Hirota equations for the descendent potential of orbifold $\mathbb{C}P^1$ and the Lax equations for extended bigraded Toda hierarchy

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Outline

1. KP and Toda
2. Hirota equations for the total descendent potential (after Milanov-Tseng)
3. From Hirota to Lax equations

§1. KP and Toda

Reductions

KP ('ΨDOs')

- Lax: $L_{KP} = \partial + u_1\partial^{-1} + \dots$

$$\downarrow$$

- Gelfand-Dickey (GD): $(L_{KP}^{n+1})_- = 0$
- $L = \partial^{n+1} + \cdots + w_0$
- n dependent variables
- Bi-Hamiltonian: $\{,\}_1$, $\{,\}_2$
- Casimirs of $\{,\}_1$ generate n sequences of Hamiltonians in involution

2D Toda ('ΔOs')

- Lax: pairs (L_1, L_2) of difference operators

$$\downarrow$$

- Bigraded Toda (BTH): $L_1^k = L_2^m$
- $L = \Lambda^k + \cdots + u_{-m}\Lambda^{-m}$
- $k + m$ dependent variables
- Bi-Hamiltonian
- Casimirs of $\{,\}_1$ generate $k + m - 1$ sequences of Hamiltonians in involution
- a common Casimir $\implies$ a missing sequence of flows!

Extended bigraded Toda hierarchy (EBTH)

Lax operator:

$$L = \Lambda^k + w_{k-1}\Lambda^{k-1} + \cdots + w_{-m}\Lambda^{-m}$$

flows coming from reduction of 2D Toda flows, $p \geqslant 0$:

$$\frac{\partial L}{\partial t^{\alpha,p}} = [(L^{p+1-\alpha/k})_+, L] \quad \alpha = k-1, \ldots, 0$$

$$\frac{\partial L}{\partial t^{\alpha,p}} = [(L^{p+1+\alpha/m})_-, L] \quad \alpha = -1, \ldots, -m+1$$

extended flows:

$$\frac{\partial L}{\partial t^{-m,p}} = [(L^p \log L)_+, L] \quad \alpha = 0, \ldots, -m+1$$

$$\log L = \sum_{l \in \mathbb{Z}} v_l \Lambda^l, \quad [\log L, L] = 0$$

with v_l uniquely determined elements in $\mathbb{C}[w_i^{(n)}, \log w_{-m}, w_{-m}^{\pm 1/m}][[\epsilon]]$.
[C., Dubrovin, Zhang (2004); C. (2006)]

Bi-Hamiltonian formulation

Two compatible Poisson brackets

$$
\begin{aligned}
\{w_i(x), w_j(y)\}_1 &= c_{ij}\big(w_{i+j}(x)\delta(x-y+i\epsilon) - w_{i+j}(x-j\epsilon)\delta(x-y-j\epsilon)\big)\\
\{w_i(x), w_j(y)\}_2 &= \sum_{l<j}[w_{i+j-l}(x)w_l(x+(i-l)\epsilon)\delta(x-y+(i-l)\epsilon)\\
&\quad - w_l(x)w_{i+j-l}(x+(l-j)\epsilon)\delta(x-y+(l-j)\epsilon)]\\
&\quad + w_i(x)\left(\frac{(\Lambda^i-\Lambda^k)(\Lambda^{-j}-1)}{1-\Lambda^k}w_j(x)\delta(x-y)\right)
\end{aligned}
$$

§2. Hirota for descendent potential of $\mathcal{C}_{k,m}$

Frobenius manifold

Landau-Ginzburg model

$M_{k,m}$ - Frobenius manifold of degree (k,m) Laurent polynomials:

$$\lambda(x) = x^k + \cdots + w_{-m} x^{-m}$$

local algebra: $\mathbb{C}[x, x^{-1}]/ < \partial_x \lambda >$
flat metric: $< \partial', \partial'' > = \mathrm{Res}_{x_i} \frac{\partial' \lambda \partial'' \lambda}{x \partial_x \lambda} \frac{dx}{x}$

