On Osserman condition in pseudo-Riemannian geometry

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- $M = (M, g \equiv \langle \cdot\, , \cdot \rangle)$ be a pseudo-Riemannian manifold

- ∇ its Levi-Civita connection,

- $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ its Riemannian curvature operator, which satisfy the standard symmetries,

$$R(X, Y) + R(Y, X) = 0,$$
$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$
$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle.$$

- The Jacobi operator, $R_X : Y \mapsto R(Y, X)X$, is a self-adjoint (symmetric) endomorphism of the tangent bundle TM.

- In Riemannian case, if M is flat or a rank one symmetric space, then local isometries of M act transitively on the unit sphere bundle, and thus the eigenvalues of Jacobi operator are constant on SM.

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• In Riemannian case, if M is flat or a rank one symmetric space, then local isometries of M act transitively on the unit sphere bundle, and thus the eigenvalues of Jacobi operator are constant on $\mathcal{S}M$.

- **Osserman conjecture.** Osserman wondered if the converse is also true: are globally (i.e. the eigenvalues do not depend on the point p) Osserman manifolds necessarily flat or locally isometric to a rank one symmetric spaces.

- Osserman conjecture have been starting point for investigations of manifolds based on properties of the spectra of natural operators defined by the Riemannian curvature tensor.

- A Riemannian manifold (M, g) is **pointwise Osserman** *if the eigenvalues of the Jacobi operator* $R_X(\cdot) = R(\cdot, X)X$ *do not depend on the unit vector* $X \in T_pM$*, for every point* $p \in M$ *(the eigenvalues may vary from point to point).*

- The natural approach to the Osserman conjecture:

1. using the pointwise Osserman condition find the algebraic curvature tensor at a point $p \in M$ and so possible existence of an additional algebraic structure (Clifford structure, in Riemannian case)
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• Notions of pointwise and globally Osserman conditions are not equivalent, for example in dimension 4: K3-surface (compact), quaternionic Kahler orbifolds (local, non-zero scalar curvature), hyperKaehler examples.

• Let $\mathcal{S}^+M$ and $\mathcal{S}^-M$ be the unit pseudo-sphere bundles of spacelike and timelike vectors, $\mathcal{S}M = \mathcal{S}^+M \cup \mathcal{S}^-M$.

• M is (globally) spacelike, timelike or null Osserman if the characteristic polynomial of R_X is constant for $X \in S^+M$, $X \in \mathcal{S}^-M$, or $X \in \mathcal{N}M$, respectively.

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1.2 Algebraic curvature tensors

• Let V be a finite dimensional real vector space which is equipped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ of signature (p,q), $p+q=n$. If a tensor $\mathcal{R}\in(V^*)^{\otimes 4}$ satisfies well-known symmetries

$$\begin{aligned}&\mathcal{R}(X,Y)+\mathcal{R}(Y,X)=0,\\&\mathcal{R}(X,Y)Z+\mathcal{R}(Y,Z)X+\mathcal{R}(Z,X)Y=0,\\&\langle\mathcal{R}(X,Y)Z,W\rangle=\langle\mathcal{R}(Z,W)X,Y\rangle.\end{aligned}\tag{1}$$

then we say that it is an algebraic curvature tensor on V.

• One says $\mathcal{R}$ is an *Osserman (Jordan-Osserman) algebraic curvature tensor* if the associated Jacobi operator has characteristic polynomial (Jordan-form) constant on the unit pseudospheres $S^-(V)$ and $S^+(V)$

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• One says $\mathcal{R}$ is an *Osserman (Jordan-Osserman) algebraic curvature tensor* if the associated Jacobi operator has characteristic polynomial (Jordan-form) constant on the unit pseudospheres $S^-(V)$ and $S^+(V)$

• A pseudo-Riemannian manifold M is said to be a *geometric realization* of $\mathcal{R}$ at a point $P \in M$ if there exists an isometry $\Psi : T_P M \longrightarrow V$ such that for all tangent vectors X, Y, Z, and W in $T_P M$, hold:

$$\langle \Psi X, \Psi Y \rangle = \langle X, Y \rangle,$$

$$\mathcal{R}(\Psi X, \Psi Y, \Psi Z, \Psi W) = \tilde{\mathcal{R}}_P(X, Y, Z, W),$$

where $\tilde{\mathcal{R}}_P$ is the curvature tensor of M restricted to $P \in M$.

Theorem 1 (Gilkey)

Every algebraic curvature tensor on a vector space V of signature (p, q) is geometrically realizable.

• This P. Gilkey's result enable us to work in algebraic settings, i.e., define a model $\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, \mathcal{R})$ and work in this model, and later pass to the geometrical context.

