Lie pseudo-groups and zero-curvature representations of differential equations

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Zero-curvature representations of differential equations

G.M. Kuz'mina. On a possibility to reduce a system of two partial differential equations of the first order to a single equation of the second order. // Proc. Moscow State Pedagogical Institute, 1967, Vol. 271, 67–76 (in Russian)

$$u_{yy} = u_{tx} + u\, u_{xx} + u_x^2 \qquad \text{(dispersionless KP)}$$

Covering

$$\begin{cases} v_t = (v^2 - u)\, v_x - u_y - v\, u_x, \\ v_y = v\, v_x - u_x \end{cases}$$

Excluding u: define w such that $w_x = v$ and $w_y = \frac{1}{2}\, v^2 - u$, then

$$w_{yy} = w_{tx} + \left(\tfrac{1}{2}\, w_x^2 - w_y\right)\, w_{xx} \qquad \text{(modified dKP)}$$

The central idea: to apply Cartan's structure theory of Lie pseudo-groups

Zero-curvature representations of differential equations

- Trivial bundle
  $\pi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad \pi \colon (x^i, u) \mapsto (x^i), \quad i \in \{1, ..., n\}$
- The bundle of infinite jets of sections of π: $J^\infty(\pi)$,
  coordinates $(x^i, u, u_i, u_{ij}, ..., u_I, ...)$, $I = (i_1, i_2, ..., i_m)$
- Total derivatives

$$D_k = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} u_{Ik} \frac{\partial}{\partial u_I}, \qquad [D_i, D_j] = 0$$

- Contact 1-forms

$$\vartheta_I = du_I - u_{Ij}\, dx^j, \qquad d\vartheta_I = dx^j \wedge \vartheta_{Ij}$$

- Differential equation $\mathcal{E}$: $\quad F(x^i, u, u_I) = 0$
- Infinitely prolonged differential equation $\mathcal{E}^\infty \subset J^\infty(\pi)$:
  $D_{k_1} \circ D_{k_2} \circ \cdots \circ D_{k_m}(F) = 0$
- Restricted total derivatives

$$\bar{D}_k = D_k|_{\mathcal{E}^\infty}, \qquad [\bar{D}_i, \bar{D}_j] = 0$$

- Covering over $\mathcal{E}^\infty$:

$$\tau \colon \widetilde{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \mathcal{V} \to \mathcal{E}^\infty, \qquad \mathcal{V} = \{(v^\kappa) \mid 0 \le \kappa \le \infty\}$$

- Extended total derivatives

$$\widetilde{D}_i = \bar{D}_i + \sum_\kappa P_i^\kappa(x^j, u_I, v^\rho)\,\frac{\partial}{\partial v^\kappa},$$

$$[\widetilde{D}_i, \widetilde{D}_j] = 0 \iff (x^i, u_I) \in \mathcal{E}^\infty$$

- Extended contact forms (Wahlquist-Estabrook forms)

$$\widetilde{\vartheta}_0^\kappa = dv^\kappa - P_i^\kappa(x^j, u_I, v^\rho)\,dx^i,$$

$$\widetilde{\vartheta}_I^\kappa = \widetilde{D}_I(\widetilde{\vartheta}_0^\kappa),$$

$$d\widetilde{\vartheta}_I^\kappa \equiv 0 \mod \widetilde{\vartheta}_K^\rho, \vartheta_K, \qquad d\widetilde{\vartheta}_I^\kappa \not\equiv 0 \mod \widetilde{\vartheta}_K^\rho$$

Example: Liouville's equation

$$u_{xy} = e^u$$

- Covering: the fibre coordinate v, extended total derivatives:

$$\widetilde{D}_x = \bar{D}_x + (u_x + e^v)\,\frac{\partial}{\partial v}, \quad \widetilde{D}_y = \bar{D}_y - \frac{1}{2}\,e^{u-v}\,\frac{\partial}{\partial v}$$

- Bäcklund transformation:

$$\begin{cases} v_x = u_x + e^v, \\ v_y = -\frac{1}{2}\,e^{u-v} \end{cases}$$

- Excluding u: Clairin's equation

$$v_{xy} + e^v\,v_y = 0.$$

- Wahlquist–Estabrook form:

$$\widetilde{\vartheta} = dv - (u_x + e^v)\,dx + \frac{1}{2}\,e^{u-v}\,dy$$

Lie pseudo-groups

A pseudo-group $\mathfrak{G}$ on a manifold M is a set of local diffeomorphisms $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$, $\Phi\colon x \mapsto \hat{x}$ such that

1) if $\Phi \in \mathfrak{G}$, $\Psi \in \mathfrak{G}$, and their composition $\Psi \circ \Phi$ is defined, then $\Psi \circ \Phi \in \mathfrak{G}$;
2) $\Phi \in \mathfrak{G} \Rightarrow \Phi^{-1} \in \mathfrak{G}$;
3) $\mathrm{id}_M \in \mathfrak{G}$;
4) if $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$ belongs to $\mathfrak{G}$, then for any open subset $\mathcal{V} \subset \mathcal{U}$ $\Phi|_{\mathcal{V}} \in \mathfrak{G}$;
5) $\Phi_1\colon \mathcal{U}_1 \to \hat{\mathcal{U}}_1$ $\Phi_2\colon \mathcal{U}_2 \to \hat{\mathcal{U}}_2$ belong to $\mathfrak{G}$, and $\Phi_1|_{\mathcal{U}_1\cap\mathcal{U}_2} = \Phi_2|_{\mathcal{U}_1\cap\mathcal{U}_2}$, then $\Phi\colon \mathcal{U}_1 \cup \mathcal{U}_2 \to \hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2$ defined by $\Phi|_{\mathcal{U}_1} = \Phi_1$ and $\Phi|_{\mathcal{U}_2} = \Phi_2$ belongs to $\mathfrak{G}$.

