On the N-wave type equations and their gauge equivalent

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1. V.S. Gerdjikov, GGG, N. A. Kostov - Eur. Phys. Journal B, **29** (2002), 243–248 (E-print: `nlin.SI/0111027`)
2. GGG, M. Condon - J. Nonl. Math. Phys. **15** (2008), Suppl. 3, 197–208 (E-print: `arXiv:0710.330`).
3. GGG - J. Math. Phys. **53** (2012), no.7, 073512 (E-print: `arXiv:1109.5108`).

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Motivation

- Invariance of the Lax representation w.r. to the group of Gauge transformations

$$[L(\lambda), M(\lambda)] = 0 \quad \rightarrow \quad [\tilde{L}(\lambda), \tilde{M}(\lambda)] = 0$$

- NLS equation and HF equation

$$iu_t + u_{xx} + 2|u^2|u(x,t) = 0 \quad \textbf{(NLS)}$$

$$iS_t^{(0)} = \frac{1}{2}[S^{(0)}(x,t), S_{xx}^{(0)}] \quad \textbf{(HF)}$$

$$S^{(0)}(x,t) = g^{(0)-1}\sigma_3 g^{(0)}(x,t); \quad (S^{(0)})^2 = \mathbb{1}$$

Zakharov V. E., Takhtajan L. A., Teor. Mat. Fiz., **38** (1979), 26–35.

M. Lakshmanan, Phys. Lett. A 61, 53–54 (1977).

Generalisations to simple Lie algebras

- $g^{(0)}$ is determined by $u(x,t)$ through

$$i\frac{dg^{(0)}}{dx} + q^{(0)}(x,t)g^{(0)}(x,t) = 0,$$

$$q^{(0)}(x,t) = \left(\begin{array}{cc} 0 & u \\ -u^* & 0 \end{array} \right), \qquad \lim_{x\to\infty} g^{(0)}(x,t) = \mathbb{1}.$$

- Both equations are infinite dimensional completely integrable Hamiltonian systems.
- Generalized Zakharov-Shabat system related to arbitrary simple Lie algebra $\mathfrak{g}$ (of rank $r > 1$):

$$L(\lambda)\psi \equiv \left(i\frac{d}{dx} + q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0,$$

where $q(x,t), J \in \mathfrak{g}$.

Fixing up the gauge

- **Fixing the gauge 1**: $J \in \mathfrak{h}$- constant, regular
  $L(\lambda) \to g_0^{-1} L(\lambda) g_0, \quad g_0(x,\lambda) \in \mathfrak{G}$ (g_0 and J commute)
  $\to \quad q(x,t) = [J, Q(x,t)]$, J - regular: $q(x,t) \in \mathfrak{g} \backslash \mathfrak{h}$.

$$Q(x,t) = \sum_{\alpha \in \Delta^+} (q_\alpha(x,t) E_\alpha + p_\alpha(x,t) E_{-\alpha})$$

  $E_{\pm\alpha}$ - root vectors of $\mathfrak{g}$, Δ_+ - positive roots: $\Delta = \Delta_+ \cup (-\Delta_+)$.
  H_i - Cartan generators, $\{H_i, E_{\pm\alpha}\}$ - Cartan-Weyl basis for $\mathfrak{g}$
  $\to$ N-wave type equations on $\mathfrak{g}$ ($I \in \mathfrak{h}$):

$$i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0,$$

  $L(\lambda)$ and $M(\lambda)$ - Lax pair for N-wave eqn's:

$$M(\lambda)\psi \equiv \left(i\frac{d}{dt} + [I, Q(x,t)] - \lambda I \right) \psi(x,t,\lambda) = 0.$$

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Fixing up the gauge ... (pole gauge)

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dx} - \lambda \mathcal{S}(x,t) \right) \tilde{\psi}(x,t,\lambda) = 0,$$

where $\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda)$,

$$\mathcal{S}(x,t) = \mathrm{Ad}_g \cdot J \equiv g^{-1}(x,t) J g(x,t).$$

and $g(x,t) = \psi(x,t,0)$- the Jost sol's at $\lambda = 0$.

$$\tilde{M}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dt} - \lambda f(\mathcal{S}) \right) \tilde{\psi}(x,t,\lambda) = 0,$$

where $f(\mathcal{S})$ is a function (in fact, a polynomial) to be determined.

