Symmetry approach to classification of integrable PDEs

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Remark

There are three types of integrable equations of the form

$$u_t = F(u, u_x, u_{xx}, u_{xxx}):$$

- $$u_t = a u_{xxx} + b,$$
- $$u_t = \frac{a}{(u_{xxx} + b)^2},$$

  and
- $$u_t = \frac{2a + b}{\sqrt{a u_{xxx}^2 + b u_{xxx} + c}} + d,$$

where the coefficients depend on u, u_x, u_{xx}.

Part 2. Integrable vector systems

Main concepts of the symmetry approach can be generalized to systems of evolution equations (A. Mikhailov, A. Shabat, R. Yamilov). However, component-wise computations in this case are very tedious. The only one serious classification problem has been solved: all systems of the form

$$u_t = u_2 + F(u,\, v,\, u_1,\, v_1), \qquad u_t = -v_2 + G(u,\, v,\, u_1,\, v_1)$$

possessing higher conservation laws were listed.

Examples: Well-known NLS-equation

$$u_t = u_2 + u^2 v, \qquad v_t = -v_2 - v^2 u;$$

one of the versions of the Boussinesq equation

$$u_t = u_2 + (u+v)^2, \qquad v_t = -v_2 + (u+v)^2;$$

and Landau-Lifshitz equation

$$u_t = u_2 - \frac{2u_1^2}{u+v} - \frac{4\,(p(u,v)\,u_1 + r(u)\,v_1)}{(u+v)^2}$$
$$v_t = -v_2 + \frac{2v_1^2}{u+v} - \frac{4\,(p(u,v)\,v_1 + r(-v)\,u_1)}{(u+v)^2},$$

where $r(y) = c_4 y^4 + c_3 y^3 + c_2 y^2 + c_1 y + c_0$ and

$$p(u,v) = 2c_4 u^2 v^2 + c_3(uv^2 - vu^2) - 2c_2 uv + c_1(u-v) + 2c_0,$$

are basic models in a very long list of integrable systems.

But there exist several classes of systems where we can avoid the component-wise computations.

Class 1. Matrix equations.

The matrix KdV equation has the following form

$$U_t = U_{xxx} + 3\,(U U_x + U_x U),$$

where $U(x,t)$ is unknown $N \times N$-matrix. The simplest higher symmetry of this equation can be also written in the matrix form:

$$U_\tau = U_{xxxxx} + 5\,(U U_{xxx} + U_{xxx} U) + 10\,(U_x U_{xx} + U_{xx} U_x) +$$
$$+10\,(U^2 U_x + U U_x U + U_x U^2).$$

In general, we consider equations of the form

$$U_t = F(U,\, U_1,\, \ldots,\, U_n), \qquad U_i = \frac{\partial^i U}{\partial x^i}, \tag{1}$$

where F is a (non-commutative) polynomial. Integrability means the existence of generalized symmetries of the same form

$$U_\tau = G(U,\, U_1,\, \ldots,\, U_m).$$

Class 2. Equations on non-associative algebras.

Equations of the form

$$U_t = F(U, U_1, \ldots, U_n), \qquad U_i = \frac{\partial^i U}{\partial x^i},$$

where F is a non-commutative, non-associative polynomial. In this case F is fixed but the multiplication in the algebra is unknown.

For example. we may consider the KdV equation

$$U_t = U_{xxx} + 3U \circ U_x,$$

where $\circ$ is a multiplcation in some algebra A. The main question is: for which algebras this equation is integrable?

Class 3. Vector equations.

Integrable vector evolution equations have the following form:

$$\mathbf{u}_t = f_n\,\mathbf{u}_n + f_{n-1}\,\mathbf{u}_{n-1} + \cdots + f_1\,\mathbf{u}_1 + f_0\,\mathbf{u}, \qquad \mathbf{u}_i = \frac{\partial^i \mathbf{u}}{\partial x^i}. \quad (2)$$

Here $\mathbf{u}$ is N-component vector, the (scalar) coefficients f_i depend on scalar products between $\mathbf{u}, \mathbf{u}_x, ..., \mathbf{u}_{n-1}$.

Examples. The following vector mKdV-systems:

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\,\mathbf{u}_x.$$

and

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\,\mathbf{u}_x + (\mathbf{u}, \mathbf{u}_x)\,\mathbf{u}$$

are integrable for any N.