A pseudo-group $\mathfrak{G}$ is called a Lie pseudo-group, if

6) the functions $\hat{x} = \Phi(x)$ are local analytic solutions of a system of PDEs (Lie equations of the pseudo-group $\mathfrak{G}$)

$$R\left(x, \Phi(x), \frac{\partial \Phi(x)}{\partial x}, \ldots, \frac{\partial^{\#I}\Phi(x)}{\partial x^I}\right) = 0.$$

Lie pseudo-groups

EXAMPLE: Lie groups = finite Lie pseudo-groups

EXAMPLE: $\mathrm{SDiff}(\mathbb{R}^2)$:

$\Phi\colon (x, y) \mapsto (\hat{x}, \hat{y})$, $\quad \Phi^*(d\hat{x} \wedge d\hat{y}) = dx \wedge dy$,

Lie equations: $$\frac{\partial \hat{x}}{\partial x}\frac{\partial \hat{y}}{\partial y} - \frac{\partial \hat{x}}{\partial y}\frac{\partial \hat{y}}{\partial x} = 1$$

Lie's infinitesimal method

- Infinitesiaml generators of the Lie pseudo-group $\mathfrak{G}$:

$$\hat{x} = \Phi(x) = x + \varepsilon\,\varphi(x) + \ldots$$

- Infinitesimal defining system (linearized Lie equations):

$$\Phi \in \mathfrak{G} \Longleftrightarrow L\left(x, \varphi(x), \frac{\partial \varphi(x)}{\partial x}, \ldots, \frac{\partial^{\# I} \varphi(x)}{\partial x^{I}}\right) = 0$$

- **Integration**

Cartan's method

PROPOSITION: For a Lie pseudo-group $\mathfrak{G}$ there exist a manifold N, a bundle $P \to M$ such that for each $x \in M$ the fibre of this bundle over x is a finite - dimensional Lie group H_x, $\dim H_x = \mathrm{const}$, and a collection of 1-forms

$$\omega^i \in \Omega^1(P \times N), \qquad i \in \{1, ..., \dim M + \dim N\},$$

such that a local diffeomorphism Φ on M, $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$ belongs to $\mathfrak{G}$ whenever there exists a fibre-preserving diffeomorphism Ψ on $P \times N$, $\Psi\colon \mathcal{W} \to \hat{\mathcal{W}}$, satisfying the following requirements:

- Φ is the projection of Ψ w.r.t. $P \times N \to M$;
- $\Psi^* \left(\omega^i|_{\hat{\mathcal{W}}}\right) = \omega^i|_{\mathcal{W}}$.

DEFINITION: The forms ω^i from the proposition are called Maurer–Cartan forms of the Lie pseudo-group $\mathfrak{G}$.

Structure equations of a Lie pseudo-group $\mathfrak{G}$:

$$d\omega^i = A^i_{\alpha j}(U^\sigma)\,\pi^\alpha \wedge \omega^j + B^i_{jk}(U^\sigma)\,\omega^j \wedge \omega^k, \qquad B^i_{jk} = -B^i_{kj},$$

$$dU^\kappa = C^\kappa_j(U^\sigma)\,\omega^j,$$

$U^\sigma \colon M \to \mathbb{R}$, $\sigma \in \{1, \dots, s\}$, $s < \dim M$, — invariants of the pseudo-group $\mathfrak{G}$,

$$\Phi^*\left(U^\kappa|_{\hat{\mathfrak{U}}}\right) = U^\kappa|_{\mathfrak{U}},$$

- π^α — depend on differentials of coordinates on H_x;
- involutivity conditions are satisfied,
- compatibility conditions are satisfied.

Involutivity conditions:

$$r^{(1)} = n \, \dim \, H_x - \sum_{k=1}^{n-1} (n-k) \, \sigma_k,$$

where $n = \dim M + \dim N$, $r^{(1)}$ is the dimension of the linear space of coefficients z^{α}_{j} such that the replacement $\pi^{\alpha} \mapsto \pi^{\alpha} + z^{\alpha}_{j} \, \omega^{j}$ preserves the structure equations;

$$\sigma_k = \max_{u_1, \dots, u_k} \, \mathrm{rank} \, \mathbb{A}_k(u_1, \dots, u_k) - \sum_{j=1}^{k-1} \sigma_j,$$

$$\mathbb{A}_1(u_1) = \left(A^{i}_{\alpha j} \, u^{j}_{1} \right),$$

$$\mathbb{A}_q(u_1, \dots, u_q) = \left(\begin{array}{c} \mathbb{A}_{q-1}(u_1, \dots, u_{q-1}) \\ A^{i}_{\alpha j} \, u^{j}_{q} \end{array} \right), \; q \in \{2, \dots, n-1\}.$$

Compatibility conditions:

- $d(d\omega^i) = 0 = d\left(A^i_{\alpha j}\,\pi^\alpha \wedge \omega^j + B^i_{jk}\,\omega^j \wedge \omega^k\right)$
- $d(dU^\kappa) = 0 = d(C^\kappa_j\,\omega^j)$

$\Longrightarrow$

- over-determined system for the coefficients $A^i_{\alpha j}$, B^i_{jk}, C^κ_j;
- $d\pi^\alpha = W^\alpha_{\lambda j}\,\chi^\lambda \wedge \omega^j + X^\alpha_{\beta\gamma}\,\pi^\beta \wedge \pi^\gamma + Y^\alpha_{\beta j}\,\pi^\beta \wedge \omega^j + Z^\alpha_{jk}\,\omega^j \wedge \omega^k.$

THEOREM (*Third fundamental Lie's theorem in Cartan's form*): For a Lie pseudo-group there exists a collection of Maurer–Cartan forms with involutive and compatible structure equations.

