Introducing a new notion of Algebraical Integrability.

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International Workshop
"Geometric Structures in Integrable Systems",
30 October 2012

In the talk we present results obtained recently with V. M. Buchstaber.

Let us consider the general homogeneous quadratic dynamical system in an n-dimensional space.

We will call it *algebraically integrable* by given functions $h_1, \dots h_n$ if the set of roots of the equation

$$\xi^n - h_1\xi^{n-1} + \cdots + (-1)^n h_n \equiv 0$$

solves the dynamical system.

Quadratic dynamical systems

In the space $\mathbb{C}^n$ with coordinates $\xi = (\xi_1, \ldots, \xi_n)^\top$ the general quadratic dynamical system is

$$\xi_k'(t) = A_k^{i,j}\xi_i(t)\xi_j(t), \quad A_k^{i,j} = A_k^{j,i}, \quad k = 1, \ldots, n. \tag{1}$$

We can identify the space of such systems with the $\frac{1}{2}n^2(n+1)$-dimensional linear space of tensors

$$A = (A_k^{i,j})\colon A_k^{i,j} = A_k^{j,i}.$$

System (1) defines a linear operator

$$L = L(A)\colon \mathbb{C}[\xi_1, \ldots, \xi_n] \to \mathbb{C}[\xi_1, \ldots, \xi_n]$$

as

$$L = A_k^{i,j}\xi_i(t)\xi_j(t)\frac{\partial}{\partial \xi_k}.$$

Lotka-Volterra type system

$$\xi_k' = \xi_k \left(\sum_{l=1}^{n} \xi_l - 2\xi_k \right), \quad k = 1, \dots, n.$$

For $n = 3$ this system was considered by S. Kovalevskaya. She showed that in this case the system has two independant quadratic integrals

$$\sum_{i \neq j} a_k \xi_i \xi_j, \qquad \sum a_k = 0.$$

For $n = 4$ this system has two independant quadratic integrals

$$(\xi_1 - \xi_3)(\xi_2 - \xi_4) \quad \text{and} \quad (\xi_1 - \xi_2)(\xi_3 - \xi_4).$$

For $n > 4$ there are no quadratic integrals.

Representation of $\mathrm{GL}(n, \mathbb{C})$

The change of variables $\eta = B\xi$ by a matrix $B = (B_i^j)$ leads to a representation of $\mathrm{GL}(n, \mathbb{C})$ on the space of tensors $A = (A_k^{i,j})$:

$$A_k^{i,j} \mapsto A_p^{q,r} B_k^p (B^{-1})_q^i (B^{-1})_r^j.$$

Definition (1)

A system (1) is called *symmetric* if the symmetric group $\mathfrak{S}_n$ lies in the stabilizer subgroup of (1), that is the system does not change if $B \in \mathfrak{S}_n$.

Definition

The system (1) is said to be *quasi-symmetric* with respect to $B \in \mathrm{GL}(n, \mathbb{C})$ if this system in the coordinates $\eta = B\xi$ is symmetric.

Symmetric quadratic dynamical systems.

$$\xi^n - h_1\xi^{n-1} + \cdots + (-1)^n h_n \equiv 0.$$

On the space $\mathcal{M}^n = \{\xi \in \mathbb{C}^n \colon \xi_i \neq \xi_j \ \forall i, j \colon i \neq j\}$ there is a free action of the group $\mathfrak{S}_n$. The projection on the space of orbits is a covering induced by the universal algebraic map

$$S \colon \mathbb{C}^n \to \mathbb{C}^n \colon (\xi_1, \ldots, \xi_n) \mapsto (h_1(\xi_1, \ldots, \xi_n), \ldots, h_n(\xi_1, \ldots, \xi_n)).$$

Here h_k is the k-th elementary symmetric function in $\xi_1, \ldots, \xi_n$. Denote by Sym the ring of symmetric polynomials in $\xi_1, \ldots, \xi_n$.

Definition (2)

A system (1) is called *symmetric* if $S_* L$ takes Sym into itself.

Lemma

The definitions 1 and 2 are equivalent.