Theorem (A.Meshkov, VS 2002).

- i). If equation (2) possesses an infinite series of generalized symmetries of the form

$$\mathbf{u}_\tau = g_m\,\mathbf{u}_m + g_{m-1}\,\mathbf{u}_{m-1} + \cdots + g_1\,\mathbf{u}_1 + g_0\,\mathbf{u}, \qquad (3)$$

  then there exists a formal series

$$L = a_1\,D + a_0\, + a_{-1}\,D^{-1} + a_{-2}\,D^{-2} + \cdots\,, \qquad (4)$$

  satisfying the operator relation

$$L_t = [A,\, L], \qquad A = \sum_0^n f_i\,D^i. \qquad (5)$$

  Here f_i are the coefficients of equation (2).

- ii). The following functions

$$\rho_{-1} = \frac{1}{a_1}, \qquad \rho_0 = \frac{a_0}{a_1}, \qquad \rho_i = \operatorname{res} L^i, \qquad i \in \mathbb{N} \tag{6}$$

  are conserved densities for equation (2).

- iii). If equation (2) possesses an infinite series of conserved densities, then there exist a series L satisfying (5), and a series S of the form

$$S = s_1 D + s_0 + s_{-1} D^{-1} + s_{-2} D^{-2} + \cdots ,$$

  such that

$$S_t + A^t S + S A = 0, \quad S^t = -S, \quad L^t = -S^{-1} L S,$$

  where the upper index t stands for a formal conjugation.

- iiii). Under the conditions of item iii) densities (6) with $i = 2k$ are of the form $\rho_{2k} = D(\sigma_k)$ for some functions σ_k.

Integrable vector equations of the form

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2\mathbf{u}_{xx} + f_1\mathbf{u}_x + f_0\mathbf{u} \tag{7}$$

were studied by A.Meshkov and VS.

Equation (7) is called isotropic if the coefficients f_i are scalar functions in the following six variables:

$$(\mathbf{u},\, \mathbf{u}),\ (\mathbf{u}, \mathbf{u}_x),\ (\mathbf{u}_x, \mathbf{u}_x),\ (\mathbf{u}, \mathbf{u}_{xx}),\ (\mathbf{u}_x, \mathbf{u}_{xx}),\ (\mathbf{u}_{xx}, \mathbf{u}_{xx}). \tag{8}$$

It is clear that isotropic models are invariant with respect to the group $SO(N)$.

We consider equations (7) that are integrable for arbitrary dimension N. In virtue of the arbitrariness of N, variables (8) can be regarded as independent.

Isotropic equations on the sphere

Let us additionally assume that $\mathbf{u}^2 = 1$. Then $(\mathbf{u}, \mathbf{u}_x) = 0$ and $(\mathbf{u}, \mathbf{u}_{xx}) = -(\mathbf{u}_x, \mathbf{u}_x)$. Therefore we have only three independent scalar products

$$(\mathbf{u}_x, \mathbf{u}_x), \quad (\mathbf{u}_x, \mathbf{u}_{xx}), \quad (\mathbf{u}_{xx}, \mathbf{u}_{xx})$$

in the coefficients of the equation.

Theorem (Meshkov-VS 2002). Suppose that an equation of the form (7) on the sphere $\mathbf{u}^2 = 1$ has an infinite series of generalized symmetries or conserved densities; then this equation belongs to the following list:

List of integrable isotropic equations on the sphere

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_{xx} + \frac{3}{2}\frac{u_{[2,2]}}{u_{[1,1]}}\,\mathbf{u}_x,$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_{xx} + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2\,(1 + a\,u_{[1,1]})}\right)\mathbf{u}_x,$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} + \frac{3}{2}\left(\frac{a^2\,u_{[1,2]}^2}{1 + a\,u_{[1,1]}} - a\,(u_{[2,2]} - u_{[1,1]}^2) + u_{[1,1]}\right)\mathbf{u}_x + 3\,u_{[1,2]}\,\mathbf{u},$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{(q+1)\;u_{[1,2]}}{2\,q\,u_{[1,1]}}\,\mathbf{u}_{xx} + 3\,\frac{(q-1)\;u_{[1,2]}}{2\,q}\,\mathbf{u}$$

$$+\frac{3}{2}\left(\frac{(q+1)\,u_{[2,2]}}{u_{[1,1]}} - \frac{(q+1)\,a\,u_{[1,2]}{}^2}{q^2 u_{[1,1]}} + u_{[1,1]}\,(1-q)\right)\mathbf{u}_x,$$

where $u_{[i,j]} := (\mathbf{u}_i, \mathbf{u}_j)$ and $q = \sqrt{1 + a\,u_{[1,1]}}$.

