Integrable WZNW σ- model and string model of WZNW type with $SU(n)$, $SO(n)$, $SP(n)$ constant torsion and infinite hydrodynamic chains for $n \to \infty$

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Abstract

The integrability of WZNW model and string model of WZNW model type with constant $SU(2)$, $SO(3)$, $SP(2)$ and $SU(3)$ torsion is investigated. The closed boson string model in the background gravity and antisymmetric B-field is considered as integrable system in terms of initial chiral currents. The model is considered under assumption that internal torsion related with metric of Riemann-Cartan space and external torsion related with antisymmetric B-field are (anti)coincide. New equation of motion and exact solution this equation was obtained for string model with constant $SU(2)$, $SU(3)$ torsion. New equations of motion and new Poisson brackets(PB) for infinite dimensional hydrodynamic chains was obtained for string model with constant $SU(n)$, $SO(n)$, $SP(n)$ torsion for $n \to \infty$.

String

$$L = \frac{1}{2}\int\limits_0^{2\pi}[\sqrt{-g}g^{\alpha\beta}g_{ab}(\phi)\frac{\partial\phi^a}{\partial x^\alpha}\frac{\partial\phi^b}{\partial x^\beta} + \epsilon^{\alpha\beta}B_{ab}\frac{\partial\phi^a}{\partial x^\alpha}\frac{\partial\phi^b}{\partial x^\beta}]dx,$$

where $g_{ab}(\phi(x)) = g_{ba}(\phi(x))$, $B_{ab}(\phi(x)) = -B_{ba}(\phi(x))$ $g_{\alpha\beta} = e^{\varphi}\eta_{\alpha\beta}$ $g_{ab}(\phi) = e_a^\mu(\phi)e_b^\nu(\phi)g_{\mu\nu}$ $g_{ab}(\phi)$-metric tensor of curve n-dimensional space, $a, b = 1, 2, \ldots, n$. $g^{\mu\nu}$-metric tensor of flat space, tangent space to curve space, $\mu, \nu = 1, 2, \ldots, n$. Both metric have the arbitrary signature.

$g_{\alpha\beta}$- metric tensor of curve 2-d space.

Equation of motion

$$g_{ab}(\phi)g^{\alpha\beta}\frac{\partial^2\phi^a}{\partial x^\alpha \partial x^\beta} + \Gamma_{abc}g^{\alpha\beta}\frac{\partial\phi^b}{\partial x^\alpha}\frac{\partial\phi^c}{\partial x^\beta} + H_{abc}\epsilon^{\alpha\beta}\frac{\partial\phi^b}{\partial x^\alpha}\frac{\partial\phi^c}{\partial x^\beta} = 0,$$

$$\Gamma_{abc} = \frac{1}{2}(\frac{\partial g_{ab}}{\partial\phi^c} + \frac{\partial g_{ac}}{\partial\phi^b} - \frac{\partial g_{bc}}{\partial\phi^a}),$$

connection

$$H_{abc} = \frac{\partial B_{ab}}{\partial\phi^c} + \frac{\partial B_{ca}}{\partial\phi^b} + \frac{\partial B_{bc}}{\partial\phi^a}.$$

torsion

In the terms of repers the connection $\Gamma^c_{ab}(\phi) = \frac{e^c_\mu}{2}[\frac{\partial e^\mu_a}{\partial\phi^b} + \frac{\partial e^\mu_b}{\partial\phi^a}]$ is symmetric on a, b. The function H_{abc} is total antisymmetric function on a, b, c.

New variables
Let us introduce new variables to obtain first order equation instead of second it.

$$J_0^\mu(\phi) = e_\mu^a(\phi)[p_a - B_{ab}(\phi){\phi'}^b]$$

$$J_1^\mu(\phi) = e_a^\mu {\phi'}^a,$$

Momentum

$$p_a(t,x) = g_{ab}(\phi)\dot{\phi}^b + B_{ab}{\phi'}^b$$

Equations of motion

$$\partial_0 J_1^\mu - \partial_1 J_0^\mu = C^\mu_{\nu\lambda}(\phi) J_0^\nu J_1^\lambda, \ \partial_0 J_0^\mu - \partial_1 J_1^\mu = -H^\mu_{\nu\lambda}(\phi) J_0^\nu J_1^\lambda$$

