Integrable non-homogeneous systems of hydrodynamic type.

Vladimir Sokolov

Landau Institute for Theoretical Physics, Moscow, Russia, sokolov@itp.ac.ru

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The integrability theory for $1+1$-dimensional homogeneous systems of hydrodynamic type of the form

$$u_{i,t} = \sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,x}, \qquad i = 1, ..., n,$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ was developed by B. Dubrovin, S. Novikov, S. Tsarev.

In contrast to the homogeneous case, there is no satisfactory criteria of integrability for non-homogeneous hydrodynamic type systems of the form

$$u_{i,t} = \sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,x} + b_i(\mathbf{u}). \qquad (1)$$

The following example

$$u_t = vu_x + \frac{1}{v-u}, \qquad v_t = uv_x + \frac{1}{u-v} \quad (2)$$

found by J. Gibbons and S. Tsarev indicates the existence of integrable systems (1) having properties unusual for $1+1$-dimensional integrable models.

In particular, system (2) has only a few local infinitesimal symmetries and therefore the symmetry approach to classification of integrable $1+1$-dimensional systems is not applicable to (2).

It turns out that this system possesses infinite non-commutative hierarchy of nonlocal symmetries explicitly depending on x and t.

The solving of (2) can be reduced to a construction of a conformal mapping of a slit domain. The construction is related to the existence of the following Lax representation

$$\Psi_x = \frac{\lambda - u - v}{(\lambda - u)(\lambda - v)}\Psi_\lambda,$$

$$\Psi_t = \frac{1}{(\lambda - u)(\lambda - v)}\Psi_\lambda$$

for (2). Here λ is a spectral parameter.

Similar Lax representations were considered by S. Burtsev, V. Zakharov, and A. Mikhailov.

Possibly the technique developed S.Manakov and P. Santini for solving of dispersionless systems can be adopted for systems with such Lax pairs.

We consider systems (1) having Lax representations of the form

$$\Psi_x = f(\mathbf{u}, \lambda)\Psi_\lambda, \qquad \Psi_t = g(\mathbf{u}, \lambda)\Psi_\lambda. \quad (3)$$

Note that the compatibility conditions for these equation equation can be written as

$$\frac{\partial f}{\partial t} - \frac{\partial g}{\partial x} + [f, g] = 0,$$

where $[f, g] = f\frac{\partial g}{\partial \lambda} - g\frac{\partial f}{\partial \lambda}$ is the bracket in an algebra of vector fields.

To generalize example (2) we assume that both functions f and g in (3) have simple poles at $\lambda = u_1, ..., u_n$. We do not restrict ourselves by functions f, g rational in λ but all examples in this talk are rational.

Classification in the case $n = 2$

Proposition 1. The system (1) has the form

$$u_t = a(v)u_x + h(u,v), \quad v_t = a(u)v_x + h(v,u);$$

The functions f, g from the Lax representation are give by

$$f(u,v,\lambda) = \frac{h(\lambda,u) - h(\lambda,v)}{a(u) - a(v)},$$

$$g(u,v,\lambda) = \frac{a(v)h(\lambda,u) - a(u)h(\lambda,v)}{a(u) - a(v)};$$

To find h and a one has to solve the following functional equation:

$$h(\lambda,v)h(\lambda,u)_\lambda - h(\lambda,u)h(\lambda,v)_\lambda +$$

$$h(u,v)h(\lambda,u)_u - h(v,u)h(\lambda,v)_v -$$

$$\frac{h(u,v)a'(u) - h(v,u)a'(v)}{a(u) - a(v)}(h(\lambda,u) - h(\lambda,v)) = 0.$$

Let us present several solutions of this functional equation.

Example 1. The functions

$$h(u,v) = \frac{S_3(u)\,S_3(v)}{W_2(v)(u-v)}, \qquad a(u) = \frac{P_2(u)}{W_2(u)},$$

where S_3, P_2 and W_2 are arbitrary polynomials of degree 3, 2 and 2 correspondingly, satisfy the function equation.

For equation (2) we have $S_3(u) = 1$, $W_2(u) = -1$, $P_2(u) = -u$. $\square$

Example 2. Let

$$h(u,v) = \frac{Q(u)\,Q(v)P(u)(P(v)^2P(u) + (u-v)Q(v))}{R(v)(u-v)},$$

$$a(u) = \frac{S(u)}{R(u)},$$

where $P(u) = p_1u + p_0$, $Q(u) = q_1u + q_0$ are arbitrary polynomials and the coefficients of polynomials $S(u) = s_2u^2 + s_1u + s_0$ and $R(u) = r_2u^2 + r_1u + r_0$ satisfy the same linear equation

$$2p_0q_0x_2 - (p_0q_1 + p_1q_0)x_1 + 2p_1q_1x_0 = 0.$$

Notice that it means that the double ratio (pq, z_1z_2) equals -1, where p, q, z_1, z_2 are roots of the polynomials $P(u), Q(u)$ and $x_2u^2+x_1u+x_0$ correspondingly. $\square$

Example 3. The functions

$$h(u,v) = \frac{Q(u)\,Q(v)R(u)(R(v)R(u) + k(u-v))}{T(v)(u-v)},$$

$$a(u) = \frac{S(u)}{T(u)},$$

where Q, R, S, T are arbitrary polynomials of first degree and k is an arbitrary constant, satisfy the functional equation. $\square$

It is easy to verify that the classes of solutions described in Examples 1-3 are invariant with respect to the transformations

$$u \to \frac{k_1 u + k_2}{k_3 u + k_4}, \qquad v \to \frac{k_1 v + k_2}{k_3 v + k_4}$$

and

$$x \to r_1 x + r_2 t, \qquad t \to r_3 x + r_4 t,$$

where k_i and r_i are arbitrary constants.