Notice that if $a = 0$ and therefore $q = \pm 1$ then the latter equation yields

$$\mathbf{u}_t = \mathbf{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_{xx} + 3\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\mathbf{u}_x,$$

and

$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3\,u_{[1,1]}\,\mathbf{u}_x + 3\,u_{[1,2]}\,\mathbf{u}.$$

Another solved classification problem: equations of the form

$$\mathbf{u}_t = D(\,\mathbf{u}_{xx} + f_1\mathbf{u}_x + f_0\mathbf{u}\,)$$

Anisotropic equations

Consider the following equation (I. Golubchik-VS 2000):

$$\mathbf{u}_t = \left(\mathbf{u}_{xx} + \frac{3}{2}(\mathbf{u}_x,\ \mathbf{u}_x)\mathbf{u}\right)_x + \frac{3}{2}\langle \mathbf{u}, \mathbf{u}\rangle \mathbf{u}_x, \qquad \mathbf{u}^2 = 1. \qquad (9)$$

Here $\langle \mathbf{a}\,, \mathbf{b}\rangle = (\mathbf{a},\ R\mathbf{b})$, where R is an arbitrary constant symmetric matrix R. One can assume that $R = \mathrm{diag}(r_1, \ldots, r_N)$. Equation (9) has a Lax pair whose spectral parameter lies on an algebraic curve of genus $1 + (N-3)2^{N-2}$. If $N = 3$, then (9) is a symmetry for the famous Landau-Lifshitz equation.

In this case the coefficients in

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2\mathbf{u}_{xx} + f_1\mathbf{u}_x + f_0\mathbf{u}$$

depends on two different independent scalar products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$.

Theorem (Meshkov-VS 2002). Suppose equation (7) on the sphere $(\mathbf{u},\mathbf{u})=1$ with

$$f_i = f_i(u_{[1,1]},\, u_{[1,2]},\, u_{[2,2]},\, v_{[0,0]},\, v_{[0,1]},\, v_{[1,1]}),$$

where $v_{[i,j]} := \langle \mathbf{u}_i, \mathbf{u}_j \rangle$. has an infinite series of symmetries or conserved densities; then this equation is one of the above or belongs to the following list:

$$\mathbf{u}_t = \mathbf{u}_3 - \frac{3\,u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_2 + \left(\frac{3\,u_{[2,2]}}{2\,u_{[1,1]}} + \frac{3\,u_{[1,2]}^2}{2\,u_{[1,1]}^2} + \frac{c\,v_{[1,1]}}{u_{[1,1]}} \right) \mathbf{u}_1,$$

$$\mathbf{u}_t = \mathbf{u}_3 + \left(v_{[0,0]} + \frac{3}{2}\,u_{[1,1]} \right) \mathbf{u}_1 + 3\,u_{[1,2]}\,\mathbf{u}_0,$$

$$\mathbf{u}_t = \mathbf{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\,\mathbf{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{(v_{[0,0]}+a)\,u_{[1,2]}^2}{q\,u_{[1,1]}^2} - \right.$$

$$\left. -2\frac{v_{[0,1]}u_{[1,2]}}{q\,u_{[1,1]}} + \frac{v_{[1,1]}}{u_{[1,1]}} - \frac{v_{[0,1]}^2}{q\,u_{[1,1]}} \right) \mathbf{u}_1,$$

where $q = u_{[1,1]} + v_{[0,0]} + a$.

The classification of anisotropic equations on the sphere with

$$f_i = f_i(u_{[1,1]},\, u_{[1,2]},\, u_{[2,2]},\, v_{[0,0]},\, v_{[0,1]},\, v_{[1,1]},\, v_{[0,2]},\, v_{[1,2]},\, v_{[2,2]})$$

was completed by M. Balakhnev and A. Meshkov in 2005.

