On Hamiltonian geometry of the associativity equations

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The associativity equations

The Witten–Dijkgraaf–Verlinde–Verlinde equations (the WDVV equations) or the associativity equations:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\lambda \partial t^\nu \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\gamma \partial t^\beta \partial t^\mu} \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\lambda \partial t^\nu \partial t^\alpha},$$

$F(t^1, \ldots, t^N)$ is a function of N variables $(t^1, \ldots, t^N)$, the matrix

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta}, \quad \eta^{\alpha\mu}\eta_{\mu\beta} = \delta^\alpha_\beta,$$

is constant and non-degenerate.
The WDVV equations are an overdetermined non-linear integrable system.
B. Dubrovin, Geometry of 2D topological field theories, 1994.

N-parametric deformations of Frobenius algebras

For any $t = (t^1, \ldots, t^N)$ the functions

$$c^{\alpha}_{\beta\gamma}(t) = \eta^{\alpha\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\beta \partial t^\gamma}$$

define the structure of an commutative associative algebra $\mathcal{A}(t)$ in N-dimensional linear space with basis $e_1, \ldots, e_N$ and multiplication

$$e_\beta \circ e_\gamma = c^{\alpha}_{\beta\gamma}(t) e_\alpha.$$

The condition of associativity

$$(e_\alpha \circ e_\beta) \circ e_\gamma = e_\alpha \circ (e_\beta \circ e_\gamma),$$

which is equivalent to the relation

$$c^{\mu}_{\alpha\beta}(t) c^{\nu}_{\mu\gamma}(t) = c^{\nu}_{\alpha\mu}(t) c^{\mu}_{\beta\gamma}(t),$$

results in the WDVV equations.

N-parametric deformations of Frobenius algebras

e_1 is always a unit in the algebra $\mathcal{A}(t)$:

$$e_1 \circ e_\mu = c^\nu_{1\mu}(t) e_\nu = \eta^{\nu\lambda}\eta_{\lambda\mu} e_\nu = e_\mu.$$

Moreover, the algebra $\mathcal{A}(t)$ is equipped with a non-degenerate symmetric bilinear form

$$\langle e_\alpha, e_\beta \rangle = \eta_{\alpha\beta}$$

that is *invariant* with respect to multiplication in the algebra, that is,

$$\langle e_\alpha \circ e_\beta, e_\gamma \rangle = \langle e_\alpha, e_\beta \circ e_\gamma \rangle.$$

A finite-dimensional commutative associative algebra possessing a unit and equipped with a non-degenerate invariant symmetric bilinear form is called a *Frobenius algebra*. The associativity equations describe N-parametric deformations of N-dimensional Frobenius algebras.

The variable t^1 is an initially fixed variable and the assignment of an arbitrary constant non-degenerate symmetric matrix $\eta_{\alpha\beta}$ for the function F simply determines the dependence of this function on the variable t^1 (by virtue of its definition the function F is always considered up to quadratic polynomials in the variables $t^1, \dots, t^N$):

$$F = \frac{1}{6}\eta_{11}(t^1)^3 + \sum_{\beta=2}^{N} \frac{1}{2}\eta_{1\beta}(t^1)^2 t^\beta + \sum_{\alpha\geq 2}\sum_{\beta>\alpha} \eta_{\alpha\beta} t^1 t^\alpha t^\beta + \\ + \sum_{\alpha\geq 2} \frac{1}{2}\eta_{\alpha\alpha} t^1 (t^\alpha)^2 + f(t^2, \dots, t^N).$$

Correspondingly, for the given metric $\eta_{\alpha\beta}$ the equations of associativity are equivalent to a system of non-linear partial differential equations for the function $f(t^2, \dots, t^N)$.

The first equations of associativity arise for $N = 3$. For $N = 3$ the following two essentially different types of dependence of the function F on the fixed variable t^1 were considered by Dubrovin: $F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + f(t^2, t^3)$, which corresponds to the antidiagonal metric $(\eta_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $F = \frac{1}{6}(t^1)^3 + t^1 t^2 t^3 + f(t^2, t^3)$, which corresponds to the metric with the same signature but of the form $(\eta_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The corresponding equations of associativity reduce to the following two non-linear equations of the third order for the function of two independent variables $f = f(x, t)$ (here we use the notation $x = t^2$, $t = t^3$):

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}, \qquad (1)$$

$$f_{xxx} f_{ttt} - f_{xxt} f_{xtt} = 1, \qquad (2)$$

respectively.