Here $C^{\mu\nu\lambda}$ is the torsion:

$$C^\mu_{\nu\lambda} = \frac{\partial e_a^\mu}{\partial x^b}(e_\nu^b e_\lambda^a - e_\nu^a e_\lambda^b) = (\frac{\partial e_a^\mu}{\partial x^b} - \frac{\partial e_b^\mu}{\partial x^a}) e_\nu^b e_\lambda^a. \tag{1}$$

Commutation relations for new variables

$$\{J_0^\mu(x), J_0^\nu(y)\} = C_\lambda^{\mu\nu}(\phi) J_0^\lambda(x)\delta(x-y) + H_\lambda^{\mu\nu}(\phi) J_1^\lambda(x)\delta(x-y)$$
$$\{J_0^\mu(x), J_1^\nu(y)\} = C_\lambda^{\mu\nu}(\phi) J_1^\lambda(x)\delta(x-y) + g^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y)$$
$$\{J_1^\mu(x), J_1^\nu(y)\} = 0$$

Chiral variables

Let us introduce chiral variables

$$U^{\mu} = g^{\mu\nu} J_{0\nu} + J_1^{\mu}, \;\; V^{\mu} = g^{\mu\nu} J_{0\nu} - J_1^{\mu}$$

$$\{U^{\mu}(x), U^{\nu}(y)\} = \frac{1}{2}[(3C_{\lambda}^{\mu\nu} + H_{\lambda}^{\mu\nu})U^{\lambda} - (C_{\lambda}^{\mu\nu} + H_{\lambda}^{\mu\nu})V^{\lambda}]\delta(x-y) + \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$

$$\{V^{\mu}(x), V^{\nu}(y)\} = \frac{1}{2}[(3C_{\lambda}^{\mu\nu} - H_{\lambda}^{\mu\nu})V^{\lambda} - (C_{\lambda}^{\mu\nu} - H_{\lambda}^{\mu\nu})U^{\lambda}]\delta(x-y) - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$

$$\{U^{\mu}(x), V^{\nu}(y)\} = \frac{1}{2}[(C_{\lambda}^{\mu\nu} + H_{\lambda}^{\mu\nu})U^{\lambda} + (C_{\lambda}^{\mu\nu} - H_{\lambda}^{\mu\nu})V^{\lambda}]\delta(x-y).$$

Here function $H_{\mu\nu\lambda}(\phi)$ is additional external torsion.

These PB's form algebra if:

$$1)\ C_\lambda^{\mu\nu} = 0,\ H_\lambda^{\mu\nu} = 0$$

and functions $U^\mu(x)$ are abelian currents;

$$2)\ C_\lambda^{\mu\nu},\ H_\lambda^{\mu\nu}$$

are structure constants $f_\lambda^{\mu\nu}$ of Lie algebra, and the functions $U^\mu(x)$ are chiral currents. Here are two possibilities to simplify this algebra:

$$a)\ H_\lambda^{\mu\nu} = -C_\lambda^{\mu\nu},$$

$$\{U^\mu(x), U^\nu(y)\} = C_\lambda^{\mu\nu} U^\lambda \delta(x-y) + \delta^{\mu\nu} \tfrac{\partial}{\partial x} \delta(x-y),$$

$$\{V^\mu(x), V^\nu(y)\} = C_\lambda^{\mu\nu} (2V^\lambda - U^\lambda) \delta(x-y) - \delta^{\mu\nu} \tfrac{\partial}{\partial x} \delta(x-y),$$

$$\{U^\mu(x), V^\nu(y)\} = C_\lambda^{\mu\nu} V^\lambda \delta(x-y).$$

$b)$ $H_{\lambda}^{\mu\nu} = C_{\lambda}^{\mu\nu}$,

$$\{U^{\mu}(x), U^{\nu}(y)\} = C_{\lambda}^{\mu\nu}(2U^{\lambda} - V^{\lambda})\delta(x-y) + \delta^{ab}\frac{\partial}{\partial x}\delta(x-y),$$

$$\{V^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu}V^{\lambda} - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$

$$\{U^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu}U^{\lambda}\delta(x-y).$$