Corollary

Each symmetric quadratic dynamical system has the form

$$\xi_k'(t) = \alpha\xi_k^2 + \beta\xi_k \sum_{i\neq k} \xi_i + \gamma \sum_{i\neq k} \xi_i^2 + \delta \sum_{i<j, i\neq k, j\neq k} \xi_i\xi_j, \quad k = 1, \ldots, n.$$

Generic symmetric quadratic dynamical systems

For $\deg \xi_k = -4$, $\deg t = 4$ system (1) is homogeneous.
For a homogeneous multiplicative basis $a_1, \ldots, a_n$ in Sym with $\deg a_k = -4k$ system (1) implies

$$a_k'(t) = L a_k(t), \qquad k = 1, \ldots, n. \tag{2}$$

Using the grading we obtain

$$a_k'(t) = c_k a_{k+1}(t) + g_{k+1}(a_1(t), \ldots, a_k(t)), \quad k = 1, \ldots, n,$$

where $c_n = 0$ and $\deg g_{k+1} = -4(k+1)$.

Definition

A symmetric system (1) is *generic* if $c_k \neq 0$ for $k = 1, \ldots, n-1$.

Lemma

A system being generic does not depend on the choice of multiplicative basis.

Algebraic integrability

For the equation

$$\xi^n - h_1\xi^{n-1} + \cdots + (-1)^n h_n \equiv 0 \tag{3}$$

let $\Delta \subset \mathbb{C}^n$ be the discriminant variety.

Definition

System (1) is *algebraically integrable in* $U \subset \mathbb{C}$ *by a set of functions* $(h_1(t), \dots, h_n(t))$ if $(h_1(t), \dots, h_n(t)) \notin \Delta$ for any $t \in U$ and the set of roots $(\xi_1(t), \dots, \xi_n(t))$ of equation (3) is a solution of (1) for any $t \in U$.

Problem of Algebraic integrability

Find an ordinary differential equation of degree n for h and differential polynomials $h_2, \ldots, h_n$ in h such that in the neighbourhood $U \subset \mathbb{C}$ the set of functions $h_1(t) = h(t), h_2(t), \ldots, h_n(t)$ algebraically integrates system (1).

Theorem

For each generic symmetric system (1) *with the initial data* $\xi(t_0) = (\xi_1(t_0), \ldots, \xi_n(t_0)) \in \mathcal{M}^n$ *there is a solution to the problem of algebraic integrability in the vicinity of* $\xi(t_0)$.

Consider system (2) with $a_k = h_k$ being the elementary symmetric functions.
Under the conditions of the theorem $c_k \neq 0$, $k = 1, \dots, n-1$.
Hence, $h_j(t)$ with $j = 2, \dots, n$ can be expressed as polynomials in $h_1(t), \dots, h_{j-1}(t)$ and their derivatives from the $(j-1)$-th equation of system (2).
Thus, the equation

$$a_n'(t) = g_{n+1}(a_1(t), \dots, a_n(t))$$

gives a homogeneous differential equation for $h_1(t)$:

$$h^{(n)} + \alpha h h^{(n-1)} + \dots + \omega h^n = 0 \tag{4}$$

with constant coefficients $\alpha, \dots, \omega$.

Therefore, $(h_1(t), \dots, h_n(t))$ algebraically integrates system (2).

The initial conditions in (4) are as follows:

$$h_1(t_0) = \xi_1(t_0) + \cdots + \xi_n(t_0)$$

and

$$h_1^{(k)}(t_0) = (L^k h_1)(t_0).$$

We have reduced the problem of integrability of a symmetric quadratic dynamical system to the question of solving an ordinary differential equation of the form

$$h^{(n)} + \alpha h h^{(n-1)} + \cdots + \omega h^n = 0 \qquad (5)$$

with constant coefficients $\alpha, \ldots, \omega$.

Problem

Classify the non-linear ordinary differential equations (5) obtained from generic quadratic dynamical systems.

Fuchs–Poincaré problem

Definition

A point is called a *singularity* of a function if this function is not analytic (possibly not defined) in that point.

Definition

A singularity of a function is called *a critical singularity* if going around this singularity changes the value of the function.

