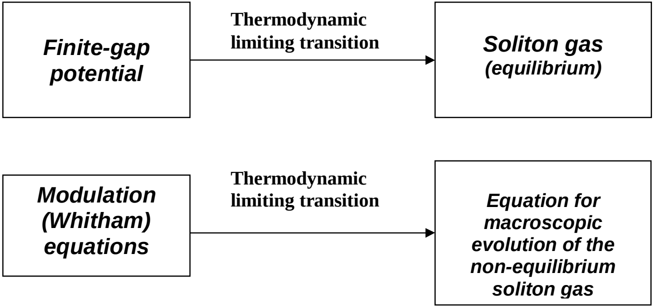
- If $k_j \to 0$ for some particular j: appearance of a soliton in the solution.
- If $k_j \to 0$ for all $j = 1, \ldots, N$: the entire N-phase solution degenerates into the N-soliton solution.

The total 'density of waves' in the N-phase solution is $\rho_N = \sum_{j=1}^{N} k_j$

We want to take the limit $k_j \to 0,\ j = 1, \ldots, N,\ N \to \infty$, such that ρ_N would converge to some nonzero value: the thermodynamic-type limit.

It is clear that the limit would exist only for a special band/gap distribution (scaling).

Thermodynamic limit of the Whitham equation – the other side of the story *(GE, Phys. Lett. A 2003)*



Modulations of finite-gap potentials: the Whitham equations

- Finite-gap solution of the KdV equation

$$u_N(x,t) = \Phi(\theta_1, \dots, \theta_N | \lambda_1, \dots, \lambda_{2N+1}), \qquad (1)$$

$$\theta_j = k_j x - \omega_j t + \theta_j^0, \quad j = 1, \dots, N, \quad \mathbf{k} = \mathbf{k}(\lambda); \ \omega = \omega(\lambda)$$

- Let $X = \epsilon x, \ T = \epsilon t, \ \epsilon << 1$ and $\lambda_j = \lambda_j(X,T)$. Let the function (1) be the principal term of the asymptotic as $\epsilon \to 0$ solution of the KdV equation.
- Then, to first order in ϵ we obtain equations for $\lambda_j(X,T)$ – the Whitham equations *(Flaschka, Forest & McLaughlin 1980)*

$$\frac{\partial \lambda_j}{\partial T} + V_j(\lambda_1 \dots, \lambda_{2N+1}) \frac{\partial \lambda_j}{\partial X} = 0, \qquad j = 1, \dots, 2N+1,$$

where the characteristic velocities $V_j(\lambda_1 \dots, \lambda_{2N+1})$ are certain combination of complete hyperelliptic integrals

The Whitham equations

The generating equation for the Whitham system is *(Flaschka, Forest, McLaughlin 1980)*

$$\partial_T dp_N = \partial_X dq_N \,,$$

where dp_N and dq_N are the quasimomentum and quasienergy differentials

$$dp_N(\lambda) = \frac{\lambda^N + b_{N-1}\lambda^{N-1} + \cdots + b_0}{R(\lambda)} d\lambda \,,$$

$$dq_N(\lambda) = 12\frac{\lambda^{N+1} + c_N\lambda^N + \cdots + c_0}{R(\lambda)} d\lambda \,, \qquad c_N = -\frac{1}{2}\sum_{j=1}^{2N+1} \lambda_j$$

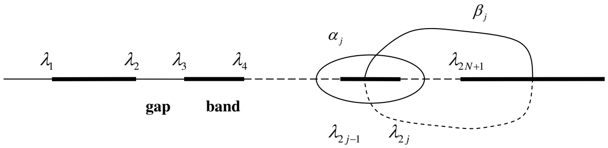
on the hyperelliptic Riemann surface of genus N (the branch points are the endpoints of spectral bands):

$$R^2(\lambda) = \prod_{j=1}^{2N+1} (\lambda - \lambda_j) \,, \qquad \lambda \in \mathbb{C}, \quad \lambda_j \in \mathbb{R} \,.$$

The Whitham equations

Coefficients b_j, c_j are found from the normalisation over the β-cycles:

$$\oint_{\beta_i} dp_N(\lambda) = 0\,, \qquad \oint_{\beta_i} dq_N(\lambda) = 0\,, \qquad i = 1, \ldots, N\,; \qquad c_0 = -\frac{1}{2}\sum_{j=1}^{2N+1} \lambda_j\,.$$



Then the wavenumbers k_j and frequences ω_j are found as

$$k_j(\lambda_1, \ldots, \lambda_{2N+1}) = \oint_{\alpha_j} dp_N(\lambda)\,, \quad \omega_j(\lambda_1, \ldots, \lambda_{2N+1}) = \oint_{\alpha_j} dq_N(\lambda)\,,$$

Note: $k_j > 0$, $\omega_j > 0$.