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Theorem 2 (Fiedler and Gilkey)

The spanning sets of $\mathcal{C}(V)$ *are sets* $\mathcal{S}(V) = \{\mathcal{R}^{\phi} : \phi = \phi^*\}$ *and* $\mathcal{A}(V) = \{\mathcal{R}^{\theta} : \theta = -\theta^*\}$*, where*

$$\mathcal{R}^{\phi}(X,Y)Z = \langle \phi Y, Z\rangle \phi X - \langle \phi X, Z\rangle \phi Y$$

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for ϕ *and* θ *are a symmetric and a skew-symmetric endomorphism of* V*, respectively.*

- *More precisely, for an arbitrary algebraic curvature tensor* $\mathcal{R}$ *there exist sets* $\Phi = \{\phi_1, \phi_2 \ldots, \phi_s\}$ *and* $\Theta = \{\theta_1, \theta_2 \ldots, \theta_t\}$ *of symmetric and skew-symmetric endomorphisms of* V*, respectively, such that* $\mathcal{R}$ *has representations*

$$\mathcal{R} = \sum_{i=1}^{s} \varepsilon_{\phi_i} \mathcal{R}^{\phi_i} = \sum_{j=1}^{t} \varepsilon_{\theta_j} \mathcal{R}^{\theta_j} \qquad \text{where } \varepsilon_{\phi_i}, \varepsilon_{\theta_j} \in \{1, -1\},$$

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• The duality principle, in Riemannian case, is the following property of the eigenvalue λ of the Jacobi operators: let X and Y be unit vectors, then

$$R_X(Y) = \lambda Y \qquad \text{if and only if} \qquad R_Y(X) = \lambda X.$$

• We extend notion of duality principle for definite vectors of a pseudo-Riemannian Osserman manifold: let R be an algebraic curvature tensor. For $\lambda \in \mathbb{R}$ we say that it satisfies the duality principle if for all mutually orthogonal unit vectors X, Y holds

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Theorem 3 (Andrejić, Rakić)

Let M be a diagonalizable Osserman pseudo-Riemannian manifold.

(i1) *If there is no null eigenvector of R_X for all definite X then M satisfies the duality principle.*

(i2) *M satisfies the duality principle in the case when R_X has all different eigenvalues.*

(i3) *If M is a four-dimensional, then duality principle holds.*

• **Total duality.** Let $\mathcal{R}$ be an algebraic curvature tensor. We say that non-zero vectors $X, Y \in V$ are totaly dual if the following equivalence holds

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Proposition 4

Let $\mathcal{R}$ be an algebraic curvature tensor on a vector space V of signature (p, q). The following conditions are equivalent:

(1) *If $p > 1$, then $\mathrm{Tr}(R_X)$ is constant on $S^-(V)$.*

(2) *If $q > 1$, then $\mathrm{Tr}(R_X)$ is constant on $S^+(V)$.*

(3) *There exists a constant c such that $\rho(x, y) = c\langle x, y\rangle$.*

Specially, if M is a (timelike) Osserman manifold with the metric of arbitrary signature (p, q) then M is Einstein space, and all three conditions define the notion of the Einstein algebraic curvature tensor.

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• We say that $\mathcal{R}$ is k-stein if there exist constants c_j for $1 \leq i \leq k$ so that $\mathrm{Tr} R_X^i = c_i \langle X, X \rangle^i$, for all $X \in V$.

Proposition 5

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• Proposition 5 shows (for $k = n = \dim M$) that notions of timelike Osserman algebraic curvature tensor, spacelike Osserman algebraic curvature tensor and n-stein algebraic curvature tensor are equivalent.

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Let $\mathcal{R}$ be an algebraic curvature tensor on a vector space V of signature (p, q), where $q \geq 2$. Let $0 \neq X$ be a null vector of V. If $\mathcal{R}$ is n-stein, then $\mathcal{R}_X$ is nilpotent.

Theorem 7 (GiSwVa, AlBoBlRa).

Let M be a four dimensional pseudo-Riemannian manifold. Then, the following conditions are equivalent:

(1) *M is pointwise Osserman.*

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and they are related with non-trivial decomposition of the tangent bundle $T(S^{n-1})$.