L. Fuchs remarked that differential equation solutions can have movable singularities, that is singularities whose *location depends* on the initial conditions of the solution.
In 1884 L. Fuchs and H. Poincaré have considered the problem of integrating differential equations and came to the conclusion that it is closely connected to the problem of defining new functions by means of non-linear ordinary differential equations.

In 1888 S. Kovalevskaya solved the classical precession of a top under the influence of gravity problem.

S. Kovalevskaya's approach to the problem is based on finding solutions with no movable critical singularities.

She proved that there exists only three cases with such solutions. Two of them are the famous Euler and Lagrange tops.

In the third case (now named Kovalevskaya top) she found new solutions and thus was first to discover the advantages of solving differential equations whose solutions have no movable critical singularities.

Painlevé property

The property of a differential equation that its solutions have no movable critical singularities is well known now as the *Painlevé property*.

The general solution of equations with Painlevé property lead to the single-valued function.

All *linear* ordinary differential equations have the Painlevé property, but it turns out that this property is rare for *non-linear* differential equations.

Around 1900, P. Painlevé studied second order explicit non-linear differential equations

$$y'' = F(t, y, y')$$

with F being a rational function of y and y' and a locally-analytical function of t.

It turned out that among such equations up to certain transformations only fifty equations have the Painlevé property, and among them six are not integrable in terms of previously known functions.

P. Painlevé and B. Gambier have introduced new special functions, now known as *Painlevé transcendents*, as the general solutions to this equations.

Chazy equations

In 1910 J. Chazy considered the problem of classification of all third-order differential equations of the form

$$y''' = F(t, y, y', y''),$$

where F is a polynomial in y, y', and y'' and locally analytic in t, having the Painlevé property.

The most known autonomous Chazy equations are

Chazy-3 equation: $y''' = 2yy'' - 3(y')^2,$

Chazy-12 equation: $y''' = 2yy'' - 3(y')^2 - \frac{4}{k^2 - 36}(6y' - y^2)^2,$

where $k \in \mathbb{N}$, $k > 1$, $k \neq 6$.

It was recently shown by M. J. Ablowitz, S. Chakravarty, R. Halburd that this Chazy equations are reductions of the self-dual Yang-Mills equations with an infinite-dimensional gauge algebra.

The relation of considered equations to mathematical physics arises from an observation by M. J. Ablowitz and H. Segur that reductions of a partial differential equation of the soliton type give rise to ordinary differential equations that posess the Painlevé property.

Two-dimensional systems

The general two-dimensional symmetric quadratic dynamical system has the form

$$\begin{aligned} \xi_1' &= \alpha\xi_1^2 + \beta\xi_1\xi_2 + \gamma\xi_2^2, \\ \xi_2' &= \gamma\xi_1^2 + \beta\xi_1\xi_2 + \alpha\xi_2^2. \end{aligned} \tag{6}$$

It is generic for $\beta \neq \alpha + \gamma$.

In the coordinates $\eta = B\xi$ where $B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ we obtain

$$\eta_1' = (\alpha + \gamma + \beta)\eta_1^2 + (\alpha + \gamma - \beta)\eta_2^2, \qquad \eta_2' = (\alpha - \gamma)\eta_1\eta_2.$$

Therefore, B establishes a one-to-one correspondence between quasi-symmetric quadratic dynamical systems

$$\eta_1' = a\eta_1^2 + b\eta_2^2, \quad \eta_2' = c\eta_1\eta_2. \tag{7}$$

for constant a, b, c and symmetric quadratic dynamical systems.

One can check that the normalizer of dynamical systems of the form (7) is the diagonal torus in $\mathrm{GL}(2, \mathbb{C})$.

The conjugation by the matrix

$$B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

takes this torus to the group of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.
This group brings the space of two-dimensional symmetric quadratic dynamical systems into itself.