- $[\tilde{L}(\lambda), \tilde{M}(\lambda)] = 0 \quad \rightarrow$

$$\mathcal{S}_t - \frac{d}{dx} f(\mathcal{S}) = 0.$$

Gauge equivalence

- $L(\lambda)$ and $\tilde{L}(\lambda)$ have equivalent spectral properties and spectral data

V.E. Zakharov, A.V. Mikhailov, Commun. Math. Phys. **74**, 21– (1980);

V.E. Zakharov, A.V. Mikhailov, Sov. Phys. JETP **47**, 1017–1027 (1978).

$\rightarrow$ the classes of NLEE related to $L(\lambda)$ and $\tilde{L}(\lambda)$ are also equivalent.

Applying the gauge transformation

- $f(\mathcal{S}) = g^{-1}(x,t) I g(x,t)$ is uniquely determined by I. Both J and I belong to the Cartan subalgebra $\mathfrak{h}$ so they have common set of eigenspaces.
- $\mathfrak{g} \simeq \mathbf{A}_r = sl(n)$, $n = r+1$: $\operatorname{tr} I = \operatorname{tr} J = 0$

$$J = \operatorname{diag}(J_1, ..., J_n) \quad I = \operatorname{diag}(I_1, ..., I_n)$$

The projectors on the common eigensubspaces of J and I are given by:

$$\pi_k(J) = \prod_{s \neq k} \frac{J - J_s}{J_k - J_s} = \operatorname{diag}(0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0).$$

Next we note that

$$I = \sum_{k=1}^{n} I_k \pi_k(J).$$

Applying the gauge transformation ... (cont'd)

Apply the gauge transformation to I:

$$f(\mathcal{S}) = \sum_{k=1}^{n} I_k \pi_k(\mathcal{S}),$$

i.e., $f(\mathcal{S})$ is a polynomial of order $n-1$. Obviously $\mathcal{S}$ is restricted by:

$$\prod_{k=1}^{n} (\mathcal{S} - J_k) = 0, \qquad \operatorname{tr} \mathcal{S}^k = \operatorname{tr} J^k,$$

for $k = 2, \ldots, n$.

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Applying the gauge transformation: $\mathfrak{g} \simeq \mathbf{B}_r - \mathbf{D}_r$

- $\mathfrak{g} \simeq \mathbf{B}_r - \mathbf{D}_r$

$$J = \sum_{k=1}^{r} J_k H_{e_k}, \qquad I = \sum_{k=1}^{r} I_k H_{e_k}.$$

The odd powers of H_{e_k} also belong to $\mathfrak{g}$ while the even powers do not. The projectors $f_k(J)$ onto H_{e_k} then can be written down as:

$$f_k(J) = \frac{J}{J_k} \prod_{s \neq k} \frac{J^2 - J_s^2}{J_k^2 - J_s^2} = H_{e_k} \in \mathfrak{h}.$$

- Therefore

$$I = \sum_{k=1}^{r} I_k f_k(J),$$

and applying the gauge transformation we get:

$$f(\mathcal{S}) \equiv g^{-1}(x,t) I g(x,t) = \sum_{k=1}^{r} I_k f_k(\mathcal{S}).$$

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Applying the gauge transformation: Nonlinear constraints

- Nonlinear constraints on $\mathcal{S}$:

$$\mathcal{S}^{\kappa_0} \prod_{k=1}^{r} (\mathcal{S}^2 - J_k^2) = 0,$$

where $\kappa_0 = 0$ if $\mathfrak{g} \simeq \mathbf{C}_r$ or $\mathbf{D}_r$ and $\kappa_0 = 1$, if $\mathfrak{g} \simeq \mathbf{B}_r$. To construct the others we use the typical representation of $\mathfrak{g}$. It this settings we see that all even powers of H_{e_k} have trace equal to 2. Thus:

$$\operatorname{tr}(J^{2k}) \equiv 2 \sum_{p=1}^{r} J_p^{2k} = \operatorname{tr}(\mathcal{S})^{2k},$$

for $k = 1, \dots, r$.