THEOREM (*Third inverse fundamental Lie's theorem in Cartan's form*): For a given involutive and compatible system of structure equations there exists a collection of 1-forms ω^1, ..., ω^n and functions U^1, ..., U^s satisfying this system. The forms ω^1, ..., ω^m are Maurer–Cartan forms of a Lie pseudo-group, and the functions U^1, ..., U^s are invariants of this pseudo-group.

- Cartan É. Œuvres Complètes. Paris: Gauthier - Villars, 1953
- Vasil'eva M.V. Structure of Infinite Lie Groups of Transformations. Moscow: MSPI, 1972 (in Russian)
- Stormark O. Lie's Structural Approach to PDE Systems. Cambridge: CUP, 2000

Contact transformations

- Trivial bundle: $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\pi\colon (x^i, u) \mapsto (x^i)$
- the second order jets: $J^2(\pi)$, (x^i, u, u_i, u_{ij}), $u_{ij} = u_{ji}$
- contact forms : the ideal $\mathcal{C} =< \vartheta_0, \vartheta_i >$ generated by the forms $\vartheta_0 = du - u_j\, dx^j$, $\vartheta_i = du_i - u_{ij}\, dx^j$.
- The pseudo-group of contact transformations $\mathrm{Cont}(J^2(\pi))$: local diffeomorphisms $\Psi\colon J^2(\pi) \to J^2(\pi)$ such that $\Psi^*\mathcal{C} \subset \mathcal{C}$.

Maurer–Cartan forms of the pseudo-group $\mathrm{Cont}(J^2(\pi))$:

$$\Theta_0 = a\,(du - u_i\,dx^i),$$
$$\Theta_i = a\,B_i^j\,(du_j - u_{jk}\,dx^k) + g_i\,\Theta_0,$$
$$\Theta_{ij} = a\,B_i^k\,B_j^l\,(du_{kl} - u_{klm}\,dx^m) + s_{ij}\,\Theta_0 + w_{ij}^k\,\Theta_k,$$
$$\Xi^i = b_j^i\,dx^j + c^i\,\Theta_0 + f^{ij}\,\Theta_j,$$

where $\quad b_k^i\,B_j^k = \delta_j^i, \quad f^{ik} = f^{ki},\ s_{ij} = s_{ji}, \quad w_{ij}^k = w_{ji}^k,$

$u_{klm} = u_{lkm} = u_{kml}.$

Structure equations

$$d\Theta_0 = \Phi_0^0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,$$
$$d\Theta_i = \Phi_i^0 \wedge \Theta_0 + \Phi_i^k \wedge \Theta_k + \Xi^k \wedge \Theta_{ik},$$
$$d\Theta_{ij} = \Phi_i^k \wedge \Theta_{kj} - \Phi_0^0 \wedge \Theta_{ij} + \Upsilon_{ij}^0 \wedge \Theta_0 + \Upsilon_{ij}^k \wedge \Theta_k + \Xi^k \wedge \Theta_{ijk},$$
$$d\Xi^i = \Phi_0^0 \wedge \Xi^i - \Phi_k^i \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k$$

Symmetry pseudo-groups of differential equations

- A PDE of the second order: $\iota\colon \mathcal{E} \to J^2(\pi)$
- Contact symmtries of $\mathcal{E}$ — contact transformations that map $\mathcal{E}$ into itself: $\mathrm{Cont}(\mathcal{E}) \subset \mathrm{Cont}(J^2(\pi))$,
- **Maurer-Cartan forms of the pseudo-group $\mathrm{Cont}(\mathcal{E})$ can be found from the restrictions $\theta_0 = \iota^*\,\Theta_0$, $\theta_i = \iota^*\,\Theta_i$, $\theta_{ij} = \iota^*\,\Theta_{ij}$, $\xi^i = \iota^*\,\Xi^i$ of Maurer-Cartan forms of the pseudo-group $\mathrm{Cont}(J^2(\pi))$** on $\mathcal{E}$ **algorithmically by means of Cartan's method of equivalence**
- Details:
    - M. Fels, P.J. Olver., Moving coframes I. A practical algorithm. // Acta Appl. Math., 1998, Vol. 51, pp. 161–213
    - M., Moving coframes and symmetries of differential equations. // J. Phys. A: Math. Gen., 2002, Vol. 35, pp. 2965 – 2977