In the generic case system (6) is algebraically integrable by the set of functions $(h_1(t), h_2(t))$ where h_1 is a solution of equation

$$h'' + \lambda_1 h' h + \lambda_2 h^3 = 0 \qquad (8)$$

with $\lambda_1 = (\gamma - 3\alpha - \beta)$, $\lambda_2 = (\alpha - \gamma)(\alpha + \beta + \gamma)$, and

$$h_2 = \frac{1}{2(\beta - \alpha - \gamma)} h_1' - \frac{\alpha + \gamma}{2(\beta - \alpha - \gamma)} h_1^2.$$

The initial conditions for h_1 in (8) corresponding to the generic case are $(\beta + \alpha + \gamma) h_1^2(t_0) \neq 2 h_1'(t_0)$.

Special cases

The general solution to (8) is

For $\lambda_1 = 0$: $$h(t) = k_2 \mathrm{sn}\left(\left(\sqrt{\frac{\lambda_2}{2}}t + k_1\right)k_2, i\right).$$

For $\lambda_2 = 0$: $$h(t) = \frac{\sqrt{2k_1}}{\sqrt{\lambda_1}}\tanh\left(\sqrt{\frac{k_1\lambda_1}{2}}(t + k_2)\right).$$

For $\lambda_1^2 = 9\lambda_2$: $$h(t) = \frac{6(k_1 t + k_2)}{\lambda_1(k_1 t^2 + 2k_2 t + 2)}.$$

Here k_1, k_2 are constants, sn is the Jacobi sine.

Three-dimensional systems

The general three-dimensional symmetric quadratic dynamical system has the form

$$\begin{aligned}
\xi_1' &= \alpha\xi_1^2 + \beta\xi_1(\xi_2+\xi_3) + \gamma(\xi_2^2+\xi_3^2) + \delta\xi_2\xi_3,\\
\xi_2' &= \alpha\xi_2^2 + \beta\xi_2(\xi_3+\xi_1) + \gamma(\xi_3^2+\xi_1^2) + \delta\xi_3\xi_1,\\
\xi_3' &= \alpha\xi_3^2 + \beta\xi_3(\xi_1+\xi_2) + \gamma(\xi_1^2+\xi_2^2) + \delta\xi_1\xi_2.
\end{aligned} \tag{9}$$

It is generic for $2\alpha - 2\beta + 4\gamma - \delta \neq 0$ and $\alpha - \beta - \gamma + \delta \neq 0$.

In the coordinates $\eta = B\xi$ where

$$B = \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon & \varepsilon^2 & 1 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix}, \quad \varepsilon^3 = 1, \varepsilon \neq 1,$$

we obtain

$$\begin{aligned} \eta_1' &= k_1\eta_1^2 + k_2\eta_2\eta_3, \\ \eta_2' &= k_3\eta_3^2 + k_4\eta_1\eta_2, \\ \eta_3' &= k_3\eta_2^2 + k_4\eta_1\eta_3, \end{aligned} \tag{10}$$

where

$$k_1 = \frac{1}{3}(\alpha + 2\beta + 2\gamma + \delta), \qquad k_2 = \frac{1}{3}(2\alpha - 2\beta + 4\gamma - \delta),$$

$$k_3 = \frac{1}{3}(\alpha - \beta - \gamma + \delta), \qquad k_4 = \frac{1}{3}(2\alpha + \beta - 2\gamma - \delta).$$

Therefore, B establishes a one-to-one correspondence between systems of the form (9) and (10).

The maximal subgroup of $\mathsf{GL}(3,\mathbb{C})$ that takes the space of systems of the form (10) into itself is generated by matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad b^3 = c^3, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the subgroup of matrices in $\mathsf{GL}(3,\mathbb{C})$ obtained from this subgroup by conjugation by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \varepsilon & \varepsilon^2 & 1 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix}, \quad \varepsilon^3 = 1, \quad \varepsilon \neq 1,$$

brings the space of three-dimensional symmetric quadratic dynamical systems into itself.