- These are precisely r independent algebraic constraints on $\mathcal{S}$. Solving for them we conclude that the number of independent coefficients in $\mathcal{S}$ is equal to the number of roots $|\Delta|$ of $\mathfrak{g}$.

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Example: $\mathfrak{g} \simeq so(5)$

Parametrization for J and $q(x)$

$$q(x,t) = \begin{pmatrix} 0 & q_{10} & q_{11} & q_{12} & 0 \\ p_{10} & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & q_{10} \\ 0 & p_{12} & -p_{11} & p_{10} & 0 \end{pmatrix}; \quad J = \begin{pmatrix} J_1 & 0 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J_2 & 0 \\ 0 & 0 & 0 & 0 & -J_1 \end{pmatrix}$$

The corresponding 4-wave system has the form:

$$\begin{aligned} & i(J_1 - J_2) q_{10,t} - i(I_1 - I_2) q_{10,x} + 2\kappa q_{11} q_{01}^* = 0, \\ & iJ_2 q_{01,t} - iI_2 q_{01,x} + \kappa (q_{11}^* q_{12} + q_{11} q_{10}^*) = 0, \\ & iJ_1 q_{11,t} - iI_1 q_{11,x} + \kappa (q_{12} q_{01}^* - q_{10} q_{01}) = 0, \\ & i(J_1 + J_2) q_{12,t} - i(I_1 + I_2) q_{12,x} - 2\kappa q_{11} q_{01} = 0. \end{aligned}$$

where $I = \operatorname{diag}(I_1, I_2, 0, -I_2, -I_1)$ and $\kappa = J_1 I_2 - J_2 I_1$.

Example: $\mathfrak{g} \simeq so(5)$ ... (cont'd)

The gauge equivalent equations are:

$$\mathcal{S}_t - f_1 \mathcal{S}_x - f_3 (\mathcal{S}^3)_x = 0, \qquad f_1 = \frac{I_2 J_1^3 - I_1 J_2^3}{J_1 J_2 (J_1^2 - J_2^2)}, \qquad f_3 = \frac{I_1 J_2 - I_2 J_1}{J_1 J_2 (J_1^2 - J_2^2)},$$

where the matrix $\mathcal{S}(x,t) \in so(5)$ is constrained by:

$$\operatorname{tr} \mathcal{S}^2 = 2(J_1^2 + J_2^2), \qquad \operatorname{tr} \mathcal{S}^4 = 2(J_1^4 + J_2^4), \qquad \mathcal{S}(\mathcal{S}^2 - J_1^2)(\mathcal{S}^2 - J_2^2) = 0.$$

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Direct scattering problem: Jost solutions and scattering matrix

- The direct scattering problem is based on the Jost solutions:

$$\lim_{x\to\infty}\psi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty}\phi(x,\lambda)e^{i\lambda Jx} = \mathbb{1},$$

  and the scattering matrix:

$$T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda).$$

- The FAS $\xi^{\pm}(x,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\lambda \gtrless 0$ and are related to the Jost solutions by

$$\xi^{\pm}(x,\lambda) = \phi(x,\lambda)S^{\pm}(\lambda) = \psi(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda),$$

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Riemann-Hilbert problem for FAS

- Gauss decomposition for the scattering matrix:

$$T(\lambda) = T^{-}(\lambda)D^{+}(\lambda)\hat{S}^{+}(\lambda)$$
$$= T^{+}(\lambda)D^{-}(\lambda)\hat{S}^{-}(\lambda).$$

- On the real axis $\xi^{+}(x,\lambda)$ and $\xi^{-}(x,\lambda)$ are related by

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda)G_0(\lambda), \quad G_0(\lambda) = \hat{S}^{-}(\lambda)S^{+}(\lambda),$$

and the function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues.