EXAMPLE: Liouville's equation $u_{xy} = e^u$

Maurer–Cartan forms of the symmetry pseudo-group:

$$
\begin{aligned}
&\theta_0 = du - u_x dx - u_y dy,\\
&\theta_1 = q^{-1}\,(du_x - u_{xx} dx - e^u dy),\\
&\theta_2 = q\,e^{-u}\,(du_y - e^u dx - u_{yy} dy),\\
&\theta_{11} = q^{-2}\,\big(du_{xx} - u_x du_x + (u_x u_{xx} + q^3 r_1)\,dx\big),\\
&\theta_{22} = q^2\,e^{-2u}\,\big(du_{yy} - u_y du_y + (u_y u_{yy} + e^{3u} q^{-3} r_2) dy\big),\\
&\xi^1 = q\,dx,\\
&\xi^2 = q^{-1} e^u dy,\\
&\eta_1 = q^{-1}\,(dq - u_x\,\xi^1),\\
&\eta_2 = dr_1 - 3\,r_1\,\eta_1 + q^{-2}(u_{xx} + u_x^2)\,(\theta_1 + \xi^2) + 3\,q^{-1} u_x\,\theta_{11} + r_3\,\xi^1,\\
&\eta_3 = dr_2 + 3r_2(\eta_1 + \theta_0) + \tfrac{q^2}{e^{2u}}(u_{yy} + u_y^2)(\theta_2 + \xi^1) + \tfrac{3q}{e^u} u_y \theta_{22} + r_4\,\xi^2.
\end{aligned}
$$

- the emphasized Maurer-Cartan forms:

$$\eta_1 = \frac{dq}{q} - u_x\,dx, \quad \xi^1 = q\,dx, \quad \xi^2 = \frac{e^u}{q}\,dy,$$

- denote $q = e^v$, take the linear combination:

$$\widetilde{\vartheta} = \eta_1 - \xi^1 + \frac{1}{2}\,\xi^2 = dv - (u_x + e^v)\,dx + \frac{1}{2}\,e^{u-v}\,dy$$

- **This is the Wahlquist–Estabrook form of the aforementioned covering**

Contact integrable extensions of symmetry pseudo-groups

Bryant R.L., Griffiths P.A. Characteristic Cohomology of Differential Systems (II): Conservation Laws for a Class of Parabolic Equations // Duke Math. J., **78**, 531–676 (1995):

$n = 2$, finite-dimensional coverings

Definition 1. Let

$$d\omega^i = A^i_{\alpha j}\,\pi^\alpha \wedge \omega^j + B^i_{jk}\,\omega^j \wedge \omega^k, \tag{1}$$

$$dU^\kappa = C^\kappa_j\,\omega^j \tag{2}$$

be structure equations of a Lie pseudo-group $\mathfrak{G}$. Its coefficients are supposed to be functions of the invariants U^σ of $\mathfrak{G}$. Consider the system

$$\begin{aligned} d\tau^q = {} & D^q_{\rho r}\,\eta^\rho \wedge \tau^r + E^q_{rs}\,\tau^r \wedge \tau^s + F^q_{r\beta}\,\tau^r \wedge \pi^\beta \\ & + G^q_{rj}\,\tau^r \wedge \omega^j + H^q_{\beta j}\,\pi^\beta \wedge \omega^j + I^q_{jk}\,\omega^j \wedge \omega^k, \end{aligned} \tag{3}$$

$$dV^\epsilon = J^\epsilon_j\,\omega^j + K^\epsilon_q\,\tau^q, \tag{4}$$

with unknown 1-forms τ^q, $q \in \{1, ..., Q\}$, η^ρ, $\rho \in \{1, ..., R\}$, and unknown functions V^ϵ, $\epsilon \in \{1, ..., S\}$, $Q, R, S \in \mathbb{N}$. The coefficients of this system are supposed to be functions of U^σ and V^ϵ.

System (3), (4) is called an integrable extension of system (1), (2), if equations (1) – (4) are simultaneously compatible and involutive.

Suppose system (3), (4) is an integrable extension of system (1), (2). Then, in accordance with the third inverse fundamental theorem of Lie, system (1)–(4) defines a Lie pseudo-group $\mathfrak{H}$.

Definition 2. The integrable extension (3), (4) is called trivial , if there exists a change of variables on the manifold of action of the pseudo-group $\mathfrak{H}$ such that in the new variables equations (3), (4) do not contain the forms ω^j, π^β, and the coefficients of (3), (4) do not depend on U^q. Otherwise, the integrable extension is called non-trivial .

Let θ^α_K, ξ^j be Maurer–Cartan forms of the pseudo-group $\mathrm{Cont}(\mathcal{E})$ of symmetries for a PDE $\mathcal{E}$ such that θ^α_K are contact forms (their restrictions on each solution of the equation $\mathcal{E}$ are equal to 0), and ξ^j are horizontal forms ($\xi^1 \wedge \ldots \wedge \xi^n \neq 0$ on each solution).

Definition 3. Nontrivial integrable extension of the structure equations of the pseudo-group $\mathrm{Cont}(\mathcal{E})$

$$d\omega^q = \Pi^q_r \wedge \omega^r + \xi^j \wedge \Omega^q_j \tag{5}$$

is called contact integrable extension when

- $\Omega^q_j \equiv 0 \pmod{\theta^\alpha_K, \omega^q_j}$ for a set of additional forms ω^q_j;
- $\Omega^q_j \not\equiv 0 \pmod{\omega^q_j}$;
- coefficients of Ω^q_j and Π^q_r depend on the invariants of $\mathrm{Cont}(\mathcal{E})$ and, maybe, on a set of additional functions W^ρ;
- In the latter case there exist functions $P^{I\rho}_\alpha$, Q^ρ_q, $R^{j\rho}_q$, S^ρ_j such that

  $$\zeta^\rho = dW^\rho - P^{I\rho}_\alpha\,\theta^\alpha_I - Q^\rho_q\,\omega^q - R^{j\rho}_q\,\omega^q_j - S^\rho_j\,\xi^j = 0.$$