Whitham equations as the equations for the spectral measure

Assume that the finite-band part of the spectrum λ lies in $[-1, 0]$.
We integrate the Whitham system $\partial_t dp_N(\lambda) = \partial_x dq_N(\lambda)$ on the real line from -1 to $-\eta^2 \in [-1, 0]$ and take the real part to obtain:

$$\partial_t \mathcal{N}_N(-\eta^2) = \partial_x \mathcal{V}_N(-\eta^2)\,.$$

Here

$$\mathcal{N}_N(\lambda) = \frac{1}{\pi} Re \int_{-1}^{\lambda} dp_N(\lambda')$$

is the **integrated density of states** (*Johnson & Moser (1982)*), and
$\mathcal{V}_N(\lambda) = \frac{1}{\pi} Re \int_{-1}^{\lambda} dq_N(\lambda')$ – its temporal analog.

Importantly, $d\mathcal{N}_N(-\eta^2)$ is a measure supported on the spectrum (*Johnson & Moser (1982)*).

Thermodynamic limit

The total density of states

$$\rho_N = \frac{1}{\pi} Re \int_{-1}^{0} dp_N(\lambda') = \frac{1}{2\pi} \sum_{j=1}^{N} k_j$$

In the thermodynamic limit, $\forall k_j \to 0$ but $\lim_{N\to\infty} \rho_N = O(1)$.
This is achieved by the following (thermodynamic) spectral scaling

$$|\text{gap}_j| \sim \frac{1}{\varphi(\eta_j)N} \qquad |\text{band}_j| \sim \exp\{-\gamma(\eta_j)N\}, \quad j = 1, \dots, N$$

where $\phi(\eta)$, $\gamma(\eta)$ are some continuous positive functions on $[0, 1]$.

- it Venakides (1989) The continuum limit of theta functions;

Note: in the thermodynamic limit $|\text{band}_j|/|\text{gap}_j| \to 0$ as $N \to \infty$ $\forall j$ i.e. **"infinite-soliton" limit**.

The thermodynamic limit of the Whitham equations

- The modulation system $\partial_t d\mathcal{N}_N(-\eta^2) = \partial_x d\mathcal{V}_N(-\eta^2)$
- In the thermodynamic limit as $N \to \infty$:
  - $d\mathcal{N}_N \to \pi f(\eta) d\eta > 0$, $\quad d\mathcal{V}_N \to -\pi f(\eta) s(\eta) d\eta$
  - $s(\eta)$ and $f(\eta)$ become related via integral equation:

$$s(\eta) = -4\eta^2 + \frac{1}{\eta} \int_0^1 \ln \left| \frac{\eta - \mu}{\eta + \mu} \right| f(\mu)[s(\eta) - s(\mu)] d\mu, \qquad (1)$$

Note: $f(\eta)$ has the meaning of the distribution function.

Now, we postulate that on a larger scale, $\Delta x, \Delta t \gg 1$:

$$f(\eta) = f(\eta, x, t), \qquad s(\eta) = s(\eta, x, t)$$

Then the modulation system transforms into

$$f_t = (fs)_x \, , \qquad (2)$$

Equations (2), (1) form a closed system : the kinetic equation for the KdV soliton gas of **finite density** (*El 2003*)

Kinetic equation for solitons: small-density expansion

We replace $s \to -s$. Now s is the velocity of soliton gas.

$$f_t + (fs)_x = 0\,, \tag{1}$$

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_0^\infty \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[s(\eta)-s(\mu)]d\mu, \tag{2}$$

The small-density, $\rho = \int_0^\infty f d\eta \ll 1$, expansion of (2), yields

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_0^\infty \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[4\eta^2-4\mu^2]d\mu + \mathcal{O}(\rho^2), \tag{3}$$

– the velocity of a 'trial' soliton in a rarefied soliton gas (Zakharov 1971). So Eqs. (1), (2) represent a generalisation of Zakharov's kinetic equation for a rarefied soliton gas to the case of the gas of finite density.