Example (Balakhnev-Meshkov 2005)

$$\mathbf{u}_t = \mathbf{u}_3 - 3\frac{v_{[0,1]}}{v_{[0,0]}}\mathbf{u}_2 - 3\left(\frac{v_{[0,2]}}{v_{[0,0]}} - 2\frac{v_{[0,1]}^2}{v_{[0,0]}^2}\right)\mathbf{u}_1 + 3\left(u_{[1,2]} - \frac{v_{[0,1]}}{v_{[0,0]}}u_{[1,1]}\right)\mathbf{u},$$

Integrable hyperbolic vector equations the the sphere were studied by A. Meshkov and VS.

Example (Meshkov-VS 2012)

$$\mathbf{u}_{xy} = \frac{\mathbf{u}_x}{\langle \mathbf{u}, \mathbf{u}\rangle}\left(\langle \mathbf{u}, \mathbf{u}_y\rangle + \sqrt{1 + \langle \mathbf{u}, \mathbf{u}\rangle |\mathbf{u}_x|^{-2}}\ \phi\right) - (\mathbf{u}_x, \mathbf{u}_y)\mathbf{u},$$

$$\text{where} \qquad \phi = \sqrt{\langle \mathbf{u}, \mathbf{u}_y\rangle^2 + \langle \mathbf{u}, \mathbf{u}\rangle(1 - \langle \mathbf{u}_y, \mathbf{u}_y\rangle)}$$

Part 3. Non-associative algebraic structures and multi-component polynomial integrable models.

Main results have been obtained by S. Svinolupov.

- **Svinolupov S.I.**, *On the analogues of the Burgers equation*, Phys. Lett. A, **135**(1), 32–36, 1989.
- **Svinolupov S.I.**, *Jordan algebras and generalized Korteweg-de Vries equations.*, Theor. and Math. Phys., **87**(3), 391–403, 1991.
- **Svinolupov S.I.**, *Generalized Schrödinger equations and Jordan pairs,*, CMP, **143**(1), 559–575, 1992.
- **Svinolupov S.I.**, *Jordan algebras and integrable systems,* Func. analiz i pril., **27**(4), 40–53, 1993.
- **Sokolov V.V., Svinolupov S.I.**, *Vector-matrix generalizations of classical integrable equations,*, Theor. and Math. Phys., **100**(2), 959– 962, 1994.

Jordan KdV systems.

The KdV equation is given by

$$u_t = u_{xxx} + u u_x.$$

Consider the following N-component system

$$u^i_t = u^i_{xxx} + C^i_{jk} u^k u^j_x, \qquad i, j, k = 1, \ldots, N, \tag{10}$$

where C^i_{jk} are constants.

Let A be an N-dimensional algebra with a basis $\mathbf{e}_1, ..., \mathbf{e}_N$ such that

$$\mathbf{e}_j \circ \mathbf{e}_k = C^i_{jk} \mathbf{e}_i.$$

If $U = \sum_k u^k \mathbf{e}_k$ then the algebraic form of (10) is given by

$$U_t = U_{xxx} + U \circ U_x, \tag{11}$$

where $\circ$ denotes the multiplication in A.

Theorem. Equation (11) has a generalized symmetry of the form

$$U_\tau = U_m + f(U, ..., U_{m-1}),$$

where $m \geqslant 5$, f a (non-commutative, non associative) homogeneous differential polynomial iff A is a Jordan algebra.

Definition. Algebra A is called Jordan if the following identities

$$X \circ Y = Y \circ X, \qquad X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X.$$

hold.

If $*$ is a multiplication in any associative algebra then $X \circ Y = X * Y + Y * X$ is a Jordan multiplication.

Definition. A system (10) is called irreducible if it does not contain any subsystem of the same form.

Proposition. A system (10) is irreducible iff the corresponding algebra A is simple.

Examples of simple Jordan algebras:

a) The set of all $N \times N$ matrices w.r.t. $X \circ Y = XY + YX$;
b) The set of all symmetric $N \times N$ matrices w.r.t. the same operation;
c) The set of all N-dimensional vectors w.r.t.

$$X \circ Y = < X, C > Y + < Y, C > X - < X, Y > C,$$

$\langle \cdot, \cdot \rangle$ is a scalar product, C is a given constant vector;
d) A special Jordan algebra $H_3(O)$ of dimension 27.