$M_{k,m}$ coincides with the F.mfd. on the orbit space on extended affine Weyl group $\widetilde{\mathcal{W}}^{(k)}(A_{k+m-1})$ [Dubrovin, Zhang (1998)]

Givental total descendent potential

To a calibrated s.s. Frobenius manifold one associates

$$\mathcal{D}^{M_{k,m}} = c\hat{S}^{-1}\hat{\psi}\hat{R}e^{\widehat{U/z}}\prod_{i=1}^{k+m}\mathcal{D}_{pt}(\epsilon^2\Delta_i; \sqrt{\Delta_i}q_i)$$

$\mathcal{D}_{pt}(q_i)$ - KW potential $\in \mathbb{C}((\epsilon))[[q_{0,i}, q_{1,i}, q_{2,i}, \dots]]$ with dilaton shift $q_{1,i} \to q_{1,i} + 1$

u_i - canonical coords, $U = \mathrm{diag}(u_1, \dots, u_{k+m})$

Δ_i, ψ related to the change from canonical to flat coords

$\hat{S}$, $\hat{R}$ are quantized elements of twisted loop group $\mathcal{L}^{(2)}GL(H)$

Vertex operators and Hirota equation

Let

$$f^a = \sum_{l \in \mathbb{Z}} I_a^{(l)}(\lambda)(-z)^l$$

the vertex operator is

$$\Gamma^a = e^{\widehat{(f^a)_-}} e^{\widehat{(f^a)_+}}$$

The quantization rules give

$$\widehat{(f^a)_-} = -\epsilon^{-1} \sum_{n \geqslant 0} (I_a^{(-n-1)})_\alpha q_n^\alpha$$

$$\widehat{(f^a)_+} = \epsilon \sum_{n \geqslant 0} (-1)^n (I_a^{(n)})^\alpha \frac{\partial}{\partial q_n^\alpha}$$

Hirota quadratic equations (for ancestor potential)

$$(\Gamma^{\#\delta} \otimes \Gamma^\delta) \left(\sum_{a \in A} c_a(\lambda) \Gamma^a \otimes \Gamma^{-a} \right) (\mathcal{A} \times \mathcal{A}) d\lambda \quad \text{is regular } \lambda \sim \infty$$

Vertex operators: ∂_x-valued

D - algebra of differential operators of the form $\sum_{k=0}^{n} a_k(x,\epsilon)\partial_x^k$

- introduce D-valued vertex operators Γ^{δ}, $\Gamma^{\delta\#}$

- for the descendent Hirota, they have the form:

$$\Gamma_{\infty}^{\delta} = \exp\Big(- \sum_{n>0} \frac{\lambda^n}{n!} q_n^k \partial_x \Big) \cdot \exp(x \frac{\partial}{\partial q_0^k})$$

$$\Gamma_{\infty}^{\delta\#} = \exp(x \frac{\partial}{\partial q_0^k}) \cdot \exp\Big(\sum_{n>0} \frac{\lambda^n}{n!} q_n^k \partial_x \Big)$$

Hirota quadratic equations

(descendent) Hirota quadratic equation

$$(\Gamma_\infty^{\delta\#} \otimes \Gamma_\infty^{\delta}) \left(\frac{1}{k} \sum_{a=1}^{k} \lambda_a^{\frac{1-k}{k}} \Gamma_\infty^{a} \otimes \Gamma_\infty^{-a} - \frac{1}{m} \sum_{b=k+1}^{k+m} \lambda_b^{-\frac{1+m}{m}} \Gamma_\infty^{b} \otimes \Gamma_\infty^{-b} \right) (\tau \otimes \tau) d\lambda$$

is regular for $(q_0^k)'' - (q_0^k)' = \epsilon r$ for $r \in \mathbb{Z}$.