Equation (1) describes quantum cohomology, i.e., the deformations of the cohomology ring of the projective plane $\mathbf{P}^2$. The solution of the problem on the number $n(d)$ of rational curves of degree d on $\mathbf{P}^2$ which pass through $3d-1$ generic points, as Kontsevich showed, is also connected to this equation. The series

$$\phi(x,t) = \sum_{d=1}^{\infty} n(d) \frac{t^{3d-1}}{(3d-1)!} e^{dx}$$

must satisfy equation (1), whence Kontsevich's recursion relations for the numbers $n(d)$ are obtained: $n(1) = 1$, and for $d \geq 2$

$$n(d) = \sum_{\substack{k, l \geq 1, \\ k+l=d}} n(k)n(l)k^2 l \left[l \begin{pmatrix} 3d-4 \\ 3k-2 \end{pmatrix} - k \begin{pmatrix} 3d-4 \\ 3k-1 \end{pmatrix} \right].$$

Generalizations of this problem to other varieties are also connected to the associativity equations (Kontsevich, Manin et al.)

Theorem (M., 1994)

Equations (1) and (2) are equivalent to integrable non-diagonalizable homogeneous systems of hydrodynamic type

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & 2b & -a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_x ,$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(1+bc)/a^2 & c/a & b/a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_x ,$$

respectively.

We introduce the new variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$, $d = f_{ttt}$. The compatibility conditions have the form $a_t = b_x$, $b_t = c_x$, $c_t = d_x$. Moreover, equations (1) and (2) are equivalent to the relations $d = b^2 - ac$ and $d = (1 + bc)/a$, respectively. Thus, in the new variables equations (1) and (2) become homogeneous 3×3 systems of hydrodynamic type:

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = (b^2 - ac)_x,$$

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = \big((1 + bc)/a\big)_x,$$

respectively.

Both these systems (1) and (2) are non-diagonalizable. This fact can be verified in different ways. There is an effective criterion for diagonalizability found by Haantjes (1955).
Let $v^i_j(u)$ be an *affinor*, that is, a tensor of type $(1,1)$ on a manifold M with local coordinates $(u^1, \ldots, u^N)$. The *Nijenhuis tensor* $N^k_{ij}(u)$ is the *curvature tensor of the affinor* $v^i_j(u)$:

$$N^k_{ij} = v^s_i \frac{\partial v^k_j}{\partial u^s} - v^s_j \frac{\partial v^k_i}{\partial u^s} + v^k_s \frac{\partial v^s_i}{\partial u^j} - v^k_s \frac{\partial v^s_j}{\partial u^i} .$$

The *Haantjes tensor*:

$$H^i_{jk} = v^i_s v^s_r N^r_{jk} - v^i_s N^s_{rk} v^r_j - v^i_s N^s_{jr} v^r_k + N^i_{sr} v^s_j v^r_k.$$

If a tensor $v^i_j(u)$ has complete set of eigenfunctions (in particular, if all its eigenvalues are real and distinct), then the tensor can be diagonalized in a neighbourhood of a point on the manifold if and only if its Haantjes tensor vanishes.

The associativity equations are connected to the following spectral linear problem (Dubrovin):

$$\frac{\partial \phi_\alpha}{\partial t^\beta} = z c^\gamma_{\alpha\beta} \phi_\gamma, \qquad z = \mathrm{const.}$$

For this linear system the compatibility conditions have the form

$$z \frac{\partial c^\gamma_{\alpha\beta}}{\partial t^\lambda} \phi_\gamma + z^2 c^\gamma_{\alpha\beta} c^\varepsilon_{\gamma\lambda} \phi_\varepsilon = z \frac{\partial c^\gamma_{\alpha\lambda}}{\partial t^\beta} \phi_\gamma + z^2 c^\gamma_{\alpha\lambda} c^\varepsilon_{\gamma\beta} \phi_\varepsilon,$$

that is, they are equivalent to the associativity equations.
In our “hydrodynamical” variables a, b, c the linear problem can be rewritten in the following form:

$$\Psi_x = z \begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix} \Psi, \ \Psi_t = z \begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ b^2 - ac & c & 0 \end{pmatrix} \Psi.$$

For the associativity equations (1), the first Poisson structure generated by a flat metric was found by M. and Ferapontov (1995):

$$(M_1^{ij}) = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2}a & b \\ \frac{1}{2}a & b & \frac{3}{2}c \\ b & \frac{3}{2}c & 2(b^2 - ac) \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \frac{1}{2}a_x & b_x \\ 0 & \frac{1}{2}b_x & c_x \\ 0 & \frac{1}{2}c_x & (b^2 - ac)_x \end{pmatrix},$$

and the corresponding Hamiltonian of hydrodynamic type is $H = \int c\, dx$.