Theorem 1. Any solution of the functional equation such that $h(u, v)$ has a simple pole on the diagonal $u = v$ belongs to one of the three above classes up to transformations

$$u \to \phi(u), \qquad v \to \phi(v).$$

Examples in the case $n = 3$

In the case $N = 3$ integrable systems have the form

$$u_t = a_1(v,w)u_x + b_1(u,v,w),$$
$$v_t = a_2(u,w)v_x + b_2(u,v,w),$$
$$w_t = a_3(u,v)w_x + b_3(u,v,w),$$

where

$$a_1(u,v) = a_2(u,v) = a_3(u,v) = \frac{B(u) - B(v)}{A(u) - A(v)},$$

$$b_1(u,v,w) = \frac{X(v,u) - X(w,u)}{A(v) - A(w)},$$

$$b_2(u,v,w) = \frac{X(w,v) - X(u,v)}{A(w) - A(u)},$$

$$b_3(u,v,w) = \frac{X(u,w) - X(v,w)}{A(u) - A(v)}.$$

The following group of affine transformations is admissible:

$$A \to c_1 A + c_2 B + c_3, \qquad B \to c_4 A + c_5 B + c_6,$$

Let

$$X(u,\lambda) = \frac{R(u)S(\lambda)}{\lambda - u}$$

for some functions R, S.

Example 4.

$$R = \frac{S_3}{W}, \qquad S = S_3,$$

where S_3 and W are arbitrary polynomials of degree 3 and 2. Functions A and B are linear combinations of $\dfrac{u^i}{W}$, $i = 0, 1, 2$ (cf. Ex. 1).

Example 5.

$$R = 1, \quad S = S_4, \quad A = \frac{A_4}{S_4}, \quad B = \frac{B_4}{S_4},$$

where S_4, A_4, B_4 are arbitrary polynomials of fourth degree.

Example 6. The following example has a more complicated structure:

$$X(u,z) = \frac{(u-a)^2(z-a)(z-b)R(u,z)}{(u-z)u},$$

where

$$R(u,z) = (a+b)uz - 2b^2z - 2abu + b^2(a+b),$$

a and b are arbitrary parameters. The functions $A(u)$ and $B(u)$ are any linear combinations of $u, 1, u^{-1}$.

The case of arbitrary n

Proposition 3. The homogeneous part of the system is a diagonal weakly non-linear semi-Hamiltonian system:

$$u_{i,t} = a_i u_{i,x}, \qquad \partial_i a_i = 0, \quad i = 1, ..., n \quad (4)$$

where

$$\partial_j \frac{\partial_i a_k}{a_i - a_k} = \partial_i \frac{\partial_j a_k}{a_j - a_k}.$$

Here $\partial_i = \frac{\partial}{\partial u_i}$.

For such a system the coefficient a_i has the form (Ferapontov):

$$a_i = \frac{\det \Delta_{n,i}}{\det \Delta_{n-1,i}},$$

where

$$\Delta = \begin{pmatrix} 1 & ... & 1 \\ q_{1,1}(u_1) & ... & q_{1,n}(u_n) \\ .... & ... & .... \\ q_{n-1,1}(u_1) & ... & q_{n-1,n}(u_n) \end{pmatrix}.$$

The functions f and g in the Lax pair have the form

$$f = \frac{\det P}{\det \Delta}, \qquad g = \frac{\det Q}{\det \Delta},$$

where

$$P = \begin{pmatrix} h_1(\lambda, u_1) & ... & h_n(\lambda, u_n) \\ & \Delta_{n-1} & \end{pmatrix},$$

$$Q = \begin{pmatrix} h_1(\lambda, u_1) & ... & h_n(\lambda, u_n) \\ & \Delta_n & \end{pmatrix}.$$

The non-homogeneous part of the system is given by

$$b_i = \frac{\det Q_{n,i}|_{\lambda=u_i}}{\det \Delta_{n-1,i}}, \qquad i = 1, ..., n.$$

Example 7. The system

$$u_{i,t} = \sum_{j\neq i} u_j \; u_{i,x} + \prod_{j\neq i} (u_i - u_j)^{-1}, \qquad i = 1, ..., n.$$

has appeared in a paper by Ferapontov and Fordy, where a different problem was studied. If $n = 2$, then this system coincides with the Gibbons-Tsarev system.

Main construction

Suppose that $f = S(\lambda) f_1,\ g = S(\lambda) g_1$, where $S(\lambda)$ is a polynomial with constant coefficients and with pairwise distinct roots $\lambda_1, ..., \lambda_p$. We assume that $n \leq 2p - 3$ and $p \leq n + 1$.

At first glance this is not true for the Gibbons-Tsarev system (2). Nevertheless, under the transformation

$$\lambda \to \bar{\lambda} = \frac{a\lambda + b}{c\lambda + d}$$

a common multiplier $(c\lambda + d)^3$ in f and g appears.

Lemma 1. Set $\mu_i = f_1|_{\lambda=\lambda_i}$, and $\nu_i = g_1|_{\lambda=\lambda_i}$. Then

$$\frac{\partial \mu_i}{\partial t} = \frac{\partial \nu_i}{\partial x}.$$

Theorem 2. Let

$$h_i(\lambda, u) = \frac{S(\lambda)}{\lambda - u}, \qquad q_{j,i}(u) = u^j,$$

if $i = 1, ..., n, \; j = 1, ..., p-4$ and

$$q_{p-4+j,i}(u) = \sum_{k=1}^{p} \frac{c_{k,j}}{u - \lambda_k} \tag{5}$$

if $i = 1, ..., n, \; j = 1, ..., n-p+3$. Here $c_{i,j}$ are arbitrary constants such that the matrix $(c_{i,j})$ has rank $n+3-p$. Then formulas for a_i, b_i, f, g define a system (4) possessing a Lax representation (3).