Integrable string models

To construct integrable dynamical system we must to have hierarchy of PB's and to find hierarchy of Hamiltonians through bi-Hamiltonity condition. Another way we must have hierarchy of Hamiltonians and to find hierarchy of PS brackets. This way is more simple if the dynamical system have some group structure. In bi-Hamiltonian approach to integrable string models with constant torsion we considered the conserved initial chiral currents $U^{\mu}(x)$ and conserved primitive invariant currents $H_n(U(x))$, as local fields of the Riemann manifold. The non-primitive local charges of invariant chiral currents form the hierarchy of new Hamiltonians.

The chiral currents U^μ in first case and V^μ in second case form Kac-Moody algebras. Equations of motion in light-cone coordinates

$$x^\pm = \frac{1}{2}(t \pm x), \frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$$

have following form:

$$H^\mu_{\nu\lambda} = -C^\mu_{\nu\lambda},\ \partial_- U^\mu = 0,\ \partial_+ V^\mu = 2C^{\mu\nu\lambda} U^\nu V^\lambda,$$

$$H^\mu_{\nu\lambda} = C^\mu_{\nu\lambda},\ \partial_+ V^\mu = 0,\ \partial_- U^\mu = -2C^{\mu\nu\lambda} U^\nu V^\lambda.$$

The chiral currents $U^\mu(x)$ are generators of translation on the curve space

$$\delta_c \phi^a(x) = \{\phi^a(x), c^\mu U_\mu(x)\} = c^\mu e^a_\mu(\phi) = c^a(\phi).$$

Simultaneously, they are generators of group transformations with structure constant $C^{\mu\nu}_\lambda$ in the tangent space.

To construct integrable dynamical system we must to have hierarchy of PB brackets and to find hierarchy of Hamiltonians through bi-Hamiltonity condition. We have used the hydrodynamic approach of Dubrovin, Novikov to integrable systems and Dubrovin solutions of WDVV associativity equation to construct new integrable string equations of hydrodynamic type on the torsionless Riemann space of chiral currents .

Invariant chiral currents

Commutation relations () and equations of motion show that currents U^μ form closed algebra. Therefore, we will consider PB of right chiral currents U^μ and Hamiltonians constructed only from right currents.

The constant torsion does not contribution to equation of motion, but it gives possibility to introduce group structure and to introduce symmetric structure constant.

Let t_a are the generators $SU(n)$, $SO(n)$, $SP(n)$ Lie algebras:

$$[t_\mu, t_\nu] = 2if_{\mu\nu\lambda}t_\lambda. \tag{2}$$

There is additional relation for generators Lie algebra in the defining matrix representation. There is following relation for symmetric double product generators of $SU(n)$ algebra:

$$\{t_\mu, t_\nu\} = \frac{4}{n}\delta_{\mu\nu} + 2d_{\mu\nu\lambda}t_\lambda,\ \mu = 1, ..., n^2 - 1. \tag{3}$$

Here $d_{\mu\nu\lambda}$ is total symmetric structure constant tensor.

The similar relation for total symmetric triple product of $SO(n)$ and $SP(n)$ algebras has form:

$$t_{(\mu}t_{\nu}t_{\lambda)} = v^{\rho}_{\mu\nu\lambda}\,t_{\rho}. \tag{4}$$

Here $v_{\rho\mu\nu\lambda}$ total symmetric structure constant tensor. The invariant chiral currents can be constructed as product of invariant symmetric tensors

$$d_{(\mu_1...\mu_n)} = d^{k_1}_{(\mu_1\mu_2}d^{k_2}_{\mu_3 k_1}...d^{k_{n-3}}_{\mu_{n-1}\mu_n)},\; d_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}$$

for $SU(n)$ group and initial chiral currents U^{μ}:

$$C_n(U(x)) = d_{(\mu_1...\mu_n)}U_{\mu_1}U_{\mu_2}...U_{\mu_n},\; C_2 = \delta_{\mu\nu}U^{\mu}U^{\nu}. \tag{5}$$