The second Poisson structure of this system, which is compatible with the first one, was constructed in the paper by Ferapontov, Galvão, M. and Nutku (1997). It is a homogeneous differential-geometric Poisson structure of the third order (Dubrovin–Novikov type) with a flat metric, and this Poisson structure cannot be reduced to a constant form by a local change of coordinates:

$$
\begin{aligned}
(M_2^{ij}) = & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a \\ 1 & -a & a^2 + 2b \end{pmatrix} \left(\frac{d}{dx}\right)^3 + \\
& + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2a_x \\ 0 & -a_x & 3(b_x + aa_x) \end{pmatrix} \left(\frac{d}{dx}\right)^2 + \\
& + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{xx} + a_x^2 + aa_{xx} \end{pmatrix} \frac{d}{dx}. \qquad (1)
\end{aligned}
$$

The Hamiltonian is non-local and has the form:

$$H_1 = -\int \left(\frac{1}{2} a \left(\left(\frac{d}{dx} \right)^{-1} b \right)^2 + \left(\left(\frac{d}{dx} \right)^{-1} b \right) \left(\left(\frac{d}{dx} \right)^{-1} c \right) \right) dx.$$

Thus, the associativity equation (1) is a bi-Hamiltonian system with compatible differential-geometric Poisson structures and for the associativity equation (2) we found no Hamiltonian representation.

Another example of the metric $\eta_{\alpha\beta}$ generating the associativity equations with similar Hamiltonian and bi-Hamiltonian representations was published by Kalayci and Nutku (1997): $F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{1}{2}(t^1)^2 t^2 + f(t^2, t^3)$, which corresponds to the metric $(\eta_{\alpha\beta}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The corresponding equations of associativity reduce to the following non-linear equation of the third order for the function of two independent variables $f = f(x, t)$ $(x = t^2,\ t = t^3)$:

$$f_{ttt} + f_{ttt} f_{xxx} - f_{ttx} f_{txx} + f_{ttt} f_{txx} - f_{xtt}^2 + f_{xxx} f_{xtt} - f_{xxt}^2 = 0. \qquad (3)$$

In the new variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$, $d = f_{ttt}$ equation (3) becomes 3×3 system of hydrodynamic type:

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = d_x,$$

where

$$d = \frac{b^2 + c^2 + bc - ac}{a + b + 1}.$$

For the associativity equations (3), the first Poisson structure is generated by a flat metric and has the following form:

$$(M^{ij}) = \begin{pmatrix} -\frac{3}{2} - 2a & \frac{1}{2}(a-3b) & b-c \\ \frac{1}{2}(a-3b) & b-c & \frac{1}{2}(3c-d) \\ b-c & \frac{1}{2}(3c-d) & 2d \end{pmatrix} \frac{d}{dx} + \\ + \begin{pmatrix} -a_x & \frac{1}{2}(a_x-b_x) & b_x \\ -b_x & \frac{1}{2}(b_x-c_x) & c_x \\ -c_x & \frac{1}{2}(c_x-d_x) & d_x \end{pmatrix}, \qquad (2)$$

the corresponding Hamiltonian of hydrodynamic type is $H = \int c\,dx$.

For an arbitrary system of hydrodynamic type

$$u^i_t = v^i_j(u) u^j_x$$

(for an arbitrary affinor $v^i_j(u)$) Bogoyavlenskij (1996) defined two metrics (may be degenerate) given by the corresponding Nijenhuis and Haantjes tensors:

$$n_{ij}(u) = N^s_{ir}(u) N^r_{js}(u), \quad h_{ij}(u) = H^s_{ir}(u) H^r_{js}(u)$$

(the Nijenhuis metric and the Haantjes metric).
Moreover, Bogoyavlenskij (1996) proved that for $N = 3$ the Haantjes tensor $H^k_{ij}(u)$ satisfies the identity

$$H(H(X_1, X_2), X_3) + H(H(X_2, X_3), X_1) + H(H(X_3, X_1), X_2) = 0,$$

and hence defines a Lie algebra $\mathcal{G}(u)$ structure in each tangent space $T_u M^3$, and the Haantjes metric $h_{ij}(u)$ is the Cartan–Killing form for the Lie algebra $\mathcal{G}(u)$.