  The structure equations for the forms ζ^ρ together with the structure equations of $\mathrm{Cont}(\mathcal{E})$ and equations (5) satisfy the involutivity conditions.

r-mdKP equation

r^{th} modified dispersionless KP equation (M. Błaszak, 2002):

$$u_{yy} = u_{tx} + \left(\frac{1}{2}\,(\kappa + 1)\,u_x^2 + u_y\right)\,u_{xx} + \kappa\,u_x\,u_{xy}$$

Special cases:

- $\kappa = 0$: mdKP,
  G.M. Kuz'mina, 1967; I.M. Krichever, 1988;
  B.A. Kupershmidt, 1990.
- $\kappa = 1$: dBKP,
  N. Dasgupta, R. Chowdhury, 1992; K. Takasaki, 1993;
  B.G. Konopelchenko, L. Martinez Alonso, 2003.
- $\kappa = -1$:
  M.V. Pavlov, 2003; M. Dunajski, 2004; E.V. Ferapontov, K.R. Khusnutdinova, 2004; V.Yu. Ovsienko, C. Roger, 2006

Contact integrable extensions:

$\kappa \notin \{-3, -1\}$:

First type:

$$
\begin{aligned}
d\omega_0 = \Big(&\omega_1 + \tfrac{1}{2}\,(\eta_1 + \theta_{22}) + \tfrac{1}{16}\,(8\,V + \kappa^2 + 13\,\kappa + 12)\,(\kappa+1)^{-1}\,\xi^3 \\
&+\tfrac{1}{2}\,((\kappa+1)^2\,U - V^2 - 2\,(\kappa^2 + 3\,\kappa + 2)\,V)\,(\kappa+1)^{-2}\,\xi^1\Big)\wedge\omega_0 \\
&+ \Big(\theta_3 - V\,(\kappa+1)^{-1}\,\theta_2 - \tfrac{1}{8}\,(\kappa-4)\,\theta_0 + (\kappa+1)^{-2}\,V^2\,\omega_1\Big)\wedge\xi^1 \\
&+\omega_1\wedge\xi^2+(\theta_2-V\,(\kappa+1)^{-1}\,\omega_1)\wedge\xi^3.
\end{aligned}
$$

Second type

$$
\begin{aligned}
d\omega_0 = \Big(&\omega_1 + \tfrac{1}{2}\,(\theta_{22} + \eta_1 + (U - W^2 + 2\,(W - V))\,\xi^1) - \tfrac{1}{16}\,(8\,W - \kappa + 12)\,\xi^3\Big) \\
&+ \Big(W^2\,\omega_1 + W\,\theta_2 - \tfrac{1}{8}\,(\kappa-4)\,\theta_0 + \theta_3\Big)\wedge\xi^1+\omega_1\wedge\xi^2+(W\,\omega_1+\theta_2)\wedge\xi^3, \\
dW = \tfrac{1}{2}\,&(2\,(Z_1+1) - \kappa\,(W-1) - V)\,\omega_0 - (V + (\kappa+1)\,W)\,\omega_1 + \eta_2 - \theta_2 \\
&+\tfrac{1}{2}\,W\,(\eta_1-\theta_{22})+\tfrac{1}{2}\,W\,(2\,W\,Z_1-U+4\,V+W\,(W-\kappa))\,\xi^1+Z_1\,\xi^2 \\
&+\tfrac{1}{16}\,(W\,(16\,Z_1+8\,W-7\,\kappa-4)+16\,V)\,\xi^3.
\end{aligned}
$$

$\kappa = -3$: additional CIE

$$
\begin{aligned}
d\omega_0 = {} & \left(\tfrac{1}{2}\left(\theta_{22} + \eta_1\right) + \tfrac{1}{8}\left(4\,U - V\left(V + 8\,W - 11\right)\right)\xi^1 + \omega_1 \right. \\
& \left. - \tfrac{1}{16}\left(4\,V + 16\,W + 5\right)\xi^3\right)\wedge\omega_0 + \left(\theta_3 - W\,\theta_0 + \tfrac{1}{2}\,V\,\theta_2 + \tfrac{1}{4}\,V^2\,\omega_1\right)\wedge\xi^1 \\
& + \omega_1\wedge\xi^2 + \left(\theta_2 + \tfrac{1}{2}\,V\,\omega_1\right)\wedge\xi^3 \\
dW = {} & \left(Z + \tfrac{3}{2}\,W + \tfrac{21}{16}\right)\,\omega_0 + \left(W + \tfrac{7}{8}\right)\,\left(\omega_1 - \theta_{22}\right) + Z\,\xi^2 \\
& - \left(U\,\left(W + \tfrac{7}{8}\right) + V\,\left(W^2 - \tfrac{203}{64}\right) - \tfrac{1}{32}\,V^2\left(8\,Z + 7\right)\right)\,\xi^1 \\
& + \tfrac{1}{64}\,\left(32\,V\left(Z + W\right) - 16\,W\left(4\,W + 3\right) + 7\left(4\,V + 1\right)\right)\,\xi^3.
\end{aligned}
$$

r-mdKP equation

$\kappa = -1$: the only CIE of the second type

$$d\omega_0 = \left(\tfrac{1}{2}\,(\theta_{22} + \eta_1 + (U - W^2 + 2\,W)\,\xi^1) - \tfrac{1}{16}\,(8\,W - 11)\,\xi^3 + \omega_1\right)\wedge\omega_0$$
$$+\left(\theta_3 + \tfrac{5}{8}\,\theta_0 + W\,\theta_2 + W^2\,\omega_1\right)\wedge\xi^1 + \omega_1\wedge\xi^2 + (\theta_2 + W\,\omega_1)\wedge\xi^3),$$
$$dW = \tfrac{1}{2}(2\,Z+W+1)\,\omega_0 - \theta_2 - \tfrac{1}{2}\,W\,(\theta_{22}+\eta_1)+\eta_2+\tfrac{1}{16}\,W\,(16\,Z+8\,W+3)\,\xi^3$$
$$+Z\,\xi^2 + \tfrac{1}{2}\,(W^2\,(2\,Z+1) + W\,(W^2 - U))\,\xi^1$$