The total descendent potential $\mathcal{D}^{M_{k,m}}$ satisfies the Hirota quadratic equation [Milanov, Tseng 2008]

For $1 \leqslant a \leqslant k$, let

$$
\begin{aligned}
f_\infty^a =& \frac{1}{k} \sum_{n \geqslant 0} \frac{\lambda^n}{n!} (\log_a \lambda - c_n) \phi^k (-z)^{-n-1} + \frac{1}{k} \sum_{n \geqslant 0} n! \lambda^{-n-1} \phi^k z^n \\
&+ \sum_{\alpha=1}^{k-1} \sum_{n \in \mathbb{Z}} \big(\frac{\alpha}{k} - 1\big)_n \lambda_a^{\alpha/k-n-1} \phi_\alpha (-z)^n.
\end{aligned}
$$

For $k+1 \leqslant b \leqslant k+m$, let

$$
\begin{aligned}
f_\infty^b =& -\frac{1}{m} \sum_{n \geqslant 0} \frac{\lambda^n}{n!} (\log_b(\lambda Q^{-m}) - c_n) dt^k (-z)^{-n-1} - \frac{1}{m} \sum_{n \geqslant 0} n! \lambda^{-n-1} dt^k z^n \\
&- \sum_{\alpha=1}^{m} \sum_{n \in \mathbb{Z}} \big(\frac{\alpha}{m} - 1\big)_n \lambda_b^{\alpha/m-n-1} \frac{\partial}{\partial t^{k+m-\alpha}} (-z)^n
\end{aligned}
$$

Remark on GW

[Milanov, Tseng 2008]

$\mathcal{C}_{k,m}$ - $\mathbb{C}P^1$ orbifold with two orbifold points of order k, m

- $M_{k,m}$ is isomorphic to the Frobenius manifold $QH^*_{orb}(\mathcal{C}_{k,m})$
- Conjecture: the total descendent potential $\mathcal{D}^{M_{k,m}}$ coincides with the generating function of orbifold Gromov-Witten invariants of $\mathcal{C}_{k,m}$

We prove: $\mathcal{D}^{M_{k,m}}$ is a tau function of EBTH

§3. From Hirota to Lax for EBTH

Symbol of dressing operator

Consider first:

$$\Gamma^{\delta\#}\Gamma^a\tau = \Gamma^{\delta\#} \cdot e^{\widehat{(f^a)_-}} \cdot e^{\widehat{(f^a)_+}}\tau$$

$$\mathcal{P}_1(\lambda_a^{1/k}) = \frac{e^{\widehat{(f^a)_+}}\tau}{\tau}$$

Explicitly

$$\mathcal{P}_1(\zeta) = \frac{1}{\tau}\exp\left(\frac{m}{k}\epsilon\sum_{n\geqslant 0} n!\zeta^{-nk-k}\frac{\partial}{\partial q_n^{k+m}} + \right.$$
$$\left. + \epsilon\sum_{n\geqslant 0}\sum_{\alpha\geqslant 1}^{k-1}\left(n-\frac{\alpha}{k}\right)_n \zeta^{\alpha-nk-k}\frac{\partial}{\partial q_n^\alpha}\right)\tau$$

- a-dependence **only** through $\lambda_a^{1/k}$!
- no shift in q_n^k: bigger arbitrarity in τ !

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- a-dependence **only** through $\lambda_a^{1/k}$!
- no shift in q_n^k: bigger arbitrarity in τ !

$$\Gamma^{\delta\#}\Gamma^{a}\tau = \exp(x\frac{\partial}{\partial q_0^k}) \cdot \exp\Big(\sum_{n>0}\frac{\lambda^n}{n!}q_n^k\partial_x\Big)\cdot$$

$$\cdot \exp\Big(-\frac{1}{\epsilon k}\sum_{n\geqslant 0}\big(\sum_{\alpha=1}^{k-1}\big(\frac{\alpha}{k}-1\big)_{-n-1}\lambda_a^{\alpha/k+n}q_n^{k-\alpha}+\frac{\lambda^n}{n!}(\log\lambda_a - c_n)q_n^k\big)\Big)\cdot$$