Remark. Suppose the functional equation holds and h has a simple pole on the diagonal. Then the following so-called Gibbons-Tsarev type system is compatible:

$$\partial_i p_j = h(p_j, p_i)\partial_i u, \qquad \partial_i v = a(p_i)\partial_i u,$$

$$\partial_i \partial_j u = \frac{h(p_i, p_j)a'(p_i) - h(p_j, p_i)a'(p_j)}{a(p_i) - a(p_j)}\partial_i u \partial_j u.$$

Here $i \neq j = 1, ..., N$; $p_1, ..., p_N, u$ are functions in $r_1, ..., r_N$, $\partial_i = \frac{\partial}{\partial r_i}$ and N is arbitrary. Note that there exist Gibbons-Tsarev type systems that have a different structure. $\square$

Proposition 2. For each Gibbons-Tsarev type system of this kind the functions h and a define a non-homogeneous gydrodynamic type system with $n = 2$ possessing Lax representation.

Proof. Set $N = 2$ in the GT-system and consider p_1, p_2 as functions of u, v. We get

$$(p_j)_u + (p_j)_v a(p_i) = h(p_j, p_i),$$

where $i \neq j = 1, 2$. It is known that each Gibbons-Tsarev type system admits so-called dispersionless Lax operator. It is a function $L(\lambda, r_1, ..., r_N)$ defined by the following system

$$\partial_i L = h(\lambda, p_i) L_\lambda \partial_i u, \quad i = 1, ..., N.$$

If $N = 2$ then L as a function of u, v, λ satisfies

$$L_u = f(p_1, p_2, \lambda) L_\lambda, \quad L_v = g(p_1, p_2, \lambda) L_\lambda.$$

Consider (1+1)-dimensional hydrodynamic type systems

$$\mathbf{r}_t = A(\mathbf{r})\mathbf{r}_x,$$

where $\mathbf{r} = (r_1, ..., r_N)^t$ and A is $N \times N$-matrix.

Suppose A is diagonal:

$$(r_i)_t = \lambda_i(r_1, ..., r_N)(r_i)_x.$$

If this system has symmetries of the same form

$$(r_i)_\tau = \mu_i(r_1, ..., r_N)(r_i)_x.$$

Then

$$\frac{\partial_j \lambda_i}{\lambda_i - \lambda_j} = \frac{\partial_j \mu_i}{\mu_i - \mu_j}, \quad i \neq j,$$

where $\partial_i = \frac{\partial}{\partial r_i}$, and

$$\partial_j \frac{\partial_i \lambda_k}{\lambda_i - \lambda_k} = \partial_i \frac{\partial_j \lambda_k}{\lambda_j - \lambda_k},$$

The latter relations are called **semi-Hamiltonian conditions**.

There exist evolution integrable 3D-equations.

Example 1. The KP-equation can be written as

$$u_y = -\frac{1}{2}u_{xx} + v_x, \qquad v_y = u_t + \frac{1}{2}v_{xx} - \frac{1}{3}u_{xxx} - uu_x.$$

The correspondent dispersionless system (DKP):

$$u_y = v_x, \qquad v_y = u_t - uu_x.$$

An N-component hydrodynamic reduction of DKP is defined by a pair of compatible (1+1)-dimensional systems

$$\begin{aligned} (r_i)_t = \lambda_i(r_1, ..., r_N)(r_i)_x, \\ (r_i)_y = \mu_i(r_1, ..., r_N)(r_i)_x \end{aligned} \tag{6}$$

and by functions $U(r_1, ..., r_N), V(r_1, ..., r_N)$ such that for any solution of (6) the functions $u = U(r_1, ..., r_N)$, $v = V(r_1, ..., r_N)$ satisfy

$$u_y = v_x, \qquad v_y = u_t - uu_x.$$

Such solutions (u, v) are called N-*phase solutions*.

Substituting U and V into the DKP, we get

$$\sum (r_i)_y \, \partial_i U = \sum (r_i)_x \, \partial_i V.$$

It follows from here that

$$\partial_i V = \mu_i \partial_i U.$$

From the second equation we get

$$\mu_i \partial_i V = \lambda_i \partial_i U - U \partial_i U$$

and

$$\lambda_i = U + \mu_i^2.$$

Now it follows from

$$\frac{\partial_j \lambda_i}{\lambda_i - \lambda_j} = \frac{\partial_j \mu_i}{\mu_i - \mu_j}.$$

that

$$\partial_j \mu_i = \frac{\partial_j U}{\mu_j - \mu_i}, \quad i \neq j. \tag{7}$$

Compatibility $\partial_k \partial_j \mu_i = \partial_j \partial_k \mu_i$ of (7) gives rise to

$$\partial_i \partial_j U = \frac{2 \partial_i U \partial_j U}{(\mu_j - \mu_i)^2}, \quad j \neq i. \tag{8}$$

Remarkably, (8) is compatible !

Recall that we have one equation more:

$$\partial_i V = \mu_i \partial_i U. \tag{9}$$

Definition. A compatible system of PDEs of the form

$$\partial_i p_j = f(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \qquad j \neq i,$$

$$\partial_i \partial_j u_1 = h(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \partial_j u_1, \qquad j \neq i,$$

$$\partial_i u_k = g_k(p_i, u_1, ..., u_n)\, \partial_i u_1, \qquad k = 1, ..., n{-}1,$$

where $i, j = 1, 2, 3$ is called *n-fields GT-system*. Here p_1, p_2, p_3, $u_1, ..., u_n$ are functions of r^1, r^2, r^3.