REMARK: $2\,Z + W + 1 = 0 \Longrightarrow$ nonremovable parameter

r-mdKP equation

- $\kappa \notin \{-2, -\frac{3}{2}, -1\}$:

$$\begin{cases} v_t = \left(\frac{1}{2\kappa+3}\, v_x^{2\,(\kappa+1)} - u_x\, v_x^{\kappa+1} + \left(\frac{1}{2}\,(\kappa+1)\, u_x^2 - u_y\right)\right)\, v_x, \\ v_y = \left(\frac{1}{\kappa+2}\, v_x^{\kappa+1} - u_x\right)\, v_x. \end{cases}$$

  - $\kappa = 0$: J.-H. Chang, M.-H. Tu, 2000;
  - $\kappa = 1$: B.G. Konopelchenko, L. Martinez Alonso, 2003.
  - M.V. Pavlov, 2006

$$v_{yy} = v_{tx} - \kappa\,(v_y\, v_x^{-1} + v_x^{\kappa})\, v_{xy}$$
$$+((\kappa+1)\, v_y^2\, v_x^{-2} - v_t\, v_x^{-1} + \kappa\, v_x^{\kappa}\, v_y + \frac{(\kappa+1)^2}{2\kappa+3}\, v_x^{2(\kappa+1)})\, v_{xx}$$

- $\kappa \in \mathbb{R}$:

$$\begin{cases} v_t = \left(\frac{1}{2}\,(\kappa+1)\, u_x^2 - u_y\right)\, v_x, \\ v_y = -u_x\, v_x. \end{cases}$$

$$v_{yy} = v_{tx} + ((\kappa+1)\, v_y^2\, v_x^{-2} - v_t\, v_x^{-1})\, v_{xx} - \kappa\, v_y\, v_x^{-1}\, v_{xy},$$

r-mdKP equation

- $\kappa = -2$:

$$\begin{cases} v_t = -v_x^{-1} - u_x - \left(\frac{1}{2}\, u_x^2 + u_y\right)\, v_x, \\ v_y = \ln|v_x| - u_x\, v_x \end{cases}$$

$$\begin{aligned} v_{yy} &= v_{tx} + 2\,(\ln|v_x| + v_y)\, v_x^{-1}\, v_{xy} \\ &-(v_t v_x + \ln|v_x|\,(\ln|v_x| - 2\, v_y + 1) + v_y^2 - v_y + 1)\, v_x^{-2}\, v_{xx} \end{aligned}$$

- $\kappa = -\frac{3}{2}$:

$$\begin{cases} v_t = -\left(\frac{1}{4}\, u_x^2 + u_y\right)\, v_x - u_x\sqrt{|v_x|} + \ln|v_x|, \\ v_y = -u_x\, v_x + 2\sqrt{|v_x|} \end{cases}$$

$$\begin{aligned} v_{yy} &= v_{tx} + \tfrac{3}{2}\,(v_y - 2\,|v_x|^{1/2})\, v_x^{-1}\, v_{xy} \\ &\quad +(v_x\,(\ln|v_x| - v_t - \tfrac{1}{2}\, v_y^2) + 3\, v_y\, |v_x|^{1/2} - 4)\, v_x^{-2}\, v_{xx} \end{aligned}$$

r-mdKP equation

- $\kappa = -3$:

$$\begin{cases} v_t = -u - (u_x^2 + u_y)\, v_x, \\ v_y = -v_x\, u_x - x \end{cases}$$

$$u = (v_y + v_x\, v_{tx} + 3\,(v_y + x)\, v_{xy} + x - v_x\, v_{yy})v_{xx}^{-1} - v_t \\ -2\,(v_y + x)^2\, v_x^{-1}$$

- $\kappa = -1$:

$$\begin{cases} v_t = -\left(u_y - \lambda\, u_x - \lambda^2\right)\, v_x, \\ v_y = -(u_x - \lambda)\, v_x, \end{cases}$$

$\lambda \in \mathbb{R}$ — nonremovable parameter
M. Dunajski, 2004

$$v_{yy} = v_{tx} - (v_t + \lambda\, v_y)\, v_x^{-1}\, v_{xx} + (v_y + \lambda\, v_x)\, v_x^{-1}\, v_{xy}$$

- $\kappa = -1$:

$$\begin{cases} v_t = -\left(u_y - v\, u_x - v^2\right)\, v_x, \\ v_y = -(u_x - v)\, v_x, \end{cases}$$

$$v_{yy} = v_{tx} - (v_t + v\, v_y)\, v_x^{-1}\, v_{xx} + (v_y + v\, v_x)\, v_x^{-1}\, v_{xy}$$

r-mdKP equation

$\kappa = 0$:

$$\begin{tikzcd}
\mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[dl, leftrightarrow] \\
& \mathcal{E}_1 \arrow[d] & \\
& \mathcal{E}_0 &
\end{tikzcd}$$