Generalised kinetic equations for soliton gases with elastic collisions *GE & Kamchatnov (PRL 2005)*

- Main ingredients: (i) **the speed of a free soliton** $S(\eta)$ and (ii) **the phase shift** $\Delta x_{\eta,\mu} = G(\eta, \mu)$ due to the soliton-soliton collision.
- Introduce the spectral distribution function $f(\eta) \equiv f(\eta, x, t)$ and the mean speed of a 'trial' η- soliton $s(\eta) \equiv s(\eta, x, t)$
- Then the self-consistent definition of the soliton velocity $s(\eta)$ in a dense soliton gas with the spectral distribution $f(\eta)$ is given by the integral equation

$$s(\eta) = S(\eta) + \int_0^\infty G(\eta, \mu)[s(\eta) - s(\mu)]f(\mu)d\mu \,,$$

- Isospectrality implies the conservation equation for the spectral distribution function $f(\eta, x, t)$:

$$f_t + (sf)_x = 0 \,.$$

Hydrodynamic reductions of the kinetic equation.

$$f_t+(sf)_x=0\,,\quad s(\eta,x,t)=S(\eta)+\int_0^\infty G(\eta,\mu)[s(\eta,x,t)-s(\mu,x,t)]f(\mu,x,t)d\mu \tag{1}$$

We introduce $u(\eta,x,t)=\eta f(\eta,x,t)$, $v(\eta,x,t)=-s(\eta,x,t)$ and consider N-component 'cold-gas' (multiflow) *ansatz*

$$u(\eta,x,t)=\sum_{i=1}^N u^i(x,t)\delta(\eta-\eta^i)\,,$$

which reduces (1) to a system of N hydrodynamic conservation laws,

$$\partial_t u^i=\partial_x(u^i v^i),\qquad i=1,\dots,N\,,$$

where the velocities $v^i=v(\eta^i,x,t)$ and the 'densities' u^i are related via

$$v^i=\xi_i+\sum_{m\neq i}\epsilon_{im}u^m(v^m-v^i)\,,\qquad \epsilon_{ik}=\epsilon_{ki}\,,$$

$$\xi_i=-S(\eta^i)\,,\qquad \epsilon_{ik}=\frac{1}{\eta^i\eta^k}G(\eta^i,\eta^k)>0\,,\qquad i\neq k.$$

Hydrodynamic reductions: $N = 2$ (*GE and Kamchatnov 2005)*

For $N = 2$ the system of hydrodynamic laws assumes the form

$$\partial_t u^1 = \partial_x(u^1 v^1), \qquad \partial_t u^2 = \partial_x(u^2 v^2)$$

$$u^1 = \frac{1}{\epsilon_{12}} \frac{v^2 - \xi_2}{v^1 - v^2}, \qquad u^2 = \frac{1}{\epsilon_{12}} \frac{v^1 - \xi_1}{v^2 - v^1}.$$

Passing to the *Riemann invariants* we obtain

$$v_t^1 = v^2 v_x^1, \qquad v_t^2 = v^1 v_x^2. \tag{1}$$

The system (1) is linearly degenerate, i.e. its characteristic velocities do not depend on the corresponding Riemann invariants.

What about $N > 2$?

N=3: explicit formulae (*GE, Kamchatnov, Pavlov & Zykov 2011*).

The three-component 'cold-gas' hydrodynamic reduction of the nonlocal kinetic equation

$$\partial_t u^i = \partial_x (u^i v^i), \qquad i = 1, 2, 3\,,$$

$$v^i = \xi_i + \sum_{k \neq i}^{3} \epsilon_{ik} u^k (v^k - v^i) \qquad \epsilon_{ik} = \epsilon_{ki}.$$

has the Riemann invariant representation

$$\partial_t r^j = V^j(\mathbf{r}) \partial_x r^j\,, \qquad j = 1, 2, 3,$$

where

$$V^1 = \frac{\zeta_2 r^2 - \zeta_3 r^3}{r^2 - r^3}, \quad V^2 = \frac{\zeta_3 r^3 - \zeta_1 r^1}{r^3 - r^1}, \quad V^3 = \frac{\zeta_1 r^1 - \zeta_2 r^2}{r^1 - r^2}$$

$$\zeta_1 = \frac{\xi_3 \epsilon_{12} - \xi_2 \epsilon_{13}}{\epsilon_{12} - \epsilon_{13}}, \quad \zeta_2 = \frac{\xi_1 \epsilon_{23} - \xi_3 \epsilon_{12}}{\epsilon_{23} - \epsilon_{12}}, \quad \zeta_3 = \frac{\xi_1 \epsilon_{23} - \xi_2 \epsilon_{13}}{\epsilon_{23} - \epsilon_{13}}$$

N=3: explicit formulae.