The Darboux-Halphen system

Let us consider the classical Darboux-Halphen system

$$\xi_1' = \xi_2\xi_3 - \xi_1\xi_2 - \xi_1\xi_3, \quad \xi_2' = \xi_1\xi_3 - \xi_1\xi_2 - \xi_2\xi_3, \quad \xi_3' = \xi_1\xi_2 - \xi_1\xi_3 - \xi_2\xi_3.$$

It implies the generic homogeneous dynamical system

$$h_1' = -h_2, \qquad h_2' = -6h_3, \qquad h_3' = h_2^2 - 4h_1h_3.$$

The Darboux-Halphen system is algebraically integrable by (h_1, h_2, h_3) where $-2h_1$ is a solution to the Chazy-3 equation

$$y''' = 2yy'' - 3(y')^2,$$

and $h_2 = -h_1'$, $h_3 = -\frac{1}{6}h_2' = \frac{1}{6}h_1''$.

General Darboux-Halphen system

$$\xi_1' = a\,(\xi_2\xi_3 - \xi_1\xi_2 - \xi_1\xi_3) + b\xi_1^2,$$
$$\xi_2' = a\,(\xi_3\xi_1 - \xi_2\xi_3 - \xi_2\xi_1) + b\xi_2^2,$$
$$\xi_3' = a\,(\xi_1\xi_2 - \xi_3\xi_1 - \xi_3\xi_2) + b\xi_3^2.$$

This system is symmetric and generic for $a \neq -2b$, $2a \neq b$.
For $a \neq b$ the function

$$y = -2(a-b)(\xi_1 + \xi_2 + \xi_3)$$

is a solution of the equation

$$y''' = 2yy'' - 3(y')^2 + c(6y' - y^2)^2 \quad \text{with} \quad c = \frac{-b^2}{4(a+2b)(a-b)}.$$

For $c = 0$ this is the Chazy-3 equation.
For $c = -\frac{4}{k^2-36}$ this is the Chazy-12 equation.

The wide-known in math-physical literature generalization of the Darboux-Halphen system

$$\eta_1' = \eta_2\eta_3 - \eta_1\eta_2 - \eta_1\eta_3 + \tau^2,$$
$$\eta_2' = \eta_3\eta_1 - \eta_2\eta_1 - \eta_2\eta_3 + \tau^2,$$
$$\eta_3' = \eta_1\eta_2 - \eta_3\eta_1 - \eta_3\eta_2 + \tau^2,$$

where

$$\tau^2 = \alpha^2(\eta_1-\eta_2)(\eta_3-\eta_1)+\beta^2(\eta_2-\eta_3)(\eta_1-\eta_2)+\gamma^2(\eta_3-\eta_1)(\eta_2-\eta_3)$$

is symmetric if and only if $\alpha^2 = \beta^2 = \gamma^2$ and in this case generic for $\alpha^2 \neq \frac{1}{4}$ and $\frac{1}{9}$.

It is the case $b = a - 1$ of the general Darboux-Halphen system in coordinates

$$\eta_i = a\xi_i - \frac{1}{2}(a-1)(\xi_j + \xi_k), \quad i \neq j \neq k$$

with $\alpha^2 = (a-1)^2/(3a-1)^2$.

Four-dimensional systems

The general four-dimensional symmetric quadratic dynamical system has the form

$$\begin{aligned}
\xi_1' &= \alpha\xi_1^2 + \beta\xi_1(\xi_2+\xi_3+\xi_4) + \gamma(\xi_2^2+\xi_3^2+\xi_4^2) + \delta(\xi_2\xi_3+\xi_2\xi_4+\xi_3\xi_4),\\
\xi_2' &= \alpha\xi_2^2 + \beta\xi_2(\xi_3+\xi_4+\xi_1) + \gamma(\xi_3^2+\xi_4^2+\xi_1^2) + \delta(\xi_3\xi_4+\xi_3\xi_1+\xi_4\xi_1),\\
\xi_3' &= \alpha\xi_3^2 + \beta\xi_3(\xi_4+\xi_1+\xi_2) + \gamma(\xi_4^2+\xi_1^2+\xi_2^2) + \delta(\xi_4\xi_1+\xi_4\xi_2+\xi_1\xi_2),\\
\xi_4' &= \alpha\xi_4^2 + \beta\xi_4(\xi_1+\xi_2+\xi_3) + \gamma(\xi_1^2+\xi_2^2+\xi_3^2) + \delta(\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3).
\end{aligned}$$