Any of this currents satisfy to equation of motion $\partial_{-}C(n)(U(x)) = 0$. The similar construction can be used for $SO(n)$, $SP(n)$ groups. The invariant chiral currents can be constructed as product of invariant symmetric constant tensor

$$v_{(\mu_1 ... \mu_{2n})} = v^{\nu_1}_{(\mu_1\mu_2\mu_3} v^{\nu_1\nu_2}_{\mu_4\mu_5} ... v^{\nu_{2n-3}\mu_{2n-2}\mu_{2n-1}\mu_{2n})}, \; v_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}.$$

and initial chiral currents U^{μ}:

$$C_{2n} = v_{\mu_1 ... \mu_{2n}} U^{\mu_1} ... U^{\mu_{2n}}, \; C_2 = \delta_{\mu_1\mu_2} U^{\mu_1} U^{\mu_2}. \tag{6}$$

The invariant chiral currents for $SU(2)$, $SO(3)$, $SP(2)$ have form:

$$C_{2n} = (C_2)^n \tag{7}$$

Another family of invariant symmetric currents J_n based on the invariant symmetric chiral currents on simple Lie groups, realized as symmetric trace of n product chiral currents $U(x) = t_\mu U^\mu$, $\mu = 1, ..., n^2 - 1$:

$$J_n = SymTr(U...U). \tag{8}$$

These invariant currents are polynomials of product basic chiral currents $C_k,\ k = 2, 3, ..., k$.

$$J_2 = 2C_2,\ J_3 = 2C_3,\ J_4 = 2C_4 + \frac{4}{n}C_2^2,\ J_5 = 2C_5 + \frac{8}{n}C_2C_3,$$

$$J_6 = 2C_6 + \frac{4}{n}C_3^2 + \frac{8}{n}C_2C_4 + \frac{8}{n^2}C_2^3,$$

$$J_7 = 2C_7 + \frac{8}{n}C_3C_4 + \frac{8}{n}C_2C_5 + \frac{24}{n^2}C_2^2C_3,$$

$$J_8 = 2C_8 + \frac{4}{n}C_4^2 + \frac{8}{n}C_3C_5 + \frac{8}{n}C_2C_6 + \frac{24}{n^2}C_2C_3^2 +$$

$$+ \frac{24}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4.$$

The commutation relations for chiral currents have form:

$$\{C_m(x), C_n(y)\} = W_{mn}(y)\frac{\partial}{\partial y}\delta(y-x) - W_{nm}(x)\frac{\partial}{\partial x}\delta(x-y).$$

Hamiltonian function $W_{mn}(x)$ for finite dimensional $SU(n)$, $SO(n)$, $SP(n)$ group has form:

$$W_{mn}(x) = \frac{n-1}{m+n-2}\sum_k a_k C_{m+n-2,k}(x), \ \sum_{k=0} a_k = mn. \tag{9}$$

Here the invariant total symmetric currents $C_{n,k}$, $k = 1, 2...$ are new currents, which are polynomials of product basic invariant currents $C_{n_1}C_{n_2}...C_{n_n}$, $n_1 + ... + n_n = n$.

They can be obtained during calculation total symmetric invariant currents J_n different replacements double product (2) for $SU(n)$ group and triple product (3) in the expression for J_n.

$$J_6 = Tr[t(tt)(tt)t] = 2C_6 + \frac{4}{n}C_3^2 + \frac{8}{n}C_2C_4 + \frac{8}{n^2}C_2^3,$$

$$J_6 = Tr[(tt)(tt)(tt)] = 2C_{6,1} + \frac{12}{n}C_2C_4 + \frac{8}{n^2}C_2^3,$$

$$J_7 = Tr[t(tt)t(tt)t] = 2C_7 + \frac{8}{n}C_3C_4 + \frac{8}{n^2}C^2C_5 + \frac{24}{n^2}C_2^2C_3, \qquad (10)$$

$$J_7 = Tr[(tt)(tt)(tt)t] = 2C_{7,1} + \frac{4}{n}C_3C_4 + \frac{12}{n^2}C^2C_5 + \frac{24}{n^2}C_2^2C_3,$$

$$J_8 = Tr[t(tt)tt(tt)t] = 2C_8 + \frac{4}{n}C_4^2 + \frac{8}{n}C_3C_5 + \frac{8}{n}C_2C_6 +$$