A.B. Shabat Functional Annal. Appl. **9**, 75–78 (1975);

A.B. Shabat, Diff. Equations **15**, 1824–1834 (1979).

The scattering data

- Time evolution of the scattering data:

$$i\frac{dS^{\pm}}{dt} - \lambda[I, S^{\pm}] = 0 \qquad i\frac{dT^{\pm}}{dt} - \lambda[I, T^{\pm}] = 0,$$

  $D^{\pm}(\lambda)$ are time-independent.
- For the gauge equiv. systems:

$$\tilde{\xi}^{\pm}(x,\lambda) = g^{-1}(x,t)\xi^{\pm}(x,\lambda)g_{-},$$

  where $g_{-} = \lim_{x\to-\infty} g(x,t) = \hat{T}(0)$.
  $\tilde{\xi}^{\pm}(x,\lambda)$ are analytic w. r. to $\lambda \leftarrow$ the scattering matrix $T(0) \in \mathfrak{H}$.
- Asymptotics of the FAS for $x \to \pm\infty$:

$$\lim_{x\to-\infty} \tilde{\xi}^{+}(x,\lambda) = T(0)S^{+}(\lambda)\hat{T}(0)$$

$$\lim_{x\to\infty} \tilde{\xi}^{+}(x,\lambda) = e^{-i\lambda Jx}T^{-}(\lambda)D^{+}(\lambda)\hat{T}(0)$$

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The scattering data for $\tilde{L}$

$$\therefore \quad \tilde{T}(\lambda) = T(\lambda)\hat{T}(0).$$

Obviously $\tilde{T}(0) = \mathbb{1}$and

$$\tilde{S}^{\pm}(\lambda) = T(0)S^{\pm}(\lambda)\hat{T}(0),$$

$$\tilde{T}^{\pm}(\lambda) = T^{\pm}(\lambda) \quad \tilde{D}^{\pm}(\lambda) = D^{\pm}(\lambda)\hat{T}(0).$$

On the real axis $\tilde{\xi}^{+}(x,\lambda)$ and $\tilde{\xi}^{-}(x,\lambda)$ are related by:

$$\tilde{\xi}^{+}(x,\lambda) = \tilde{\xi}^{-}(x,\lambda)\tilde{G}_0(\lambda),$$

$$\tilde{G}_0(\lambda) = \hat{\tilde{S}}^{-}(\lambda)\tilde{S}^{+}(\lambda) \quad \tilde{\xi}(x,0) = \mathbb{1}.$$

again $\tilde{G}_0(\lambda)$ can be considered as a minimal set of scattering data.

Zakharov-Shabat dressing

- the main idea: From a FAS $\tilde{\xi}^{\pm}_{(0)}(x,\lambda)$ of $\tilde{L}$ with potential $\mathcal{S}_{(0)} \to$ construct a new singular solution $\tilde{\xi}^{\pm}_{(1)}(x,\lambda)$ with singularities located at prescribed positions $\lambda_1^{\pm}$.
- Then the new solutions $\tilde{\xi}^{\pm}_{(1)}(x,\lambda)$ will correspond to a potential $\mathcal{S}_{(1)}$ of $\tilde{L}$ with two discrete eigenvalues $\lambda_1^{\pm}$.
- It is related to the regular one by the dressing factors $\tilde{u}(x,\lambda)$:

$$\tilde{\xi}^{\pm}_{(1)}(x,\lambda) = \tilde{u}(x,\lambda)\tilde{\xi}^{\pm}_{(0)}(x,\lambda)\tilde{u}_-^{-1}(\lambda),$$

$$\tilde{u}_-(\lambda) = \lim_{x\to-\infty} \tilde{u}(x,\lambda),$$

and the dressing factors for the gauge equivalent equations $\tilde{u}(x,\lambda)$ are related to $u(x,\lambda)$ by

$$\tilde{u}(x,\lambda) = g_{(0)}^{-1}(x,t)u^{-1}(x,\lambda=0)u(x,\lambda)g_{(0)}.$$

$$\lim_{\lambda\to 0} \tilde{u}(x,\lambda) = \mathbb{1}.$$

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Dressing for $\mathfrak{g} \simeq \mathbf{A}_r$