The Riemann invariants r^1, r^2, r^3 are expressed in terms of the densities u^1, u^2, u^3 as

$$
\begin{aligned}
r^1 &= \frac{(\epsilon_{12}-\epsilon_{13})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{23})}{[(\xi_3-\xi_1)\epsilon_{12}+(\xi_1-\xi_2)\epsilon_{13}]u^1-(\xi_2-\xi_3)(\epsilon_{12}u^2+\epsilon_{13}u^3+1)}, \\
r^2 &= \frac{(\epsilon_{23}-\epsilon_{12})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{13})}{[(\xi_1-\xi_2)\epsilon_{23}+(\xi_2-\xi_3)\epsilon_{12}]u^2-(\xi_3-\xi_1)(\epsilon_{12}u^1+\epsilon_{23}u^3+1)}, \\
r^3 &= \frac{(\epsilon_{13}-\epsilon_{23})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{12})}{[(\xi_2-\xi_3)\epsilon_{13}+(\xi_3-\xi_1)\epsilon_{23}]u^3-(\xi_1-\xi_2)(\epsilon_{13}u^1+\epsilon_{23}u^2+1)}.
\end{aligned}
$$

Hydrodynamic reductions: arbitrary N

Theorem (*GE, Kamchatnov, Pavlov & Zykov, J.Nonlin.Sci. 2011*)

N-component hydrodynamic type system

$$\partial_t u^i = \partial_x(u^i v^i), \qquad i = 1, \dots, N,$$

$$v^i = \xi_i + \sum_{i \neq k} \epsilon_{ik} u^k (v^k - v^i), \qquad \epsilon_{ik} = \epsilon_{ki},$$

where $\xi_1, \xi_2, \dots, \xi_N$ are constants and $\hat{\epsilon}$ is a constant symmetric matrix, $\epsilon_{ik} = \epsilon_{ki}$, is:

- *diagonalizable* ($\exists \{r^j(\mathbf{u})\}: \quad r^i_t = V^i(\mathbf{r}) r^i_x, \quad i = 1, \dots, N$)
- *linearly degenerate*, ($\partial_i V^i = 0, \quad i = 1, \dots, N$)
- *semi-Hamiltonian (i.e. integrable - Tsarev 1985, 1991),* $\partial_j \frac{\partial_k V^i}{V^k - V^i} = \partial_k \frac{\partial_j V^i}{V^j - V^i}, \quad i \neq j \neq k.$

for any N.

The proof is based on the theory of integrable linearly degenerate hydrodynamic type systems developed by Pavlov (1987) and Ferapontov (1991).

Linearly degenerate conservation laws

Pavlov (1987); Ferapontov (1991):
The system of conservation laws

$$u^i_t = (u^i v^i)_x, \quad v^i = v^i(\mathbf{u}(\mathbf{r})) \quad i = 1, \dots, N$$

is a semi-Hamiltonian linearly degenerate hydrodynamic type system iff the densities u^i and velocities $v^i(\mathbf{u})$ admit the representations

$$u^i = \frac{\det \Delta_i^{(1)}}{\det \Delta}(-1)^{i+1} P_i(r^i), \qquad v^i = \frac{\det \Delta_i^{(2)}}{\det \Delta_i^{(1)}}$$

in terms of the Stäkel matrix Δ via N functions r^k; here $P_i(r^i)$ are arbitrary functions.

For the N - component hydrodynamic reductions of the kinetic equation it was proved in (*GE, Kamchatnov, Pavlov & Zykov 2011*) that <u>such a parametrization exists for any N</u>.

Hence: integrability of the 'cold-gas' hydrodynamic reductions for any N.

Riemann form for arbitrary N: explicit (parametric) construction *Pavlov, Taranov, & GE 2012*

Let $\hat{\epsilon} = [\epsilon_{mn}]_{N\times N}$ be a symmetric matrix, $\epsilon_{ik} = \epsilon_{ki}$; and $\epsilon_{ii} = r^i(\mathbf{u})$.

Theorem 1

Algebraic system $v^i = \xi_i + \sum_{m=1}^{N} \epsilon_{im} u^m (v^m - v^i)$ *admits parametric solution:*

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \quad v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi_m \beta_{mi}, \qquad (*)$$

where symmetric functions $\beta_{ik}(\mathbf{r})$ *are the elements of the matrix* $\hat{\beta} = [\beta_{mn}]_{N\times N}$ *such that* $\hat{\beta}\hat{\epsilon} = -\mathbf{1}$.

Theorem 2 Under parametric representation (*) the N-flow reduction of the kinetic equation assumes the Riemann form

$$r^i_t = v^i(\mathbf{r}) r^i_x \qquad (**)$$

Since we have $u^i_t = (u^i v^i)_x$, system (**) is linearly degenerate, i.e. $\partial_i v^i = 0$ (Pavlov 1987).

$N = 3$: Similarity solutions.