$$+\frac{24}{n^2}C_2C_3^2 + \frac{24}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4,$$

$$J_8 = Tr[(tt)(tt)t(tt)t] = 2C_{8,1} + \frac{4}{n}C_4^2 + \frac{4}{n}C_3C_5 + \frac{24}{n^2}C_2C_3^2 +$$

$$+\frac{12}{n}C_2C_6 + \frac{24}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4,$$

$$J_8 = Tr[(tt)(tt)(tt)(tt)] = 2C_{8,2} + \frac{4}{n}C_4^2 + \frac{16}{n}C_2C_{6,1} + \frac{32}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4,$$

$$J_8 = Tr[t(tt)(tt)(tt)t] = 2C_{8,3} + \frac{12}{n}C_2C_6 + \frac{8}{n}C_3C_5 + \frac{24}{n^2}C_2^2C_4 + \frac{24}{n^2}C_2C_3^2 + \frac{16}{n}C_2^4,$$

This PB can be rewritten as PB of hydrodynamic type

$$\{C_m(x), C_n(y)\} = -\frac{n-1}{m+n-2}\sum_k a_k \frac{dC_{m+n-2,k}(x)}{dx}\delta(x-y)-$$

$$-\sum_k a_k C_{m+n-2,k}(x)\frac{\partial}{\partial x}\delta(x-y),$$

$$\sum_{k=0} a_k = mn. \tag{11}$$

The new chiral currents $C_{n,k}$ have form:

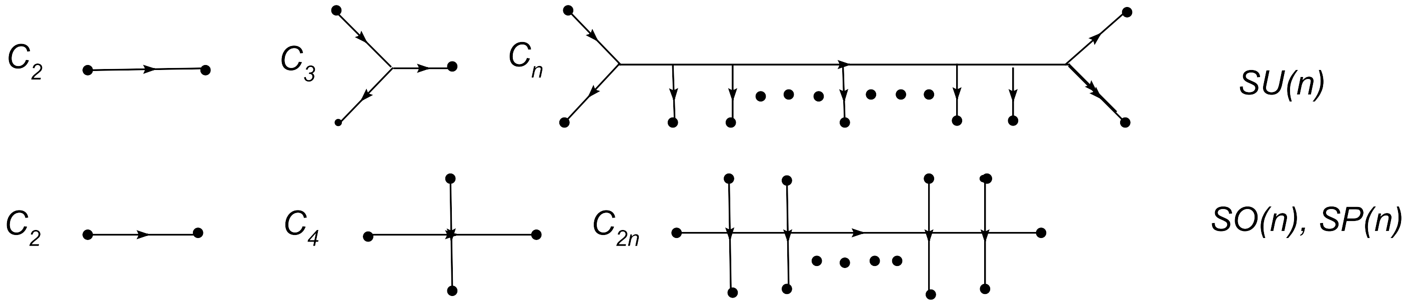
$$C_{6,1} = d^k_{\mu\nu} d^l_{\lambda\rho} d^n_{\sigma\varphi} d^{kln} (U)^{\mu\nu\lambda\rho\sigma\varphi},$$

$$C_{7,1} = d^k_{\mu\nu} d^l_{\lambda\rho} d^n_{\sigma\varphi} d^{nm}_{\tau} d^{klm} (U)^{\mu\nu\lambda\rho\sigma\varphi\tau},$$