The corresponding integrable systems :

a) The matrix KdV-equation:

$$U_t = U_{xxx} + UU_x + U_xU,$$

where U is an $N \times N$-matrix;

b) The matrix KdV-equation with $U^t = U$;

c) The vector KdV equation (Svinolupov-VS):

$$\mathbf{u}_t = \mathbf{u}_{xxx} + < C, \mathbf{u} > \mathbf{u}_x + < C, \mathbf{u}_x > \mathbf{u} - < \mathbf{u}, \mathbf{u}_x > C,$$

where $\mathbf{u}$ is an N-dimensional vector, C is a given constant vector.

Left-symmetric Burgers systems.

The Burgers equation

$$u_t = u_{xx} + uu_x$$

is homogeneous: the weights are $u \to 1,\ \frac{d}{dx} \to 1,\ \frac{d}{dt} \to 2.$

Consider the following homogeneous multi-component generalization of the Burgers:

$$u^i_t = u^i_{xx} + 2C^i_{jk}u^k u^j_x + A^i_{jkm}u^k u^j u^m, \qquad (12)$$

where $\quad i, j, k = 1, \ldots, N$

Theorem. This system has generalized symmetries iff

$$A^i_{jkm} = \tfrac{1}{3}\Big(C^i_{jr}C^r_{km} + C^i_{kr}C^r_{mj} + C^i_{mr}C^r_{jk} - C^i_{rj}C^r_{km} - C^i_{rk}C^r_{mj} - C^i_{rm}C^r_{jk}\big),$$

and

$$C^i_{jr}C^r_{km} - C^i_{kr}C^r_{jm} = C^r_{jk}C^i_{rm} - C^r_{kj}C^i_{rm} \qquad (13)$$

for any i, j, k, m (summation w.r.t. r).

Formula (13) means that C^i_{jk} are structural constants of a left-symmetric algebra A.

The algebraic form of the system is

$$U_t = U_{xx} + 2U \circ U_x + U \circ (U \circ U) - (U \circ U) \circ U,$$

where $\circ$ denotes the multiplication in A.

Definition of left-symmetric algebra:

$$As(X, Y, Z) = As(Y, X, Z),$$

where
$$As(X, Y, Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z).$$

Any associative algebra is left-symmetric.

Example of vector left-symmetric algebra.

The set of all N-dimensional vectors w.r.t.

$$X \circ Y = < X, C > Y + < X, Y > C,$$

where C is a fixed (constant) vector.

Corresponding integrable systems:

a. The matrix Burgers equation

$$U_t = U_{xx} + UU_x;$$

b. The vector Burgers equation (Svinolupov-VS)

$$\mathbf{u}_t = \mathbf{u}_{xx} + 2 < \mathbf{u}, \mathbf{u}_x > C + 2 < C, \mathbf{u} > \mathbf{u}_x +$$

$$< \mathbf{u}, \mathbf{u} >< C, \mathbf{u} > C - < \mathbf{u}, \mathbf{u} >< C, C > \mathbf{u};$$

One more example of left-symmetric algebra

(I.Golubchik-VS).

Let A be associative algebra and $R : A \to A$ satisfies the modified classical Yang-Baxter equation

$$R([R(x), y] - [R(y), x]) = [x, y] + [R(x), R(y)].$$

Then the multiplication

$$x \circ y = [R(x), y] - (xy + yx)$$

is left-symmetric.

Jordan triple systems and polynomial integrable models.

Consider mKdV-type systems of the form

$$u^i_t = u^i_{xxx} + C^i_{jkm} u^k u^j u^m_x, \quad i, j, k = 1, \ldots, N.$$

Let T be an algebraic triple system with a basis $\mathbf{e}_1, ..., \mathbf{e}_N$ such that

$$\{\mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_m\} = C^i_{jkm} \mathbf{e}_i.$$

If $U = \sum_k u^k \mathbf{e}_k$ then the algebraic form of the equation is given by

$$U_t = U_{xxx} + \{U, U, U_x\}. \qquad (14)$$

Theorem. For any triple Jordan system equation (14) has an infinite series of generalized symmetries.

Definition. A triple system T is called Jordan if the following identities

$$\{X,Y,Z\} = \{Z,Y,X\},$$

$$\{X,Y,\{V,W,Z\}\} - \{V,W,\{X,Y,Z\}\} =$$
$$\{\{X,Y,V\},W,Z\} - \{V,\{Y,X,W\},Z\}.$$

hold.