The family of the similarity solutions.

$$r^i = \frac{1}{t^\alpha} l^i\left(\frac{x}{t}\right), \quad i = 1, 2, 3,$$

is implicitly specified by the algebraic system

$$\begin{aligned}
\frac{x}{t} &= c_1\zeta_1(l^1)^\gamma + c_2\zeta_2(l^2)^\gamma + c_3\zeta_3(l^3)^\gamma, \\
-1 &= c_1(l^1)^\gamma + c_2(l^2)^\gamma + c_3(l^3)^\gamma, \\
0 &= c_1(l^1)^{\gamma-1} + c_2(l^2)^{\gamma-1} + c_3(l^3)^{\gamma-1},
\end{aligned}$$

where $\gamma = -1/\alpha$ and c_1, c_2, c_3 are arbitrary constants.

N=3: Quasiperiodic solutions.

The family of the quasi-periodic (3 periods) solution is implicitly specified by the system

$$
\begin{aligned}
x &= \zeta_1 \int^{r^1} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_2 \int^{r^2} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_3 \int^{r^3} \frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \\
-t &= \int^{r^1} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int^{r^2} \frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int^{r^3} \frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \\
0 &= \int^{r^1} \frac{d\xi}{\sqrt{R_7(\xi)}} + \int^{r^2} \frac{d\xi}{\sqrt{R_7(\xi)}} + \int^{r^3} \frac{d\xi}{\sqrt{R_7(\xi)}},
\end{aligned}
$$

where

$$
R_7(\xi) = \prod_{m=1}^{7} (\xi - E_m)\,,
$$

$E_1 < E_2 < \cdots < E_7$ are arbitrary real constants.

Non-isospectral multi-flow reductions *(Pavlov, Taranov & GE, 2012)*

Consider a more general "non-isospectral" multi-flow *ansatz* for the distribution function

$$f(\eta, x, t) = \sum_{m=1}^{N} f^m(x,t)\delta(\eta - \eta^m(x,t))$$

Note: $\eta^k = \eta^k(x,t)$ Then the kinetic equation transforms into the $2N$-component hydrodynamic type system

$$u^i_t = (u^i v^i)_x, \quad \eta^i_t = v^i \eta^i_x, \qquad i = 1, \dots, N,$$

where $u^i = \eta^i f^i$, $v^i = -s(\eta^i, x, t)$; and v^i and u^i are related via the same algebraic closure

$$v^i = \xi_i + \sum_{m \neq i} \epsilon^{im} u^m (v^m - v^i), \quad \epsilon^{ik} = \epsilon^{ki}.$$

Note: $\epsilon^{km} = \epsilon^{km}(\boldsymbol{\eta}(x,t)) = \frac{G(\eta^i, \eta^k)}{\eta^i \eta^k}$ and $\xi_i = -S(\eta^i(x,t))$.

Half-diagonal form *(Pavlov, Taranov & GE, 2012)*

We define $\epsilon^{kk} = r^k$ and introduce a (admissible) parametrization:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \quad v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi_m \beta_{mi}, \qquad (*)$$

where symmetric functions $\beta_{ik}(\mathbf{r}, \boldsymbol{\eta})$ are the elements of the matrix $\hat{\beta} = [\beta_{mn}]_{N\times N}$ such that $\hat{\beta}\hat{\epsilon} = -\mathbf{1}$.

Theorem : *Under parametrization* (*) *the* $2N$*-component generalised multi-flow reduction of the kinetic equation for solitons assumes a half-diagonal form:*

$$\begin{aligned} \eta^i_t &= v^i \eta^i_x, \quad i = 1, \ldots, N; \\ r^k_t &= v^k r^k_x + \frac{1}{u^k}\left(\sum_{n\neq k} u^n (v^n - v^k) \frac{\partial \epsilon^{nk}}{\partial \eta^k} - \xi'_k \right) \eta^k_x, \quad k = 1, \ldots, N\,. \end{aligned}$$

INTEGRABILITY?

Conclusions

- The thermodynamic limit of the Whitham equations associated with hyperelliptic Riemann surfaces leads to the kinetic equations for the corresponding soliton gases.
- N-component 'cold-gas' isospectral hydrodynamic reductions of the generalized kinetic equation for solitons represent linearly degenerate semi-Hamiltonian (integrable) systems of hydrodynamic type for any N.
- N-flow non-isospectral hydrodynamic reductions of the generalized kinetic equation for solitons represent $2N$-component quasi-diagonal systems of hydrodynamic type with N Riemann invariants and multiple (double) characteristic velocities.
- Integrability of the full kinetic equation for a soliton gas is still an open question

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THANK YOU!