$\kappa \notin \{-3, 0\}$:

$$\begin{tikzcd}
\mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[dl, leftrightarrow] \\
& \mathcal{E}_1 &
\end{tikzcd}$$

$\kappa = -3$:

$$\begin{tikzcd}
\mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[dl, leftrightarrow] \\
& \mathcal{E}_1 & \mathcal{E}_4 \arrow[l]
\end{tikzcd}$$

$$
\begin{tikzcd}
 & ? \arrow[d, dotted] & & ? \arrow[d, dotted] & \\
? \arrow[r, dotted] & \mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[dl, leftrightarrow] & ? \arrow[l, dotted] \\
 & & \mathcal{E}_1 & &
\end{tikzcd}
$$

$$\mathcal{E}_1\colon \qquad u_{yy} = u_{tx} + \left(\tfrac{1}{2}\,(\kappa+1)\,u_x^2 + u_y\right)\,u_{xx} + \kappa\,u_x\,u_{xy}$$

$$\mathcal{E}_2\colon \qquad w_{yy} = w_{tx} + ((\kappa+1)\,w_y^2\,w_x^{-2} - w_t\,w_x^{-1})\,w_{xx} - \kappa\,w_y\,w_x^{-1}\,w_{xy},$$

$$\mathcal{E}_3\colon \qquad v_{yy} = v_{tx} - \kappa\,(v_y\,v_x^{-1} + v_x^{\kappa})\,v_{xy} + ((\kappa+1)\,v_y^2\,v_x^{-2} - v_t\,v_x^{-1} + \kappa\,v_x^{\kappa}\,v_y + \tfrac{(\kappa+1)^2}{2\kappa+3}\,v_x^{2(\kappa+1)})\,v_{xx}$$

$\mathcal{E}_3$: double-modified dKP equation, M.V. Pavlov, 2006, 2010

$\kappa \notin \{-2, -3/2, -1\}$

$$u_{yy} = u_{tx} - \kappa \left(\frac{u_y}{u_x} + u_x^{\kappa}\right) u_{xy}$$
$$+ \left((\kappa+1)\,\frac{u_y^2}{u_x^2} - \frac{u_t}{u_x} + \kappa\, u_x^{\kappa}\, u_y + \frac{(\kappa+1)^2}{2\kappa+3}\, u_x^{2(\kappa+1)}\right) u_{xx}$$

Three contact integrable extensions:

1) $\mathcal{E}_2 \to \mathcal{E}_3$

2)

$$v_t = \frac{(\kappa+2)^2}{2\,\kappa+3}\,v_x^{2\,\kappa+3} - (\kappa+2)\,\left(\frac{u_y}{u_x} + u_x^{\kappa+1}\right)\,v_x^{\kappa+2} + \left(\frac{u_t}{u_x} + (\kappa+2)\,u_x^{\kappa}u_y + \frac{(\kappa+1)(\kappa+2)}{2\,\kappa+3}\,u_x^{2\,\kappa+2}\right)\,v_x$$

$$v_y = -v_x^{\kappa+2} + \left(\frac{u_y}{u_x} + u_x^{\kappa+1}\right)\,v_x$$

Exclude $u \Longrightarrow$ the same equation for v.

r-mdKP equation

3)

$$w_t = \left(\frac{u_t}{u_x} - (\kappa+1)(\kappa+2)\, u_x^{\kappa} \left(u_y - \frac{u_x^{\kappa+2}}{2\,\kappa+3}\right)\right) w_x$$

$$w_y = \left(\frac{u_y}{u_x} - (\kappa+1)\, u_x^{\kappa+1}\right) w_x$$

$\kappa \notin \{-2, -3/2, -1, 0\}$:

Exclude $u \Longrightarrow$ an equation of the third order for w

$\kappa = 0 \Longrightarrow \mathcal{E}_2 \to \mathcal{E}_3$

Details: `arXiv: 1010.1828`

r-mdKP equation

$\kappa = 0$:

$$\begin{tikzcd}
\mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[r, leftrightarrow] \arrow[dl, leftrightarrow] & \mathcal{E}_3 \\
& \mathcal{E}_1 \arrow[d] & & \\
& \mathcal{E}_0 & &
\end{tikzcd}$$

r-mdKP equation

$\kappa \notin \{-3, -2, -3/2, -1, 0\}$:

$$
\begin{tikzcd}
 & & \mathcal{E}_5 \arrow[d] & \\
\mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[r, leftrightarrow] \arrow[dl, leftrightarrow] & \mathcal{E}_3 \\
 & \mathcal{E}_1 & &
\end{tikzcd}
$$

r-mdKP equation

$\kappa = -3$:

$$
\begin{tikzcd}
 & & \mathcal{E}_5 \arrow[d] & \\
\mathcal{E}_2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{E}_3 \arrow[r, leftrightarrow] \arrow[dl, leftrightarrow] & \mathcal{E}_3 \\
 & \mathcal{E}_1 & \mathcal{E}_4 \arrow[l] &
\end{tikzcd}
$$

Recursion operators

- Recursion operators as integro-differential operators
  - P.J. Olver, 1977
  - B. Fuchssteiner, 1979
  - V.E. Zakharov, V.G. Konopelchenko, 1984
  - A.S. Fokas. P.M. Santini, 1986
  - . . .
  - J.A. Sanders, J.P. Wang, 2001
- Recursion operators as Bäcklund autotransformations
  - C.J. Papachristou, B.K. Harrison, 1988
  - C.J. Papachristou, 1990
  - G.A. Guthrie, 1994
  - I.S. Krasil′shchik, P.H.M. Kersten, 1994
  - M. Marvan, 1996
  - A. Sergyeyev, 2000
  - . . .
  - C.J. Papachristou, B.K. Harrison, 2010
  - I.S. Krasil′shchik, A.M. Verbovetsky, R. Vitolo, 2011
  - M. Marvan, A. Sergyeyev, 2012