Definition. Two GT-systems are called *equivalent* if they are related by a transformation of the form

$$p_i \to \lambda(p_i, u_1, ..., u_n),$$

$$u_k \to \mu_k(u_1, ..., u_n), \qquad k = 1, ..., n.$$

Example 2. Let a_0, a_1, a_2 be arbitrary constants. Then the system

$$\partial_i p_j = \frac{a_2 p_j^2 + a_1 p_j + a_0}{p_i - p_j} \partial_i u_1,$$

$$\partial_i \partial_j u_1 = \frac{2a_2 p_i p_j + a_1(p_i + p_j) + 2a_0}{(p_i - p_j)^2} \partial_i u_1 \partial_j u_1$$

is an one-field GT-system. The original Gibbons-Tsarev system corresponds to $a_2 = a_1 = 0, a_0 = 1$. By linear transformations the polynomial $P(x) = a_2 x^2 + a_1 x + a_0$ can be reduced to one of the following canonical forms: $P = 1$, $P = x$, $P = x^2$, or $P = x(x-1)$. $\square$

There is a classification of GT-systems of the form

$$\partial_i p_j = f(p_i, p_j)\partial_i u_1, \quad \partial_i \partial_j u_1 = h(p_i, p_j)\partial_i u_1 \partial_j u_1$$

The generic GT is given by

$$\partial_i p_j = \frac{F(p_i)F(p_j)}{(p_i - p_j)Q(p_i)}\partial_i u_1,$$

where $F'''' = 0$, $Q''' = 0$.

Definition. An additional system

$$\partial_i u_k = g_k(p_i, u_1, ..., u_{n+m})\partial_i u_1, \quad k = n+1, ..., n+m,$$

compatible with the GT-system is called *an extension* of the GT-system by fields $u_{n+1}, ..., u_{n+m}$.

It turns out that

$$\partial_i u_{n+1} = f(p_i, u_{n+1}, u_1, ..., u_n)\, \partial_i u_1,$$

is an extension for any GT-system. We call it *the regular extension* by u_{n+1}.

Example 2-1. The generic case of Example 2 corresponds to $P = x(x-1)$. The sequence of regular extensions by $u_2, u_3, ...$ is given by

$$\partial_i u_2 = \frac{u_2(u_2 - 1)}{p_i - u_2}\partial_i u_1,$$

$$\partial_i u_3 = \frac{u_3(u_3 - 1)}{p_i - u_3}\partial_i u_1,$$

$$\cdots$$

Example 3. Let

$$\theta(z,\tau)=\sum_{\alpha\in\mathbb{Z}}(-1)^{\alpha}e^{2\pi i(\alpha z+\frac{\alpha(\alpha-1)}{2}\tau)},$$

$$\rho(z,\tau)=\frac{\theta_z(z,\tau)}{\theta(z,\tau)}.$$

Then

$$\partial_\alpha p_\beta=\frac{1}{2\pi i}(\rho(p_\alpha-p_\beta)-\rho(p_\alpha))\partial_\alpha\tau,$$

$$\partial_\alpha\partial_\beta\tau=-\frac{1}{\pi i}\rho'(p_\alpha-p_\beta)\partial_\alpha\tau\partial_\beta\tau,$$

where $\alpha,\beta=1,2,3,\quad \alpha\neq\beta$, is an one-field GT-system with the field τ. $\square$

Example 4. Let

$$y^2=x(x-1)(x-u)(x-v)(x-w)$$

be generic curve of genus 2. Then the formulas

$$f = \frac{p_j(p_j-1)(p_i-u)(p_i-v)(p_i-w) + y(p_i)y(p_j)}{u(u-1)(p_i-v)(p_i-w)(p_i-p_j)},$$

$$g = \frac{u(p_i+p_j) + (u-1)(p_i^2+p_j^2)}{u(u-1)(p_i-p_j)^2} +$$

$$\frac{(p_i+p_j-4u)p_ip_j}{u(u-1)(p_i-p_j)^2} +$$

$$\frac{(2vw-(p_i+p_j)(v+w)+2p_ip_j)\, y(p_i)y(p_j)}{u(u-1)(p_i-v)(p_i-w)(p_j-v)(p_j-w)(p_i-p_j)^2}$$

$$\partial_i v = \frac{v(v-1)(p_i-u)}{u(u-1)(p_i-v)}\partial_i u,$$

$$\partial_i w = \frac{w(w-1)(p_i-u)}{u(u-1)(p_i-w)}\partial_i u,$$

define a 3-field GT-system. $\square$

GT-families of (1+1)-dimensional systems

Suppose a GT-system is fixed. We should construct pairs of functions $\lambda(p,u_1,...,u_n)$ and $\mu(p,u_1,...,u_n)$ satisfying

$$\frac{\partial_i\lambda(p_j)}{\lambda(p_i)-\lambda(p_j)}=\frac{\partial_i\mu(p_j)}{\mu(p_i)-\mu(p_j)}. \qquad (10)$$

Example 2-2. Consider n-field regular extension of GT-system from Example 2-1:

$$\partial_i p_j=\frac{p_j(p_j-1)}{p_i-p_j}\partial_i w,\ \ \partial_{ij}w=\frac{2p_ip_j-p_i-p_j}{(p_i-p_j)^2}\partial_i w\partial_j w,$$

$$\partial_i u_j=\frac{u_j(u_j-1)\partial_i w}{p_i-u_j}.$$

Consider the following overdetermind system of linear PDEs:

$$\frac{\partial^2 h}{\partial u_j \partial u_k} = \frac{s_j}{u_j - u_k} \cdot \frac{\partial h}{\partial u_k} + \frac{s_k}{u_k - u_j} \cdot \frac{\partial h}{\partial u_j}, \qquad j \neq k,$$

$$\frac{\partial^2 h}{\partial u_j \partial u_j} = -\left(1 + \sum_{k=1}^{n+2} s_k\right) \frac{s_j}{u_j(u_j - 1)} \cdot h +$$

$$\frac{s_j}{u_j(u_j - 1)} \sum_{k \neq j}^{n} \frac{u_k(u_k - 1)}{u_k - u_j} \cdot \frac{\partial h}{\partial u_k} +$$

$$\left(\sum_{k \neq j}^{n} \frac{s_k}{u_j - u_k} + \frac{s_j + s_{n+1}}{u_j} + \frac{s_j + s_{n+2}}{u_j - 1}\right) \cdot \frac{\partial h}{\partial u_j}$$

Here $i, j = 1, ..., n$, and $s_1, ..., s_{n+2}$ are arbitrary parameters. The solution space for the system is $n + 1$-dimensional.