- $\mathfrak{g} \simeq \mathbf{A}_r$

$$\tilde{u}(x,\lambda) = \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(0)} - 1 \right) P_1,$$

$$c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-} \quad P_1(x) = \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle},$$

$$|n(x)\rangle = \xi_0^+(\lambda_1^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|\hat{\xi}_0^-(\lambda_1^-),$$

where $|n_0\rangle$ and $\langle m_0|$ are constant vectors and these dressing factors satisfy the equation:

$$i\frac{d\tilde{u}}{dx} - \lambda\mathcal{S}_{(1)}\tilde{u} + \lambda\tilde{u}\mathcal{S}_{(0)} = 0.$$

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Dressing for $\mathfrak{g} \simeq \mathbf{B}_r$, $\mathbf{B}_r$

- $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$

$$u(x,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1 + (c_1^{-1}(\lambda) - 1)P_{-1}$$

$$\tilde{u}(x,\lambda) = \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(0)} - 1\right)\tilde{P}_1 + \left(\frac{c_1(0)}{c_1(\lambda)} - 1\right)\tilde{P}_{-1},$$

where $P_{-1}(x) = SP_1^T(x)S^{-1}$, $P_1(x)$ is the rank 1 projector, $\tilde{P}_{\pm 1} = g_{(0)}^{-1}P_{\pm 1}g_{(0)}(x,t)$.

- The projectors $\tilde{P}_{\pm 1}$ satisfy the equations:

$$i\frac{d\tilde{P}_1}{dx} + \lambda_1^- \tilde{P}_1 \mathcal{S}_{(0)} - \lambda_1^- \mathcal{S}_{(1)}\tilde{P}_1 = 0,$$

$$i\frac{d\tilde{P}_{-1}}{dx} + \lambda_1^+ \tilde{P}_{-1} \mathcal{S}_{(0)} - \lambda_1^+ \mathcal{S}_{(1)}\tilde{P}_{-1} = 0,$$

Dressing for $\mathfrak{g} \simeq \mathbf{B}_r$, $\mathbf{B}_r$ ... (cont'd)

- The "dressed"potential can be obtained by:

$$\mathcal{S}_{(1)} = S_{(0)} + i\frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ \lambda_1^-} \frac{d}{dx}(\tilde{P}_1(x) - \tilde{P}_{-1}(x)).$$

The dressing factors can be written in the form:

$$\tilde{u}(x,\lambda) = \exp\left[\ln\left(\frac{c_1(\lambda)}{c_1(0)}\right)\tilde{p}(x)\right],$$

where $\tilde{p}(x) = \tilde{P}_1 - \tilde{P}_{-1} \in \mathfrak{g}$ and consequently $\tilde{u}(x,\lambda)$ belongs to the corresponding orthogonal group.

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Dressing in the typical $\mathbf{B}_r$

- Explicit formulas for the typical representation of $\mathbf{B}_r$:

$$\tilde{p}(x) = \frac{2}{\langle m|n\rangle} \sum_{k=1}^{r} \tilde{h}_k(x) H_{e_k} + \frac{2}{\langle m|n\rangle} \sum_{\alpha\in\Delta_+} (\tilde{P}_\alpha(x) E_\alpha + \tilde{P}_{-\alpha}(x) E_{-\alpha}),$$

where we assumed $\mathcal{S}_{(0)} = J$, $g_{(0)} = \mathbb{1}$. Thus

$$\tilde{h}_k(x,t) = n_{0,k} m_{0,k} e^{2\nu_1 y_k} - n_{0,\bar{k}} m_{0,\bar{k}} e^{-2\nu_1 y_k},$$