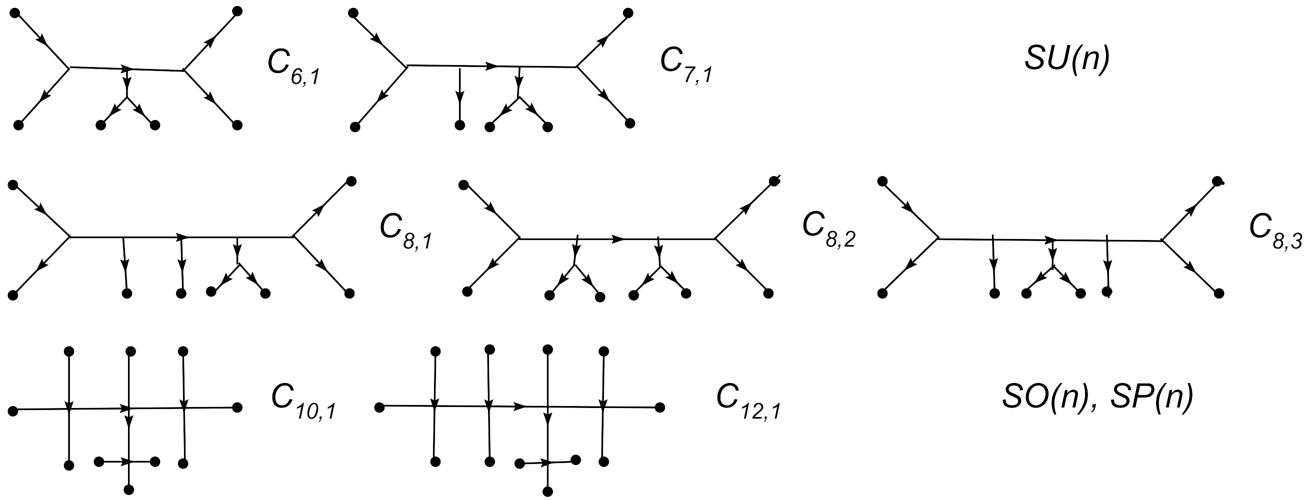
$$C_{8,1} = [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi}][d^p_{\tau\theta}] d^{nmp} (U^8)^{\mu\nu\lambda\rho\sigma\varphi\tau\theta},$$

$$C_{8,2} = [d^k_{\mu\nu}][d^l_{\lambda\rho}][d^n_{\sigma\varphi}][d^m_{\tau\theta}] d^{klp} d^{nmp} (U^8)^{\mu\nu\lambda\rho\sigma\varphi\tau\theta}.$$

$$C_{8,3} = [d^k_{\mu\nu} d^{kl}_{\lambda}][d^n_{\rho\sigma} d^{nm}_{\varphi}][d^p_{\tau\theta}] d^{lmp} (U^8) \mu\nu\lambda\rho\sigma\varphi\tau\theta.$$



monster diagram from PB



Here are only $l = n - 1$ primitive invariant tensors for $SU(n)$ algebra, $l = \frac{n-1}{2}$ for $SO(n)$ algebra and $l = \frac{n}{2}$ for $SP(n)$ algebra. Higher invariant currents C_n for $n \geq l + 1$ are non-primitive currents and they are polynomials of primitive currents. The corresponding non-primitive chiral currents the charges are not Casimir operators. The expression for these polynomials are obtained from the generating function

$$det(1 - \lambda t_\mu U^\mu) = e^{Tr(ln(1-\lambda U))} = \exp(-\sum_{n=2}^{\infty} \frac{\lambda^n}{n} J_n).$$

Here are following expression for nonprimitive currents for finite dimensional groups:

$$SU(3)$$

$$C_4 = \frac{C_2^2}{3},\ C_5 = \frac{C_2 C_3}{3},\ C_6 = \frac{C_2^3 C_3}{9},\ C_7 = \frac{C_2^2 C_3}{9},\ C_8 = \frac{C_2^4}{27}$$

$SU(4)$

$$C_5 = \frac{2C_2C_3}{3},$$

$$C_6 = \frac{1}{6}(C_3^2 + 3C_2C_4),$$

$$C_7 = \frac{1}{6}C_3(2C_2^2 + C_4),$$

$$C_8 = \frac{1}{36}C_2(7C_3^2 + 9C_2C_4)$$

Corresponding charges are not Casimir operators and there cannot be Hamiltonians in integrable systems. Here is the 1 primitive invariant tensor on $SU(2)$, $SO(3)$, $SP(2)$ algebras. The invariant non primitive tensors for $n \geq 2$ are functions of primitive tensor. Let us introduce the local chiral currents based on the invariant symmetric polynomials on $SU(2)$,$SO(3)$,$SP(2)$ Lie group:

$$C_2(U) = \delta_{ab}U^aU^b, C_{2n}(U) = (\delta_{ab}U^aU^b)^n,$$

where $n = 1, 2, ...$

$$\{C_2(x), C_2(y)\} = 2C_2(y)\partial_y\delta(y-x) - 2C_2(x)\partial_x\delta(x-y).$$

$C_2(x)$ is local field on the Riemann space of chiral currents.