For any solution h define

$$S(h,p) = \sum_{1\le i\le n} u_i(u_i-1)(p-u_1)...\widehat{i}...(p-u_n)h_{u_i} +$$

$$(1+\sum_{1\le i\le n+2} s_i)(p-u_1)...(p-u_n)h.$$

Theorem. Let h_1, h_2, h_3 be linear independent solutions. Then the functions

$$\lambda = \frac{S(h_1,p)}{S(h_3,p)}, \qquad \mu = \frac{S(h_2,p)}{S(h_3,p)} \tag{11}$$

satisfy (10).

Suppose that a GT-system and a solution $\lambda,\ \mu$ of (10) are fixed.

Proposition 1. Consider the vector space V of functions on p, genrated by $3n$ functions

$$\{g_j(p,u_1,...,u_n), \lambda(p,u_1,...,u_n)g_j(p,u_1,...,u_n),$$

$$\mu(p,u_1,...,u_n)g_j(p,u_1,...,u_n), \ ; \quad j=1,...,n\}.$$

Here $g_1 = 1$. The corresponding system

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\,\frac{\partial u_j}{\partial t}+\sum_{j=1}^{n} b_{ij}(\mathbf{u})\,\frac{\partial u_j}{\partial y}+\sum_{j=1}^{n} c_{ij}(\mathbf{u})\frac{\partial u_j}{\partial x}=0$$

consists in $n+k$ equations iff the dimension of V equals $(2n-k)$. The coefficients of the system can be found from relations

$$\sum_{j=1}^{n} (a_{ij}(\mathbf{u})\,\lambda(p,u_1,...,u_n)+b_{ij}(\mathbf{u})\,\mu(p,u_1,...,u_n)+$$

$$c_{ij}(\mathbf{u}))\,g_j(p,u_1,...,u_n)=0, \qquad (12)$$

where $i=1,...,n+k$.

Example 6. Consider the GT-system

$$\partial_j p_i = \frac{\partial_j u}{p_j - p_i}, \quad i \neq j.$$

$$\partial_i \partial_j u = \frac{2\partial_i u \partial_j u}{(p_j - p_i)^2}, \quad i \neq j.$$

$$\partial_i v = p_i \partial_i u.$$

The equation (10) has a solution $\lambda = p^2 + u, \mu = p$. Then V is generated by

$$\{p^2 + u, p^3 + up, p, p^2, 1, p\}.$$

The dimension of V equals 4. Relations (12) are rquivalent to 8 linear equations.

Matrices A, B, C, consisting from unknown coefficients of the system are defined up left and rigth multiplication by two arbitrary matrices. It is easy to verify that any solution is equivalent to $a_{11} = 1, a_{12} = a_{21} = a_{22} = 0$, $b_{11} = b_{22} = 0, b_{12} = -1, b_{21} = 1,\ c_{11} = -u, c_{12} = c_{21} = 0, c_{22} = -1$. The latter solution yields the DKP. $\square$

Dispersionless Lax pairs.

If we take

$$A=\frac{1}{2}D^2+u,\quad B=\frac{1}{3}D^3+uD+v,\quad D=\frac{d}{dx},$$

then the operator relation

$$B_y-A_t=[A,B]$$

is equivalent to

$$u_y=-\frac{1}{2}u_{xx}+v_x,\quad v_y-u_t=\frac{1}{2}v_{xx}-\frac{1}{3}u_{xxx}-uu_x.$$

If

$$A=\frac{p^2}{2}+u,\qquad B=\frac{p^3}{3}+up+v,$$

then

$$B_y-A_t=\{A,B\}$$

where $\{f,g\}=f_pg_x-f_xg_p$, is equivalent to

$$u_y=v_x,\quad v_y=u_t-uu_x.$$

Two more examples of possible A-operators:

$$A = \log(p-u), \qquad A = \sqrt{u\,(p^2+c_1)+c_2}.$$

Notice that the relation $B_y - A_t = \{A, B\}$ can be regarded as the compatibility condition $(\Phi_y)_t = (\Phi_t)_y$ for

$$\Phi_y = A(\Phi_x, u, v), \qquad \Phi_t = B(\Phi_x, u, v).$$

Thus the system

$$u_y = v_x, \quad v_y = u_t - uu_x$$

possesses the following pseudopotential representation

$$\Phi_y = \frac{\Phi_x^2}{2} + u, \qquad \Phi_t = \frac{\Phi_x^3}{3} + u\Phi_x + v.$$

Part 2. Hydrodynamic chains.

Consider integrable infinite quasi-linear chains of the form

$$u_{\alpha,t} = \phi_{\alpha,1} u_{1,x} + \cdots + \phi_{\alpha,\alpha+1} u_{\alpha+1,x}, \qquad \alpha = 1, 2, ...,$$

where $\phi_{\alpha,\alpha+1} \neq 0$, $\phi_{\alpha,j} = \phi_{\alpha,j}(u_1, ..., u_{\alpha+1})$.