Theorem (Bogoyavlenskij, 1996)

For $N = 3$, if the Haantjes tensor $H^k_{ij}(u)$ is non-zero, then the corresponding Lie algebra $\mathcal{G}$ is simple for each tangent space $T_u M^3$, so it is isomorphic either to $so(3)$ or $sl(3)$. The Cartan–Killing form $h_{ij}(u)$ defines a non-degenerate metric on the manifold M^3 that has to be conformally flat and therefore has to satisfy the classical Weyl–Schouten equations:

$$R_{ij,k} - R_{ik,j} - \frac{1}{4}(h_{ik} R_{,j} - h_{ij} R_{,k}) = 0.$$

Here R_{ij} is the Ricci tensor and R is the scalar curvature of the metric h_{ij}, the $R_{ij,k}$ and $R_{,j}$ are their covariant derivatives.

Theorem (Bogoyavlenskij, Reynolds, 2010)

For $N = 3$, a system of hydrodynamic type

$$u^i_t = v^i_j(u) u^j_x,$$

having non-degenerate Haantjes metric $h_{ij}(u)$, possesses a Hamiltonian structure with non-degenerate flat metric $g_{ij}(u)$ if and only if:

(1) The skew-symmetric $(1, 2)$-tensor

$$P^k_{ij} = Tr(B^2) T^k_{ij} + (v^k_i T^s_{jr} - v^k_j T^s_{ir} + (v^p_i \delta^k_j - v^p_j \delta^k_i) T^s_{pr}) B^r_s$$

vanishes. Here $T^k_{ij} = v^k_{i,j} - v^k_{j,i}$, $v^k_{i,j}$ is covariant derivative with respect to the Haantjes metric h_{ij}, $B^i_j = v^i_j - \frac{1}{n}(Trv)\delta^i_j$.

(2) The 1-form

$$\omega_i = \frac{1}{TrB^2} T^s_{ir} B^r_s$$

is exact. Hence there exists a function $\sigma(u)$ such that $\omega_i = \partial\sigma/\partial u^i$.

(3) The function $\sigma(u)$ satisfies the Weyl–Schouten equations:

$$\sigma_{,ij} = \sigma_{,i}\sigma_{,j} + \frac{1}{4} R h_{ij} - R_{ij} - \frac{1}{2} h_{ij}\sigma_{,s}\sigma_{,r} h^{sr}.$$

Under these conditions, the metric $g_{ij}(u) = \exp(2\sigma(u))h_{ij}(u)$ is flat and provides the required Hamiltonian structure.

Here $\sigma_{,ij}$ and $\sigma_{,i}$ are covariant derivatives with respect to the Haantjes metric h_{ij}.

M., Pavlenko, 2012

We consider the metric $(\eta_{ij}) = \begin{pmatrix} 0 & \alpha & 1 \\ \alpha & \beta & 0 \\ 1 & 0 & 0 \end{pmatrix}$. In this case, $F = \frac{1}{2}\alpha (t^1)^2 t^2 + \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}\beta t^1 (t^2)^2 + f(t^2, t^3)$ and the corresponding equations of associativity reduce to the following non-linear equation of the third order for the function of two independent variables $f = f(x, t)$ $(x = t^2,\ t = t^3)$:

$$\beta^2 f_{ttt} - \alpha f_{ttt} f_{xxx} + \alpha f_{ttx} f_{txx} + \alpha^2 f_{ttt} f_{txx} - \alpha^2 f_{xtt}^2 + f_{xxx} f_{xtt} - f_{xxt}^2 = 0.$$

In the new variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$, $d = f_{ttt}$ this associativity equation becomes 3×3 system of hydrodynamic type:

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = d_x,$$

where

$$d = \frac{b^2 + \alpha^2 c^2 - \alpha bc - ac}{\beta^2 - \alpha a + \alpha^2 b}.$$

For arbitrary α and β, this system is Hamiltonian with local Hamiltonian structure of hydrodynamic type generated by the flat metric g^{ij}:

$$\begin{pmatrix} \frac{2\alpha a}{\beta^4} - \frac{3}{2\beta^2} & \frac{a+3\alpha b}{2\beta^4} & \frac{b+\alpha c}{\beta^4} \\ \frac{a+3\alpha b}{2\beta^4} & \frac{b+\alpha c}{\beta^4} & \frac{\alpha^3 c^2+2\alpha^2 bc+\alpha(b^2-4ac)+3\beta^2 c}{2\beta^4(\beta^2+\alpha(\alpha b-a))} \\ \frac{b+\alpha c}{\beta^4} & \frac{\alpha^3 c^2+2\alpha^2 bc+\alpha(b^2-4ac)+3\beta^2 c}{2\beta^4(\beta^2+\alpha(\alpha b-a))} & \frac{2(b^2-\alpha bc+\alpha^2 c^2-ac)}{\beta^4(\beta^2+\alpha(\alpha b-a))} \end{pmatrix}$$