If $*$ is a multiplication in any associative algebra then $\{X,Y,Z\} = X*Y*Z + Z*Y*X$ is a Jordan triple product.

Examples of simple triple Jordan systems.

a) The set of all $N \times N$ matrices w.r.t.

$$\{X,Y,Z\} = XYZ + ZYX$$

b) The set of all skew-symmetric $N \times N$ matrices w.r.t. the same operation.

c) The set of all N-dimensional vectors w.r.t.

$$\{X,Y,Z\} =< X,Y > Z+ < Y,Z > X- < X,Z > Y.$$

d) The set of all N-dimensional vectors w.r.t.

$$\{X,Y,Z\} =< X,Y > Z+ < Y,Z > X.$$

There is the following generalization of Example **d**. The vector space of all $n \times m$-matrices is a Jordan triple system with respect to operation

$$\{X,Y,Z\} = XY^tZ + ZY^tX,$$

where "t"stands for transposition.

The integrable vector systems corresponding to Examples **c** and **d** are given by

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\, \mathbf{u}_x.$$

and

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\, \mathbf{u}_x + (\mathbf{u}, \mathbf{u}_x)\, \mathbf{u}.$$

The matrix triple Jordan system generates the matrix mKdV equation

$$U_t = U_{xxx} + U^2 U_x + U_x U^2.$$

Theorem. For any Jordan triple system the nonlinear Schroedinger-type system

$$U_t = U_{xx} + \{V, U, V\}, \quad V_t = -V_{xx} - \{U, V, U\},$$

and the nonlinear derivative Schroedinger-type system

$$U_t = U_{xx} + \{V, U, V\}_x, \quad V_t = -V_{xx} - \{U, V, U\}_x,$$

possess generalized symmetries.

The corresponding integrable NLS-type systems:

a. the matrix NLS equation

$$u_t = u_2 + 2\,uvu, \qquad v_t = -v_2 - 2\,vuv;$$

c the vector NLS equation 2 (Kulish-Sklyanin)

$$u_t = u_2 + 2 < u, v > u - < u, u > v,$$
$$v_t = -v_2 - 2 < u, v > v + < v, v > u;$$

d the vector NLS equation 1 (Manakov)

$$u_t = u_2 + 2 < u, v > u, \qquad v_t = -v_2 - 2 < u, v > v;$$

Part 4. Jordan triple systems and rational integrable models.

Results have been obtained by S. Svinolupov and VS.

There exist rational integrable equations like the Schwartz-KdV equation

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}.$$

What is the Jordan analog of this equation?

What is $\frac{1}{x}$? A proper answer is: it is any solution of the ODE $y' = -y^2$.

Let $\{X, Y, Z\}$ be a Jordan triple system,

$$U = \sum_k u^k \mathbf{e}_k.$$

Let $\phi(u^1, ..., u^N) = \sum_k \phi^k \mathbf{e}_k$ be a solution of the following overdetermined consistent system

$$\frac{\partial \phi}{\partial u^k} = -\{\phi,\ \mathbf{e}_k,\ \phi\}. \tag{15}$$

By definition, $U^{-1} = \phi(U)$.

If $\{X, Y, Z\} = \frac{1}{2}(XYZ + ZYX)$, then

$$\phi(U) = U^{-1}.$$

For

$$\{X, Y, Z) =< X, Y > Z+ < Y, Z > X- < X, Z > Y$$

we have

$$\phi(U) = \frac{U}{\|U\|^2}.$$

For any triple Jordan system the following equations have infinitely many generalized symmetries.

Class 1. The following Jordan equations of the Schwartz-KdV type

$$U_t = U_{xxx} - \frac{3}{2}\{U_{xx}, U_x, U_{xx}\}.$$

The correspondent matrix equation is given by

$$u_t = u_{xxx} - \frac{3}{2}u_{xx}u_x^{-1}u_{xx},$$

where $u(x,t)$ is an $N \times N$ matrix.