Example. The Benney chain

$$u_{1,t} = u_{2,x}, \qquad u_{2,t} = u_1 u_{1,x} + u_{3,x}, \cdots$$

$$u_{k,t} = (k-1) u_{k-1} u_{1,x} + u_{k+1,x}, \cdots$$

is integrable. The hydrodynamic reductions were investigated by Gibbons and Tsarev □

Two chains are called *equaivalent*, if they are related by a transformation of the form

$$u_\alpha \to \Psi_\alpha(u_1, ..., u_\alpha), \qquad \frac{\partial \Psi_\alpha}{\partial u_\alpha} \neq 0, \qquad \alpha = 1, 2, ...$$

Integrability means the existence of hydrodynamic reductions.

Definition. A semi-Hamiltonian (1+1) -dimensional system

$$(r_i)_t = \lambda_i(r_1, ..., r_N)(r_i)_x$$

and functions $u_j(r^1, ..., r^N),\ \ j = 1, 2, ...$ define an N-component reduction of the chain if $u_j = u_j(r^1, ..., r^N),\ i = 1, ...$ satisfy the chain for any solution of the system.

Example. The reductions of the Benney chains are given by the infinite triangular GT-system

$$\partial_i p_j = \frac{\partial_i u_1}{p_i - p_j}, \qquad \partial_i \partial_j u_1 = \frac{2\, \partial_i u_1\, \partial_j u_1}{(p_i - p_j)^2},$$

$$\partial_i u_m = \Big(-(m-2)u_{m-2} - \cdots - 2u_2 p_i^{m-2} -$$

$$u_1 p_i^{m-3} + p_i^{m-1})\, \partial_i u_1.$$

The semi-Hamiltonian system is defined by $\lambda_i = p_i$.

Definition. A Gibbons-Tsarev family associated with the Gibbons-Tsarev system is a (1+1)-dimensional hydrodynamic type system of the form

$$r_t^i = F(p_i, u_1, ..., u_n) r_x^i, \qquad i = 1, ..., N, \quad (13)$$

semi-Hamiltonian by virtue of the GT-system. Note that the corresponding functional equation for F does not depend on N.

Using the equivalence transformation, we can set

$$F(p, u_1, ..., u_n) = p.$$

A calcualation similar to already done for DKP-equation leads to an infinite triangular GT-system related to any integrable chain.

Definition. Compatible system of the form

$$\partial_i p_j = f(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \qquad , \; i \neq j,$$

$$\partial_i \partial_j u_1 = h(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \partial_j u_1, \qquad i \neq j,$$

$$\partial_i u_k = g_k(p_i, u_1, ..., u_k)\, \partial_i u_1, \qquad k = 1, 2, ..., ,$$

where $i, j = 1, 2, 3$, is called *triangular GT-system*. Here p_1, p_2, p_3, $u_1, u_2, ...$ are functions of r^1, r^2, r^3 and $\partial_i = \frac{\partial}{\partial r^i}$. $\square$

Substituting $u_i = u_i(r^1, ..., r^N)$, $i = 1, ...$ into the chain, we get

$$g_i(p) = \psi_{i,0} + \psi_{i,1} p + ... + \psi_{i,i-1} p^{i-1},$$

where $\psi_{i,j}$ are functions of $u_1, ..., u_i$. In particular,

$$g_2 = -\frac{p}{\phi_{1,2}} - \frac{\phi_{1,1}}{\phi_{1,2}}.$$

Proposition. The non-triangular part of the GT-system has the form

$$\partial_i p_j = \frac{P(p_i, p_j)}{p_i - p_j}\,\partial_i u_1, \qquad i \neq j, \tag{14}$$

$$\partial_i \partial_j u_1 = \frac{Q(p_i, p_j)}{(p_i - p_j)^2}\,\partial_i u_1 \partial_j u_1, \qquad i \neq j, \tag{15}$$

where P, Q - are quadratic polynomial in each of variables p_i and p_j. $\square$

Rewrite (14) as

$$\partial_i p_j = \Big(\frac{R(p_j)}{p_i - p_j} + (z_4 p_j^2 + z_5 p_j + z_6) p_i +$$

$$z_4 p_j^3 + z_3 p_j^2 + z_7 p_j + z_8\Big)\,\partial_i u_1,$$

where $R(x) = z_4 x^4 + z_3 x^3 + z_2 x^2 + z_1 x + z_0$.

Under a transformation

$$p_i = \frac{a\bar{p}_i + b}{\bar{p}_i - \psi}, \qquad i = 1, 2, 3 \qquad (16)$$

the polynomial R is changed as:

$$R(p_i) \to (p_i - \psi)^4 R(\frac{ap_i + b}{p_i - \psi}).$$

Suppose that the roots of R are distinct. It is posiible to verify that three of four roots can be sent to 0, 1 and ∞. By $u_1 \to q(u_1)$ we lead to $\lambda = u_1$.

So in the generic case we have

$$R = x(x-1)(x-u_1).$$

and finally we obtain

$$\partial_i p_j = \frac{p_j(p_j-1)(p_i-u_1)}{u_1(u_1-1)(p_i-p_j)}\partial_i u_1,$$

$$\partial_i\partial_j u_1 = \frac{Q}{u_1(u_1-1)(p_i-p_j)^2}\partial_i u_1 \partial_j u_1,$$

where

$$Q = p_i p_j(p_i+p_j)-p_i^2-p_j^2+(p_i^2+p_j^2-4p_ip_j+p_i+p_j)u_1$$

In the case of multiple roots the polynomial $R(x)$ can be reduced to one of the canonical forms : $R=0$, $R=1$, $R=x$, $R=x^2$, or $R=x(x-1)$.

Since we used the fractional linear transformation to reduce R to the canonical form, the functions F and g_2 now have the following structure:

$$g_2(p_i) = \frac{k_1 p_i + k_2}{k_3 p_i + k_4}, \qquad F(p_i) = \frac{f_1 p_i + f_2}{k_3 p_i + k_4},$$

where the coefficients are functions of u_1, u_2. So we should described first all fractional linear functions g_2.