$$\cdot\, \tau\mathcal{P}_1(\lambda_a^{1/k})$$

$$
\begin{aligned}
\Gamma^{\delta\#}\Gamma^{a}\tau = {} & \tau'\mathcal{P}_1'(\lambda_a^{1/k})\cdot \\
& \cdot \exp\Big(-\frac{1}{\epsilon k}\sum_{n\geqslant 0}\sum_{\alpha=1}^{k-1}\big(\frac{\alpha}{k}-1\big)_{-n-1}\lambda_a^{\alpha/k+n} q_n^{k-\alpha}\Big)\cdot \\
& \cdot \exp\Big(-\frac{1}{\epsilon k}\sum_{n>0}\frac{\lambda^n}{n!}(\log\lambda_a - c_n) q_n^k\Big)\cdot \\
& \cdot \lambda_a^{-\frac{q_0^k+x}{\epsilon k}} \cdot \exp\Big(\sum_{n>0}\frac{\lambda^n}{n!} q_n^k \partial_x\Big)
\end{aligned}
$$

No more $\log \lambda_a$ dependence:

$$\Gamma^{\delta\#}\Gamma^a\tau = \tau' \mathcal{P}_1'(\lambda_a^{1/k}) \cdot \exp\Big(-\frac{1}{\epsilon k} \sum_{n \geqslant 0} \sum_{\alpha=1}^{k-1} \big(\frac{\alpha}{k} - 1\big)_{-n-1} \lambda_a^{\alpha/k+n} q_n^{k-\alpha} \Big) \cdot$$

$$\cdot \exp\Big(\frac{1}{\epsilon} \sum_{n>0} \frac{\lambda^n}{n!} (\epsilon \partial_x + \frac{1}{k} c_n) q_n^k) \Big) \cdot \lambda_a^{-\frac{q_0^k + x}{\epsilon k}}$$

Define the wave function $\mathcal{W}_1$ and get:

$$\Gamma^{\delta\#}\Gamma^a\tau = \tau' \mathcal{W}_1(\lambda_a^{1/k}) \lambda_a^{-\frac{q_0^k + x}{\epsilon k}}$$

In a similar way, we prove:

$$\Gamma^{\delta}\Gamma^{-a}\tau = \lambda_a^{\frac{q_0^k+x}{\epsilon k}} \mathcal{W}_1^*(\lambda_a^{1/k})\tau',$$

$$\Gamma^{\delta\#}\Gamma^{b}\tau = \tau' \mathcal{W}_2(\lambda_b^{1/m})(\lambda_b Q^{-m})^{\frac{q_0^k+x}{\epsilon m}},$$

$$\Gamma^{\delta}\Gamma^{-b}\tau = (\lambda_b Q^{-m})^{-\frac{q_0^k+x}{\epsilon m}} \mathcal{W}_2^*(\lambda_b^{1/m})\tau'.$$

Wavefunction Hirota quadratic equation

Substitution in Hirota equation gives:

$$\frac{1}{k}\sum_{a=1}^{k}\mathcal{W}_1(\lambda_a^{1/k})\mathcal{W}_1^*(\lambda_a^{1/k})\lambda_a^{(r-k+1)/k}d\lambda-$$

$$-\frac{Q^{r+1}}{m}\sum_{b=k+1}^{k+m}\mathcal{W}_2(\lambda_b^{\frac{1}{m}})\mathcal{W}_2^*(\lambda_b^{\frac{1}{m}})\lambda_b^{(-m-r-1)/m}d\lambda$$

- change to the variable ζ
- express regularity by residues on ζ

$$\mathrm{Res}_\zeta\,\mathcal{W}_1(\zeta)\mathcal{W}_1^*(\zeta)\zeta^{ks+r}d\zeta=\mathrm{Res}_\zeta\,\mathcal{W}_2(\zeta)\mathcal{W}_2^*(\zeta)\zeta^{ms-r-2}d\zeta$$

Difference operators

$$A = \sum_s a_s \Lambda^s = \sum_s \Lambda^s \tilde{a}_s$$

left and right symbols:

$$\sigma_l(A) = \sum_s a_s \zeta^s, \quad \sigma_r(A) = \sum_s \tilde{a}_s \zeta^s$$

"Fundamental lemma":

$$\operatorname{Res}_\zeta \sigma_l(A)\sigma_r(B)\frac{d\zeta}{\zeta} = \operatorname{Res}_\Lambda AB$$

for any two difference operators A and B.