Class 2. Equation of the form

$$U_t = U_{xxx} - 3\{U_x, U^{-1}, U_{xx}\} + \frac{3}{2}\{U_x, \{U^{-1}, U_x, U^{-1}\}, U_x\}.$$

The corresponding matrix equation has the following form:

$$u_t = u_{xxx} - \frac{3}{2}u_x u^{-1}u_{xx} - \frac{3}{2}u_{xx}u^{-1}u_x + \frac{3}{2}u_x u^{-1}u_x u^{-1}u_x.$$

Class 3. The scalar representative of this class is the Heisenberg model

$$u_t = u_{xx} - \frac{2}{u+v} u_x^2, \qquad v_t = -v_{xx} + \frac{2}{u+v} v_x^2.$$

The corresponding integrable Jordan coupled equations are given by

$$u_t = u_{xx} - 2\{u_x, (u+v)^{-1}, u_x\},$$
$$v_t = -v_{xx} + 2\{v_x, (u+v)^{-1}, v_x\}$$

The matrix equation is of the form

$$u_t = u_{xx} - 2u_x(u+v)^{-1}u_x,$$

$$v_t = -v_{xx} + 2v_x(u+v)^{-1}v_x.$$

Class 4. Consider the equation

$$U_{xy} = \{U_x, U^{-1}, U_y\} \tag{16}$$

In the matrix case this coincides with the equation of the principal chiral field

$$u_{xy} = \frac{1}{2}(u_x u^{-1} u_y + u_y u^{-1} u_x).$$

It is easy to verify that for any triple Jordan system equation (16) admits the following zero-curvature representation

$$\Psi_x = \frac{2}{(1-\lambda)} L_{u_x} \Psi, \qquad \Psi_y = \frac{2}{(1+\lambda)} L_{u_y} \Psi.$$

Here we denote by L_X the left multiplication operator:

$$L_X(Y) = \{X, U^{-1}, Y\}$$

.

Remark. There exist close relationships between the Jordan algebras and the Jordan triple systems. Namely, any Jordan algebra generates a triple system by the formula

$$\{X, Y, Z\} = (X \circ Y) \circ Z + (Z \circ Y) \circ X - Y \circ (X \circ Z). \quad (17)$$

Conversely, any Jordan triple system $\{X, Y, Z\}$ yields a family of Jordan algebras with the multiplication

$$X \circ Y = \{X, \phi, Y\},$$

where ϕ is an arbitrary element.

Part 5. Deformations of algebraic structures and integrable models of geometric type.

Results have been obtained by S. Svinolupov and VS.

- **Sokolov V.V., Svinolupov S.I.**, *Deformation of non-associative algebras and integrable differential equations*, Acta Applicant Mathematica, **41**(1-2), 323-339, 1995.
- **Svinolupov S.I., Sokolov V.V.**, *Deformations of Jordan triple systems and integrable equations*, Meteor. and Mat. Fitz, **108**(3), 1160 – 1163, 1996.
- **Habibullin I.T., Sokolov V.V., Yamilov R.I.**,, *Multi-component integrable systems and non-associative structures,* in „Nonlinear Physics: theory and experiment“, World Scientific Publisher: Singapore, 139–168, 1996.

Consider multi-component systems of the form

$$u^i_t = u^i_{xxx} + a^i_{jk}(\vec{u}) u^j_x u^k_{xx} + b^i_{jks}(\vec{u}) u^j_x u^k_x u^s_x. \qquad (18)$$

This class is invariant under point transformations: $\vec{v} = \vec{\Psi}(\vec{u})$. Under these transformations, the set of functions $a^i_{jk}(\vec{u})$ are transformed as components of an affine connection Γ.

Let R and T be the curvature and the torsion tensors of Γ.

We would like to describe equations from this class having infinitely many generalized symmetries. The following equation

$$U_t = U_{xxx} - 3\{U_x, U^{-1}, U_{xx}\} + \frac{3}{2}\{U_x, \{U^{-1}, U_x, U^{-1}\}, U_x\}.$$

gives us an example of such equation.

It is convenient to rewrite the system as

$$u_t^i = u_{xxx}^i + 3\alpha_{jk}^i u_x^j u_{xx}^k +$$

$$\left(\frac{\partial \alpha_{km}^i}{\partial u^j} + 2\alpha_{jr}^i \alpha_{km}^r - \alpha_{rj}^i \alpha_{km}^r + \beta_{jkm}^i\right) u_x^j u_x^k u_x^m,$$

where $\beta_{jkm}^i = \beta_{kjm}^i = \beta_{mkj}^i$, i.e.

$$\beta(X, Y, Z) = \beta(Y, X, Z) = \beta(X, Z, Y)$$

for any vectors X, Y, Z. It is easy to verify that the set of functions β_{jkm}^i are transformed just as components of a tensor.