EXAMPLE. The universal hierarchy equation:

$$u_{yy} = u_y\, u_{tx} - u_x\, u_{ty} \tag{1}$$

- Pavlov M.V. J. Math. Phys. **44** (2003), 4134–4156
- Martínez Alonso L., Shabat A.B. Phys. Lett. A **299** (2002), 359–365; Theor. Math. Phys. **140** (2004), 1073–1085

The linearization:

$$v_{yy} = u_y\, v_{tx} + v_y\, u_{tx} - u_x\, v_{ty} - v_x\, u_{ty}. \tag{2}$$

The tangent covering: the infinite prolongation of system (1), (2) equipped with the restrictions of the extended total derivatives.

- **A section $\varphi\colon \mathcal{E} \to \mathcal{T}(\mathcal{E})$ of the tangent covering is a symmetry of $\mathcal{E}$.**
- **A recursion operator for symmetries of $\mathcal{E}$ is a Bäcklund autotransformation of $\mathcal{T}(\mathcal{E})$**

Details and examples:

- M. Marvan // Differential Geometry and Applications. Brno: Masaryk University, 1996, 393–402
- I.S. Krasil′shchik, A.M. Verbovetsky, R. Vitolo, `arXiv.org/1110.4560`
- M. Marvan, A. Sergyeyev. Inverse problems, **28** (2012), 025011 (12 pp)

The technique of the contact integrable extensions can be adapted to the problem of finding recursion operators for symmetries of PDEs.

Equations

$$\begin{cases} D_x(\psi) & = & u_x\, D_t(\varphi) - u_{tx}\, \varphi + D_y(\varphi), \\ D_y(\psi) & = & u_y\, D_t(\varphi) - u_{ty}\, \varphi. \end{cases}$$

define a recursion operator $\psi = \mathcal{R}(\varphi)$ for symmetries of the universal hierarchy equation. The inverse recursion operator $\varphi = \mathcal{R}^{-1}(\psi)$ is defined by equations

$$\begin{cases} D_t(\varphi) & = & \dfrac{D_y(\psi) + u_{ty}\, \varphi}{u_y}, \\ D_y(\varphi) & = & \dfrac{(u_y u_{tx} - u_x u_{ty})\, \varphi + u_y\, D_x(\psi) - u_x\, D_y(\psi)}{u_y}. \end{cases}$$

Details: `arXiv.org/1205.5748`

Contact symmetries of the universal hierarchy equation:

$$\varphi_{10}(A_0) = A_0\, u_t - A_{0,t}\, u, \qquad \varphi_{11}(A_1) = A_1, \qquad A_i = A_i(t),$$
$$\varphi_{20}(B_0) = B_0\, u_x + y\, B_{0,x}\, u, \quad \varphi_{21}(B_1) = B_1\, u_y, \quad B_i = B_i(x),$$
$$\varphi_3 = y\, u_y + u.$$

$$\mathcal{R}(\varphi_{10}(A_0)) = A_0(s_{1,t} - u_t^2) - A_{0,t}(s_1 - u\, u_t) + \tfrac{1}{2}\, A_{0,tt} u^2 + \varphi_{11}(A_2),$$
$$\mathcal{R}(\varphi_{11}(A_1)) = -\varphi_{10}(A_1) + \varphi_{11}(A_2),$$
$$\mathcal{R}(\varphi_{20}(B_0)) = \varphi_{20}(B_0) + \varphi_{11}(A_2),$$
$$\mathcal{R}(\varphi_{21}(B_1)) = \varphi_{11}(A_2),$$
$$\mathcal{R}(\varphi_3) = 2\, s_1 + u\, u_t,$$

where

$$\left\{ \begin{array}{lcl} s_{1,x} & = & u_y + u_t u_x, \\ s_{1,y} & = & u_t u_y. \end{array} \right.$$

$$\mathcal{R}^{-1}(\varphi_{10}(A_0)) = -\varphi_{11}(A_0) + \varphi_{21}(B_2),$$
$$\mathcal{R}^{-1}(\varphi_{11}(A_1)) = \varphi_{21}(B_2),$$
$$\mathcal{R}^{-1}(\varphi_{20}(B_0)) = u_y\, s_2 + y\, u_x B_{0,x} + \tfrac{1}{2}\, y^2 u_y B_{0,xx} + \varphi_{21}(B_2),$$
$$\mathcal{R}^{-1}(\varphi_{21}(B_1)) = \varphi_{20}(B_1) + \varphi_{21}(B_2),$$
$$\mathcal{R}^{-1}(\varphi_3) = 2\, u_y\, s_3 + y\, u_x,$$

where

$$\left\{ \begin{array}{lcl} s_{2,t} & = & (B_0\, u_{xy} + B_{0,x}\, u_y)\, u_y^{-2}, \\ s_{2,y} & = & (B_0\, (u_x u_{xy} - u_y u_{xx}) - B_{0,x} u_x u_y)\, u_y^{-2} \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} s_{3,t} & = & u_y^{-1}, \\ s_{3,y} & = & -u_x u_y^{-1} \end{array} \right.$$