In the case of four-dimensional Lotka-Volterra type system

$$\xi_k' = \xi_k \left(\sum_{l=1}^{4} \xi_l - 2\xi_k \right), \quad k = 1, \ldots, 4,$$

we obtain the equation

$$h'''' - h''' h + 5h'' h' - 4h'' h^2 - 8(h')^2 h + 4h' h^3 = 0$$

and differential polynomials

$$\begin{aligned} h_2 &= \frac{1}{4}(h' + h^2), \\ h_3 &= \frac{1}{24}(h'' + 2h' h), \\ h_4 &= \frac{1}{192}(h''' + h'' h + 2(h')^2 - 2h' h^2). \end{aligned}$$

The system

$$\xi_i' = a(\xi_j\xi_k + \xi_j\xi_l + \xi_k\xi_l) - 2a\xi_i(\xi_j + \xi_k + \xi_l) + b\xi_i^2 \qquad (11)$$

where the indices (i, j, k, l) run over the four cyclic permutations of $(1, 2, 3, 4)$ is symmetric and generic for $3a \neq b$, $a \neq -b$.

The function

$$h = \frac{3a - b}{4}(\xi_1 + \xi_2 + \xi_3 + \xi_4)$$

is a solution of the equation

$$h'''' + 20hh''' - 24h'h'' + 96h^2h'' - 144h(h')^2 + \\ + c(h' + h^2)(h'' + 6hh' + 4h^3) = 0 \qquad (12)$$

with $c = \frac{-64b^2}{(a+b)(3a-b)}$.

For $c = 64$ this equation possesses the Painlevé property. This corresponds to the cases $a = 0$ or $3a + 2b = 0$.

In the case $a = 0$ and $b = 1$ system (11) becomes the system

$$\xi_k'(t) = \xi_k^2, \quad k = 1, \ldots, n.$$

Therefore the general solution of (12) in this case has the form

$$h(t) = \frac{1}{4}\left(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{1}{t-a_3} + \frac{1}{t-a_4}\right).$$

In the case $a = 2$, $b = -3$ system (11) becomes

$$\xi_i' = 2(\xi_j\xi_k + \xi_j\xi_l + \xi_k\xi_l) - 4\xi_i(\xi_j + \xi_k + \xi_l) - 3\xi_i^2.$$

The linear change $\eta_i = -3(\xi_j + \xi_k + \xi_l)$, $i \neq j \neq k \neq l$ brings this system to the system

$$\eta_k'(t) = \eta_k^2, \quad k = 1, \ldots, n,$$

of the previous case.

Connection to heat equation solutions

Theorem

Among the following conditions each two imply the third one:
1) The function

$$\psi(z,t) = e^{r(t)}\Psi(z;\mathbf{x}(t))$$

where

$$\Psi(z;\mathbf{x}) = z^{\delta} + \sum_{k\geqslant 1} \Psi_k(\mathbf{x}) \frac{z^{2k+\delta}}{(2k+\delta)!},$$

$\deg z = 2$, $\mathbf{x} = (x_1, x_2, \dots x_{n+1})$, $\deg x_q = -4q$,
and $\Psi_k(\mathbf{x})$ *are homogeneous polynomials of degree* $-4k$,
solves the heat equation

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z,t).$$

2) The coefficients of $\Psi(z; \boldsymbol{x})$ are defined by the recursion

$$\Psi_{k+1}(\boldsymbol{x}) = 2\sum_{j=1}^{n+1} p_{j+1}(\boldsymbol{x})\frac{\partial}{\partial x_j}\Psi_k(\boldsymbol{x}) + \Psi_1(\boldsymbol{x})\Psi_k(\boldsymbol{x})$$

and the condition

$$r'(t) = \frac{1}{2}\Psi_1(\boldsymbol{x}(t)).$$

3) The functions $x_k(t)$ satisfy the homogeneous polynomial dynamical system

$$\frac{d}{dt}x_k = p_{k+1}(\boldsymbol{x}), \qquad k = 1, 2, \ldots, n+1.$$

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