In order to formulate classification results, we introduce the following tensor:

$$\sigma(X,Y,Z) = \beta(X,Y,Z) - \frac{1}{3}\delta(X,Y,Z) + \frac{1}{3}\delta(Z,X,Y),$$

where

$$\delta(X,Y,Z) = T(X,T(Y,Z)) + R(X,Y,Z) - \nabla_X(T(Y,Z)).$$

It follows from the Bianchi identity that

$$\sigma(X,Y,Z) = \sigma(Z,Y,X).$$

Notice that if $T = 0$, then

$$R(X,Y,Z) = \sigma(X,Z,Y) - \sigma(X,Y,Z),$$

$$\beta(X,Y,Z) = \frac{1}{3}\left(\sigma(X,Y,Z) + \sigma(Y,Z,X) + \sigma(Z,X,Y)\right).$$

Theorem. The system (18) has infinitely many symmetries iff

- $$\nabla_X[R(Y,Z,V)] = R(Y,X,T(Z,V)),$$

- $$\nabla_X\left[\nabla_Y(T(Z,V)) - T(Y,T(Z,V)) - R(Y,Z,V)\right] = 0,$$

- $$\nabla_X(\sigma(Y,Z,V)) = 0,$$

- $$T(X,\sigma(Y,Z,V)) + T(Z,\sigma(Y,X,V)) +$$
  $$+T(Y,\sigma(X,V,Z)) + T(V,\sigma(X,Y,Z)) = 0,$$

- $$\sigma(X,\sigma(Y,Z,V),W) - \sigma(W,V,\sigma(X,Y,Z)) +$$
  $$\sigma(Z,Y,\sigma(X,V,W)) - \sigma(X,V,\sigma(Z,Y,W)) = 0.$$

If $T = 0$, we have a symmetric space together with covariantly constant deformation of a triple Jordan system.

In the case $T \neq 0$, a generalization of the symmetric spaces appears. I do not know whether such affine connected spaces have been considered by geometers.

Theorem. For any Jordan triple system $\{\cdot, \cdot, \cdot\}$ with structural constants s^i_{jkm}, there exists a unique (up to point transformations) integrable equation (18), such that $T = 0$ and

$$\sigma^i_{jkm}(0) = s^i_{jkm}. \tag{19}$$

There is a large class of integrable systems (18) related to Jordan algebras.

Define $\alpha^i_{jk}(\vec{u})$ as a solution of the compatible PDE system

$$\frac{\partial \alpha^i_{jk}}{\partial u^m} = \alpha^i_{rk}\alpha^r_{mj} + \alpha^i_{jr}\alpha^r_{mk} - \alpha^i_{mr}\alpha^r_{jk} \tag{20}$$

with initial data $\alpha^i_{jk}(0) = C^i_{jk}$, where C^i_{jk} are structural constants of a Jordan algebra. Then we get a family of Jordan algebras with the srtuctural constants $\alpha^i_{jk}(\vec{u})$, depending on parameters $u^1, ..., u^N$. Denote the multiplication in this family by $\circ$.

Define the family of triple Jordan systems by

$$\{X, Y, Z\} = (X \circ Y) \circ Z + (Z \circ Y) \circ X - Y \circ (X \circ Z).$$

Let $\sigma^i_{jkm}(\vec{u})$ be the structural constants of this family.

Then the corresponding system (18) possesses infinitely many generalized symmetries.

For example, in the equation

$$U_t = U_{xxx} - 3\{U_x, U^{-1}, U_{xx}\} + \frac{3}{2}\{U_x, \{U^{-1}, U_x, U^{-1}\}, U_x\}$$

the term $\{U_x, U^{-1}, U_{xx}\}$ can be treated as a product $U_x \circ U_{xx}$ in a Jordan algebra whose structural constants depend on U in a special way.

Equations with initial vector Jordan triple systems cannot be obtained by this construction. They are of the form

$$u_t = u_{xxx} + \frac{3}{2}\Big(P(u, u_x)(C - \|C\|^2 u)\Big)_x + 3\frac{\lambda - 1}{\lambda + 1}\|C\|^2 P(u, u_x) u_x,$$

where $\lambda = 1$ or $\lambda = 0$, and

$$P(u, u_x) = \|u_x + \frac{< C, u_x > u}{1 - (C, u)}\|^2.$$