Waveoperators Hirota equation

$$\mathrm{Res}_\zeta\, \mathcal{W}_1(\zeta)\mathcal{W}_1^*(\zeta)\zeta^{ks+r}d\zeta = \mathrm{Res}_\zeta\, \mathcal{W}_2(\zeta)\mathcal{W}_2^*(\zeta)\zeta^{ms-r-2}d\zeta$$

"Quantize" the symbols $\mathcal{P}_i$, $\mathcal{W}_i$ to difference operators P_i, W_i (and their duals).

Wavefunction Hirota becomes

$$W_1(q',x)\Lambda^{-ks-1}W_1^*(q'',x) = W_2(q',x)\Lambda^{ms-1}W_2^*(q'',x)$$

for $(q_0^k)' = (q_0^k)''$ and $s \geqslant 0$

Consequences...

$$W_1(q',x)\Lambda^{-ks-1}W_1^*(q'',x) = W_2(q',x)\Lambda^{ms-1}W_2^*(q'',x)$$

for $(q_0^k)' = (q_0^k)''$ and $s \geqslant 0$

1. $s = 0$ and $q' = q''$ $\implies P_1\Lambda^{-1}P_1^* = P_2\Lambda^{-1}P_2^*$
   $\implies P_i^*(x-\epsilon)$ is the inverse of P_i
2. $s = 1$ and $q' = q''$ $\implies P_1\Lambda^{-k}P_1^{-1} = P_2\Lambda^m P_2^{-1} =: L$
   $\implies L = \Lambda^{-k} + \cdots + u_m\Lambda^m$
3. differentiate w.r.t. q_n^α and set $s = 0$, $q' = q''$
   $\implies$ Sato equations for EBTH

Sato equations for EBTH

$$\frac{\partial P_1}{\partial q_n^\beta} = \left(B_n^\beta\right)_{>0} P_1, \quad \frac{\partial P_2}{\partial q_n^\beta} = -\left(B_n^\beta\right)_{\leq 0} P_2,$$

$$\begin{aligned}
B_n^\beta &= \frac{1}{\epsilon k}(-\frac{\beta}{k})_{-n-1} L^{n-\frac{\beta}{k}+1} & 1 \leqslant \beta \leqslant k-1,\\
B_n^\beta &= \frac{1}{\epsilon m}(\frac{\beta-k}{m}-1)_{-n-1} L^{\frac{\beta-k}{m}+n} & k+1 \leqslant \beta \leqslant k+m,\\
B_n^k &= \frac{2}{\epsilon n!}\left(L^n\left(\log L - \frac{c_n}{2}(\frac{1}{k}+\frac{1}{m})\right)\right).
\end{aligned}$$

Conversely, with a solution of EBTH we associate a tau function that solves the MT Hirota equation.

Thanks!

Periods

functions $I_a^{(l)}(\lambda, t) \in H$, $l \in \mathbb{Z}$ on $M \times \mathbb{C}^* \backslash \Delta$ which satisfy, in particular

$$\partial_\lambda I_a^{(l)} = I_a^{(l+1)}$$

in this case:

$$(I_a^{(-p)}(\lambda))_\alpha = -\partial_\alpha \left[d^{-1} \left(\frac{(\lambda - \lambda(x))^p}{p!} \frac{dx}{x} \right)_{|x=x_a} \right] \quad p \geqslant 0$$

$x_a = x_a(t, \lambda)$ multivalued function on $M \times \mathbb{C}^* \backslash \Delta$
a - choice of point x_a in preimage $\lambda^{-1}(\lambda_0)$ at reference point (t_0, λ_0)