As Hamiltonians we choose functions

$$H_n = \frac{1}{2(n+1)} \int_0^{2\pi} C_2^{n+1}(y)dy. \tag{12}$$

The equation of motion for density of first Casimir operator has form:

$$\frac{\partial C_2}{\partial t_n} + (2n+1)(C_2)^n \frac{dC_2}{dx} = 0. \tag{13}$$

The equation for currents C_2^n is following:

$$\frac{\partial C_2^n}{\partial \tau_n} + (C_2)^n \frac{dC_2^n}{dx} = 0, \; \tau_n = (2n+1)t_n.$$

This equation is inviscid Burgers equation.

We will to find the solution in the form:

$$C_2^n(\tau_n, x) = \exp\left(a + i(x - \tau_n C_2^n(\tau_n, x))\right).$$

To obtain solution we rewrite equation of motion in following form: $Y_n = Z_n e^{Z_n}$, where $Y_n = i\tau n e^{(a+ix)}$, $Z_n = i\tau_n C_2^n$. The inverse transformation $Z_n = Z_n(Y_n)$ define by Lambert function:

$$C_2^n = \frac{1}{i\tau_n} W(i\tau_n e^{a+ix}). \qquad (14)$$

Consequently solution for first Casimir operator is:

$$C_2(t_n, x) = [\frac{1}{i(2n+1)t_n} W(i(2n+1)t_n e^{a+ix})]^{\frac{1}{n}}. \qquad (15)$$

The equation of motion for initial chiral current U^μ defined by PB (**??**) and Hamiltonian (12):

$$\frac{\partial U^\mu}{\partial t_n} = \partial_x [U^\mu (UU)^n],\ \mu = 1, 2, 3. \qquad (16)$$

It is possible to rewrite this equation as linear nonhomogeneous equation using solution (15), which diagonalize the equation (16):

$$\frac{\partial z^\mu}{\partial t_n} = \partial_x z^\mu + \partial_x f(t_n, x), \;\; z^\mu = \ln U^\mu, \;\; f = C_2^n.$$

Equation of motion for $SU(3)$ group

The invariant chiral currents $C_2(U)$, $C_3(U)$ form closed system. The nonprimitive currents have the form:

$$C_{2n} = C_2^n, \; C_{2n+1} = C_2^{n-1} C_3 \; .$$

The algebra of corresponding charges is not abelian, but charges C_{2n} form invariant subalgebra. The currents C_2 and C_3 are local coordinates on the Riemann space and invariant currents C_{2n} are densities of Hamiltonians.

Equation of motion for C_3 is following:

$$\frac{\partial C_3(x)}{\partial t_n} = -2C_2^n \partial_x C_3 - 6C_3 \partial_x C_2^n$$

In terms of variables $g = lnC_3$, $f = C_2^n$ it is linear equation

$$\frac{\partial g}{\partial_n} + 2f\partial_x g + 6\partial_x f = 0.$$

Infinite dimensional hydrodynamic chains

In the case, if dimension of matrix representation n is not ended,

the all of chiral currents are primitive currents. This easy to see from expression for new chiral currents $C_{m,k}$.

For example:

$$C_{6,1} = C_6 + \frac{2}{n}C_3^2 - \frac{2}{n}C_2C_4, \quad C_{7,1} = C_7 + \frac{4}{n}C_3C_4 - \frac{4}{n}C_2C_5,$$

$$C_{8,1} = C_8 + \frac{2}{n}C_3C_5 - \frac{2}{n}C_2C_6, \quad C_{8,3} = C_8 + \frac{2}{n}C_4^2 - \frac{2}{n}C_2C_6,$$

$$C_{8,2} = C_8 + \frac{4}{n}C_3C_5 - \frac{4}{n}C_2C_6 - \frac{4}{n^2}C_2C_3^2 + \frac{4}{n^2}C_2^2C_4.$$