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Let V be an n-dimensional vector space. A real *Clifford- module structure C* on V, where $\nu \leq n$ is a collection of linear maps J_i on V with a set of generators $\{J_1, J_2, \dots, J_\nu\}$ such that

$$J_i\, J_j + J_j\, J_i = -2\,\delta_{ij} \quad \text{for } i, j = 1, 2 \dots, \nu.$$

Theorem 9 (Gilkey, Swan, Vanhecke)

Suppose there is a $Cliff(\nu)$*- module structure on* $\mathbb{R}^n$ *and consider a set of generators* $\{J_1, J_2, \dots, J_\nu\}$ *such that* $J_i\, J_j + J_j\, J_i = -2\,\delta_{ij}$. *If* $\lambda_0,\ \lambda_1, \dots, \lambda_\nu$ *are arbitrary real numbers, then the trilinear map* $R : V \times V \times V \longrightarrow V$, *defined by*

$$R = \lambda_0\, R^0 + \frac{1}{3}\sum_{i=1}^{\nu}(\lambda_i - \lambda_0)R^{J_i} \quad \textit{where} \quad R^0(x, y)z = \langle y, z\rangle\, x - \langle x, z\rangle\, y,$$

$$R^J(x, y)z = \langle Jx, z\rangle Jy - \langle Jy, z\rangle Jx + 2\,\langle Jx, y\rangle Jz.$$

is an Osserman algebraic curvature tensor with

$$R_x(J_i x) = \lambda_i\, J_i x \qquad R_x y = \lambda_0\, y.$$

where x, y *are nonnull unit vectors on* $\mathbb{R}^n$ *with* y *orthogonal to* $\{x, J_1 x, J_2 x, \dots, J_\nu x\}$.

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Theorem 9 (Gilkey, Swan, Vanhecke)

Suppose there is a $Cliff(\nu)$*- module structure on* $\mathbb{R}^n$ *and consider a set of generators* $\{J_1, J_2, \ldots, J_\nu\}$ *such that* $J_i J_j + J_j J_i = -2\,\delta_{ij}$. *If* $\lambda_0, \lambda_1, \ldots, \lambda_\nu$ *are arbitrary real numbers, then the trilinear map* $R : V \times V \times V \longrightarrow V$*, defined by*

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Theorem 9 (Chi, Nikolayevsky)

Let M be an Osserman Riemannian manifold, then Osserman conjecture holds, except in some cases (depending on the structure of eigenvalues of Jacobi operator) in dimension $m = 16$.

Remark. *Chi proved Theorem for $m \neq 4k$, $k > 1$, and Nikolayevsky in all other cases.*

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Let $\mathcal{K}$ be a symmetric operator of Ω. Then there exists an orthonormal (main) basis in Ω such that the matrix of $\mathcal{K}$ is consequently one of the following

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Let M be a 4-dimensional pseudo-Riemannian manifold of signature $(2, 2)$. Then the following conditions are equivalent:

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Lema 13

Let M be Jordan-Osserman manifold of signature $(2,2)$.

(II) *If the minimal polynomial of R_X has a double root α, then there exists null frame such that all non-vanishing components of R are:* $R_{1441} = R_{2332} = R_{1243} = R_{3412} = \alpha$ *and* $R_{3443} = 2$.

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Example (Rakić)

Let $M = \mathbb{R}^4$, (u_1, u_2, u_3, u_4) *the Cartesian coordinates and*

$$6g = u_2^2 du_1 \otimes du_1 + u_1^2 du_2 \otimes du_2 - u_1 u_2 [du_1 \otimes du_2 + du_2 \otimes du_1]$$

$$- 3[du_1 \otimes du_4 + du_4 \otimes du_1 + du_2 \otimes du_3 + du_3 \otimes du_2].$$

Then $(\mathbb{R}^4, g)$ *is the timelike Osserman rank two homogeneous symmetric space.*

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Then $(\mathbb{R}^4, g)$ is the timelike Osserman rank two homogeneous symmetric space.

Lema 13

Let M be Jordan-Osserman manifold of signature $(2,2)$.

(II) *If the minimal polynomial of R_X has a double root α, then there exists null frame such that all non-vanishing components of R are:*
$R_{1441} = R_{2332} = R_{1243} = R_{1342} = \alpha$ *and* $R_{4334} = 2$.

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Example (Rakić)

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Let M be a 4-dimensional spacelike (timelike) Jordan-Osserman manifold then the length of ∇R vanishes, i.e. $\|\nabla R\| = 0$.

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Let M be an Osserman manifold of type (II) or (III). Then, there exists an integrable, autoparallel, isotropic 2-dimensional distribution on M.

An Osserman manifold M of type (II) is locally foliated by totally geodesic, flat, isotropic 2-dimensional submanifolds.

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Let (M, g) be a 4-dimensional Jordan-Osserman pseudo-Riemannian manifold of signature $(2, 2)$ of type (II) or (III). Then there exists a coordinate system $u = (u^1, u^2, u^3, u^4) : U \subseteq \mathbb{R}^4 \longrightarrow M$ and an null basis F_1, F_2, F_3, F_4 such that

(i)

$$g = \begin{bmatrix} 0 & 0 & g_{13} & g_{14} \\ 0 & 0 & g_{23} & g_{24} \\ g_{13} & g_{23} & 0 & g_{34} \\ g_{14} & g_{24} & g_{34} & 0 \end{bmatrix}, \quad \text{with} \quad \det \begin{bmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{bmatrix} = 1$$

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(ii)

$$F_1 = \frac{\partial}{\partial u^1}, \quad F_2 = \frac{1}{C}\frac{\partial}{\partial u^2}, \quad F_3 = CP_i\frac{\partial}{\partial u^i}, \quad F_4 = Q_i\frac{\partial}{\partial u^i},$$

where $P_i, Q_i, C \in C^\infty(U)$ and C are non-vanishing on U.