$$\langle m|n\rangle = \sum_{k=1}^{r} (n_{0,k} m_{0,k} e^{2\nu_1 y_k} + n_{0,\bar{k}} m_{0,\bar{k}} e^{-2\nu_1 y_k}) + n_{0,r+1} m_{0,r+1},$$

$$\tilde{P}_\alpha = \begin{cases} \tilde{P}_{ks}, & \text{for } \alpha = e_k - e_s \\ \tilde{P}_{k\bar{s}}, & \text{for } \alpha = e_k + e_s \\ \tilde{P}_{k,r+1}, & \text{for } \alpha = e_k \end{cases}$$

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Dressing in the typical $\mathbf{B}_r$ ... (cont'd)

- Here $1 \leq k, s \leq r$, $\mu_1 = \mathrm{Re}\, \lambda_1^+$, $\nu_1 = \mathrm{Im}\, \lambda_1^+$ and

$$\tilde{P}_{ks} = e^{i\mu_1(y_s - y_k)}(n_{0,k} m_{0,s} e^{\nu_1(y_s+y_k)} (-1)^{k+s} n_{0,\bar{s}} m_{0,\bar{k}} e^{-\nu_1(y_s+y_k)}),$$
$$y_k = J_k x + I_k t, \qquad y_{\bar{k}} = -y_k, \qquad y_{r+1} = 0.$$

- The corresponding result for the $\mathbf{D}_r$ series is obtained formally if in the above expressions with $n_{0,r+1} = m_{0,r+1} = 0$. Thus $\tilde{P}_{k,r+1} = \tilde{P}_{r+1,k} = 0$ and the last term in the right hand side of $\langle m | n \rangle$ is missing.

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Hamiltonian structures

Both classes of NLEE possess hierarchies of Hamiltonian structures.

- The phase space $\mathcal{M}_{\rm N-w}$ of the N-wave equations is the linear space of off-diagonal matrices $q(x,t)$; the hierarchy of symplectic structures is given by:

$$\Omega^{(k)}_{\rm N-w} = i\int_{-\infty}^{\infty} dx \mathrm{tr}\left(\delta q \wedge \Lambda^k [J, \delta q(x,t)]\right).$$

- The phase space $\mathcal{M}_{\mathcal{S}}$ of their gauge equivalent equations is the nonlinear manifold of all $\mathcal{S}(x,t)$ satisfying equations of the nonlinear constraints. The family of compatible 2-forms is:

$$\tilde{\Omega}^{(k)}_{\mathcal{S}} = i\int_{-\infty}^{\infty} dx \mathrm{tr}\left(\delta \mathcal{S} \wedge \tilde{\Lambda}^k [\mathcal{S}, \delta \mathcal{S}(x,t)]\right).$$

Here Λ and $\tilde{\Lambda}$ are the recursion operator of the N-wave type equations and its gauge equivalent: $\tilde{\Lambda} = g^{-1}\Lambda g(x,t)$.

- One can reformulate all fundamental properties of the NLEE (like generating(recursion) operator, Hamiltonian formulation, etc.) from one gauge to another.
- If $\mathfrak{g} \simeq so(5) \rightarrow$ the corresp. gauge equivallent system describes isoparametric hypersurfaces

Ferapontov E.V., Diff. Geom. Appl., **5** (1995), 335–369.

- The "canonical"4-wave system finds applications in nonlinear optics.

Zakharov V.E., Manakov S.V., Novikov S.P. and Pitaevskii L.I., *Theory of solitons. The inverse scattering method*, New York, Plenum, 1984.

- Open problems:
  – to study the internal structure of the soliton solutions and soliton interactions (for both types of systems);
  – to study reductions of the gauge equiv. systems.

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Good health and best of luck in the next $\sqrt{50!}$ years, Maks!


Thank you!

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