In the generic case up to $\bar{u}_2 = \sigma(u_1, u_2)$ they are :

$$\mathbf{1}: \qquad g_2(p) = \frac{u_2(u_2-1)(p-u_1)}{u_1(u_1-1)(p-u_2)};$$

$$\mathbf{2}: \qquad g_2(p) = \frac{1}{p-u_1};$$

$$\mathbf{3}: \qquad g_2(p) = \frac{u_1^{-\lambda}(u_1-1)^{\lambda-1}}{p-\lambda}, \qquad\qquad \lambda = 1, 0;$$

$$\mathbf{4}: \qquad g_2(p) = \frac{u_1-u_2}{u_1(u_1-1)}p + \frac{u_2-1}{u_1-1}.$$

The Benney chain belongs to the case $R=1$, $g_2=p$. Any GT-family has the form $F=f_1(u_1,u_2)p+f_2(u_1,u_2)$. If $f_1=1$ then $F=p+k_2u_2+k_1u_1$. The Benney case corresponds to $k_1=k_2=0$. For arbitrary k_i we get the Kupershmidt chain. In the case $f_1=A(u_1), A'\neq 0$ we obtain:

$$f_1=k_2\exp(\lambda u_1)+k_1,$$

$$f_2=k_2k_3\exp(\lambda u_1)+\lambda k_1(k_3u_1-u_2).$$

In the generic case

$$F=\exp(\lambda u_2)(S_1(u_1)p+S_2(u_1)),$$

where the functions S_i can be expressed in terms the Airy functions.

Consider the generic case. The next step is a description of fractional linear functions F with coefficients being functions of u_1, u_2. The result is the following.

Let $h_1(u_1), h_2(u_1)$ - be linearly independent solutions of the standard hypergeometric equation

$$u(u-1)\, h(u)'' + [s_1 + s_3 - (s_3 + s_4 + 2s_1)\, u]\, h(u)' +$$

$$s_1(s_1 + s_3 + s_4 + 1)\, h(u) = 0$$

and

$$h_3(u_1, u_2) = \int_0^{u_2} (t - u_1)^{s_1} t^{s_3} (t-1)^{s_4} dt.$$

Then

$$F(p, u_1, u_2) = \frac{f_1(u_1, u_2)\, p - f_2(u_1, u_2)}{p - u_2},$$

where

$$f_1 = \frac{u_2(u_2 - 1)h_1 h_{3,u_2} + u_1(u_1 - 1)(h_1 h_{3,u_1} - h_3 h_1')}{u_1(u_1 - 1)(h_1 h_2' - h_2 h_1')},$$

$$f_2 = \frac{u_1 u_2(u_2 - 1)h_1 h_{3,u_2} + u_2 u_1(u_1 - 1)(h_1 h_{3,u_1} - h_3 h}{u_1(u_1 - 1)(h_1 h_2' - h_2 h_1')}$$

For special values of s_i the hypergeometric equation can be solved explicitly. If $s_4 = -2 - s_1 - s_3$, then

$$F = \frac{(u_2 - u_1)^{s_1+1} u_2^{s_3+1} (u_2 - 1)^{-1-s_1-s_3}}{p - u_2};$$

for $s_4 = 0$ we get

$$F = \frac{(p-1)(u_2 - u_1)^{s_1+1} u_2^{s_3+1} (u_1 - 1)^{-1-s_1}}{p - u_2}. \qquad \square$$

Next, we should find the functions $g_3, g_4, ...$ in the triangular GT-system. In particular, one can choose

$$g_3(p) = -\frac{(u_1 - u_2)(u_2 - 1)p}{u_1(u_1 - 1)(p - u_2)^2},$$

$$g_i(p) = \frac{(i-3)(u_1 - u_2)(u_2 - 1)p\, u_i}{u_1(u_1 - 1)(p - u_2)^2} -$$

$$\frac{(u_1 - u_2)^{i-3}(u_2 - 1)^2 p(p - u_1)(p - 1)^{i-4}}{u_1(u_1 - 1)^{i-2}(p - u_2)^{i-1}} -$$

$$\sum_{s=1}^{i-4} \frac{(i-s-2)(u_1 - u_2)^s(u_2 - 1)^2 p(p - u_1)(p - 1)^{s-1}\, u_i}{u_1(u_1 - 1)^{s+1}(p - u_2)^{s+2}}$$

The coefficients $\phi_{i,j}$ of the corresponding chain are given by

$$F(p) = \phi_{1,1} + \phi_{1,2} g_2,$$

$$F(p) g_2 = \phi_{2,1} + \phi_{2,2} g_2 + \phi_{2,3} g_3,$$

$$F(p)g_3 = \phi_{3,1} + \phi_{3,2}g_2 + \phi_{3,3}g_3 + \phi_{3,4}g_4, ...,$$

where F si given by (**??**). These relations are equivalent to an infinite triangular system of linear triangular algebraic equations. Solving it, we get

$$\phi_{1,1} = \frac{f_1 u_1 - f_2}{u_1 - u_2},$$

$$\phi_{1,2} = -\frac{u_1(u_1 - 1)(f_1 u_2 - f_2)}{u_2(u_2 - 1)(u_1 - u_2)},$$

$$\phi_{2,1} = \frac{(u_2 - 1)(f_1 u_2 - f_2)}{(u_1 - 1)(u_1 - u_2)}, \qquad \phi_{2,2} = \frac{f_2 u_1 - f_1 u_2^2}{u_2(u_1 - u_2)},$$

and so on. $\square$

Consider now the most degenerate case $R(x) = 0$. It is easy to verify that the triangular GT-system in this case is equivalent to

$$\partial_i p_j = 0, \quad \partial_i \partial_j u_1 = 0, \quad \partial_i u_k = p_i^{k-1} u_1, \quad k = 2, 3, ...$$

Automorphisms of the GT-system

$$p_j \to p_j, \quad j = 1, ..., N, \quad u_i \to \nu u_i + \gamma_i,$$

$$p_j \to a p_j + b,$$

$$u_i \to a^{i-1} u_i + (i-1) a^{i-2} b u_{i-2} + ... + b^{i-1} u_1,$$

where $j = 1, ..., N, \quad i = 1, 2, ...$ The corresponding GT-family is of the form

$$F(p) = A(u_1, u_2) p + B(u_1, u_2).$$

The coefficients $A(x, y), B(x, y)$ should be found from the semi-Hamiltonian condition. This condition is equivalent to:

$$AB_{yy} = A_y B_y, \quad AB_{xy} = A_y B_x, \quad AB_{xx} = A_x B_x, AA_{yy}$$

This system can be solved in the elementary functions.