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Theorem 18 (Recurrent spaces)

Let (M, g) be a recurrent spacelike Osserman space of signature (2,2). Then M can be endowed locally with a coordinate system $(u^1, \cdots, u^4)$ so that the metric of M takes one of the following forms

$$(1)\ \ g = \psi(u^1, u^2)du^1 \otimes du^1 + (du^1 \otimes du^4 + du^4 \otimes du^1) + (du^2 \otimes du^3 + du^3 \otimes du^2)$$

$$(2)\ \ g = \psi du^1 \otimes du^1 + k_{22} du^2 \otimes du^2 + k_{23}(du^2 \otimes du^3 + du^3 \otimes du^2) + (du^1 \otimes du^4 + du^4 \otimes du^1),$$

$$\psi = \theta(a_{22}(u^1)^2 + 2a_{23}u^2u^3 + a_{33}(u^3)^2),$$

where θ is a function of u^1, k_{22}, k_{23} are constants such that $k_{23} \neq 0$, and $a_{\alpha\beta}$ are constants such that $\det(a_{\alpha\beta}) \neq 0$.

Corollary 19 (Recurrent spaces)

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Theorem 20 (Garcia-Rio, Kupelli, Vazquez-Lorenzo)

Let (M, g) be a nonflat 4-dimensional locally symmetric pseudo-Riemannian Osserman manifold. Then one of the following holds.

- *(M, g) is locally isometric to a rank-one symmetric space, or*
- *(M, g) is locally isometric to the rank-two symmetric space G/H, where $\{\omega^0, \omega^1, \omega^2, \omega^3, \omega^4,\}$ with*

$$d\omega^0 = d\omega^1 = 0, \;\; d\omega^2 = -\omega^0 \wedge \omega^1, \;\; d\omega^3 = \pm\omega^1 \wedge \omega^2, \;\; d\omega^4 = \mp\omega^0 \wedge \omega^2.$$

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Theorem 20 (Garcia-Rio, Kupelli, Vazquez-Lorenzo)

Let (M, g) *be a nonflat* 4*-dimensional locally symmetric pseudo-Riemannian Osserman manifold. Then one of the following holds.*

- (M, g) *is locally isometric to a rank-one symmetric space, or*
- (M, g) *is locally isometric to the rank-two symmetric space* G/H, *where* $\{\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \}$ *with*

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Theorem 21 (Gilkey, Ivanova)

Let $\mathcal{R}$ be a spacelike Jordan Osserman algebraic curvature tensor on a vector space V of signature (p, q), where $p < q$. Then R_X is diagonalizable for any $X \in S^+(V)$.

- **Conclusion.**
  - Osserman conjecture holds in Lorentzian manifolds.
  - Osserman conjecture holds in Riemannian manifolds, up to few cases in dimension $n = 16$ (which should be investigated).
  - Manifolds of signature $((p, q), \ min(p, q) \geq 2)$
    - Osserman conjecture doesn't hold,
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• The duality principle for definite vectors of a pseudo-Riemannian Osserman manifold: let R be an algebraic curvature tensor. For $\lambda \in \mathbb{R}$ we say that it satisfies the duality principle if for all mutually orthogonal unit vectors X, Y holds

$$R_X(Y) = \varepsilon_X \lambda Y \quad \Longrightarrow \quad R_Y(X) = \varepsilon_Y \lambda X \, .$$

If the duality principle holds for all $\lambda \in \mathbb{R}$ then we say that duality principle holds for the algebraic curvature tensor R.

• **Problem.** Are notions of pointwise Osserman and duality principle equivalent ?

• In 2011, this equivalence is proved in dimension less then five, by M. Brozos-Vazquezand E. Merino.

• Y. Nikolayevski and ZR prove that the Osserman condition is equivalent to the duality principle in Riemannian signature and for diagonalizable algebraic curvature tensors in pseudo-Riemannian signature. More precisely we prove,

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Theorem 22 (Nikolayevsky, Rakić (2012))

Let $\mathcal{R}$ be an algebraic curvature tensor in $\mathbb{R}^n$ with an inner product $\langle\cdot\,,\cdot\rangle$.

- *If $\langle\cdot\,,\cdot\rangle$ is Riemannian, then the Osserman condition for $\mathcal{R}$ is equivalent to the duality principle.*
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THANK YOU FOR

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