The algebra of PB for chiral currents has form:

$$\{C_m(x), C_n(y)\} = W_{mn}(y)\frac{\partial}{\partial y}\delta(y-x) - W_{nm}(x)\frac{\partial}{\partial x}\delta(x-y). \qquad (17)$$

$$W_{mn}(x) = \frac{mn(n-1)}{m+n-2}C_{m+n-2}(x) \qquad (18)$$

This PB satisfies to skew-symmetric condition

$$\{C_m(x), C_n(y)\} = -\{C_n(y), C_m(x)\}$$

. Jacobi identity impose conditions on the Hamiltonian function $W_{mn}(x)$:

$$(W_{kp}+W_{pk})\frac{\partial W_{mn}}{\partial C_k} = (W_{km}+W_{mk})\frac{\partial W_{pn}}{\partial C_k}, \;\; \frac{dW_{kp}}{dx}\frac{\partial W_{nm}}{\partial C_k} = \frac{dW_{km}}{dx}\frac{\partial W_{np}}{\partial C_k}. \qquad (19)$$

The Jacobi identity satisfies for metric tensor $W_{mn}(U)$ (19) for $m = p$ from compatibility condition Kroneckers $\delta_{m+n-2,k}$ and $\delta_{p+n-2,k}$. This PB can be rewritten as PB of hydrodynamic type and describe the hydrodynamic chain (see paper Pavlov and references therein):

$$\{C_m(x), C_n(y)\} = -\frac{mn(n-1)}{m+n-2}\frac{dC_{m+n-2}(x)}{dx}\delta(x-y) - mnC_{m+n-2}(x)\frac{\partial}{\partial x}\delta(x-y). \tag{20}$$

The algebra of charges $\int\limits_0^{2\pi} C_n(x)dx$ is abelian algebra. Let us choose as Hamiltonians the operators Casimir C_n:

$$H_n = \frac{1}{n}\int\limits_0^{2\pi} C_n(x)dx,\ n = 2, 3\dots. \tag{21}$$

The equations of motion for densities of Casimir operators are following:

$$\frac{\partial C_m(x)}{\partial t_n} = \frac{1}{n}\int_0^{2\pi}[W_{mn}(y)\partial_y\delta(y-x) - W_{nm}(x)\partial_x\delta(x-y)]dy = \qquad (22)$$

$$= \frac{m(n-1)}{m+n-2}\partial_x C_{m+n-2}.$$

We can construct equations of motion for initial chiral currents U^μ using flat PB () and Hamiltonians H_n (21), where $C_n(x)$ defined by (4) for $SU(\infty)$ group:

$$\frac{\partial U^\mu(x)}{\partial t_n} = \frac{1}{n}\int_0^{2\pi} dy\{U^\mu(x), C_n(y)\}_0,$$

$$\frac{\partial U_{\underline{\mu}}(x)}{\partial t_n} = \partial_x (d^{k_1}_{\mu_1 \mu_2} d^{k_2}_{k_1 \mu_3} ... d^{k_{n-3}}_{\mu_{n-1} \underline{\mu}} U^{\mu_1}(x) ... U^{\mu_{n-1}}(x)). \qquad (23)$$

As example we consider $n = 3$:

$$\frac{\partial U_\mu}{\partial t_3} = \partial_x (d_{\mu\nu\lambda} U^\nu U^\lambda), \quad \mu = 1, 2, ... \infty.$$

By similar manner we can obtain equation of motion for chiral currents of $SO(n)$, $SP(n)$:

$$\frac{\partial U_{\underline{\mu}}(x)}{\partial t_n} = \partial_x (v^{k_1}_{\mu_1\mu_2\mu_3}...v^{k_{2n-3}}_{\mu_{2n-2}\mu_{2n-1}\underline{\mu}} U^{\mu_1}...U^{\mu_{2n-1}}). \tag{24}$$

As example we consider $n = 4$:

$$\frac{\partial U_\mu}{\partial t_4} = \partial_x (v_{\mu\nu\lambda\rho} U^\nu U^\lambda U^\rho), \;\; \mu = 1, 2, ...\infty.$$

Conclusion

We used non-zero constant torsion in space of string coordinates. However, torsion does not appear in answers because we used total symmetrical invariant currents. Torsion help me to use total symmetric structure constant to construct Hamiltonians.