It follows from (**??**) that there exist two cases:

1 $F(p, u_1, u_2) = \exp(\lambda u_2)(a(u_1)p + b(u_1))$,

2. $F(p, u_1, u_2) = a(u_1)p + \lambda u_2 + b(u_1)$.

Subcases in the first case: $b' \neq 0$ $b' = 0$. The first subcase leads to

$$a = \sigma', \quad b = k_1\sigma \qquad \sigma(x) = c_1 \exp(\mu_1 x) + c_2 \exp(\mu_2 x$$

The second yields

$$b = c_1, \qquad a(x) = c_2 \exp(\mu x) + c_3, \qquad c_2(c_1\lambda - c_3\mu) =$$

The same subcases of the case 2 lead to

$$a = \sigma', \quad b = k_1\sigma \qquad \sigma(x) = c_1 + c_2 x + c_3 \exp(\mu x), \quad \text{w}$$

$$b = c_1, \qquad a(x) = c_2 \exp(\mu x) + c_3, \quad \text{where} \quad c_2(\lambda - c_3\mu$$

In the generic case F is equvalent to

$$F(p) = e^{u_2+u_1}(p-1) + e^{u_2-u_1}(p+1).$$

The corresponding chain is given by

$$u_{k,t} = (e^{u_2+u_1}+e^{u_2-u_1})u_{k+1,x}+(e^{u_2-u_1}-e^{u_2+u_1})u_{k,x},$$

The chain possesses an infinite hierachy of commuting flows. The next one has the form

$$u_{k,\tau} = (e^{u_2+u_1} + e^{u_2-u_1})u_{k+2,x}+$$

$$(u_3 - u_1)(e^{u_2+u_1} + e^{u_2-u_1})u_{k+1,x}+$$

$$+(e^{u_2+u_1}(u_1-u_3-1)+e^{u_2-u_1}(u_3-u_1-1))u_{k,x},$$

$k = 1, 2, 3, \ldots$ In the case 2 with $c_3 = \lambda = 0, k_1 = 1$ we get the chain

$$u_{k,t} = u_{k+1,x} + u_1 u_{k,x}, \qquad k = 1, 2, 3, \ldots$$

This chain is a generation of

$$u_{k,t} = u_{k+1,x} + u_2 u_{k,x}, \qquad k = 1, 2, 3, \ldots$$

Some of families of functions F are linerly depends on two essential parameters. Denote them γ_1, γ_2. The corresponding integrable chain

$$u_{k,t} = \gamma_1(\phi_{k,1}u_{1,x} + \cdots + \phi_{k,k+1}u_{k+1,x}) +$$

$$\gamma_2(\psi_{k,1}u_{1,x} + \cdots + \psi_{k,k+1}u_{k+1,x})$$

is also linearly depends in γ_1, γ_2. We call that (2+1)-dimensional chain

$$u_{k,t} = (\phi_{k,1}u_{1,x} + \cdots + \phi_{k,k+1}u_{k+1,x}) +$$

$$(\psi_{k,1}u_{1,y} + \cdots + \psi_{k,k+1}u_{k+1,y})$$

is integrable. In each case the reductions can be easily found. For instance, in the generic case

$$F(p) = \gamma_1 e^{u_2+u_1}(p-1) + \gamma_2 e^{u_2-u_1}(p+1).$$

The chain (**??**) has the form

$$u_{k,t} = e^{u_2+u_1}(u_{k+1,x} - u_{k,x}) + e^{u_2-u_1}(u_{k+1,y} + u_{k,y}),$$

The reductions are defined by

$$r^i_t = (e^{u_2+u_1}(p_i-1)+e^{u_2-u_1}q_i(p_i+1))r^i_x, \quad r^i_y = q_i r^i_x,$$

where

$$\partial_i u_k = p_i^{k-1} u_1, \quad k = 1, 2, ..., \qquad \partial_i \partial_j u_1 = 0, \qquad \partial_i p_j$$

$$\partial_i q_j = \frac{(q_i - q_j)((p_i + 1)(p_j - 1)e^{u_1} + q_j(p_i - 1)(p_j + 1)}{(p_i - p_j)(e^{u_1} + q_i e^{-u_1})}$$

Conjecture. Any integrable (2+1) -dimensional chain is generated by two-dimensional family of solutions of (**??**) by the construction described above.

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The general solution is given by

$$\mu_i(\mathbf{r}) = x + \lambda_i(\mathbf{r})t, \qquad i = 1, ..., N.$$

Express u_1 from this formula and substitute it to the GT-system. We get the following new one-field GT-system

$$\partial_i p_j = \frac{p_j(p_j-1)(p_i-u_1)}{u_1(u_1-1)(p_i-p_j)}\partial_i u_1,$$

$$\partial_i\partial_j u_1 = \frac{Q}{u_1(u_1-1)(p_i-p_j)^2}\partial_i u_1 \partial_j u_1,$$

where

$$Q = p_i p_j(p_i+p_j)-p_i^2-p_j^2+(p_i^2+p_j^2-4p_i p_j+p_i+p_j)u_1$$