CENTRE
DE RECHERCHES
MATHÉMATIQUES

KAM-theory for PDE with one and many space-variables

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§1. Finite dimension

A) Consider $\mathbb{R}^n_p \times \mathbb{T}^n_q$, $\omega_2 = dp \wedge dq$,
$U \in \mathbb{R}^n_p$, $h(p)$ – analytic funct. on U.
Corresponding hamiltonian system:

$$\dot{p} = 0, \qquad \dot{q} = \nabla_p h(p). \qquad (1)$$

Consider perturbed Hamiltonian
$h(p) + \varepsilon h_1(p,q)$, h_1 – analytic.
Corresponding system is:

$$\dot{p} = -\varepsilon \nabla_q h_1, \quad \dot{q} = \nabla_p h(p) + \varepsilon \nabla_p h_1 \qquad (2)$$

Tori $T_p = \{p\} \times \mathbb{T}^n$, $p \in U$, are invar. for (1) and filled in with time-QP solutions of (1)

KAM-Theorem A. For most $p \in U$, eq. (2) has an invar. n-torus T_p^ε, s.t. $\operatorname{dist}(T_p^\varepsilon, T_p) \lesssim \varepsilon^a$, $a > 0$. It is filled in with time-QP solutions of (2).

"for most p" = "for all $p \in U^\varepsilon$, where $\operatorname{mes}(U \setminus U^\varepsilon) \lesssim \varepsilon^b$, $b > 0$".

B) Consider
$(\mathbb{R}^n_x \times \mathbb{R}^n_y,\ \omega_2 = dx \wedge dy)$,
$h_0(x,y) = \frac{1}{2} \sum \lambda_j (x_j^2 + y_j^2)$,
$\lambda = (\lambda_1, \dots, \lambda_n) \in Q \Subset \mathbb{R}^n$. – parameter.
Corresponding equation:

$$\dot{x}_j = -\lambda_j y_j, \quad \dot{y}_j = \lambda_j x_j \qquad (3)$$

This is a linear ham. eq., depending on parameter. Fix any $I = (I_1, \dots, I_n) \in \mathbb{R}^n_+$.
The torus

$$T(I) = \{(x,y) \mid \tfrac{1}{2}(x_j^2 + y_j^2) = I_j \ \forall j\}$$

is invar. for (3), filled in with time-QP solutions

Consider small ham. perturbation:

$$\dot{x}_j = -\lambda_j y_j - \varepsilon \frac{\partial h_1}{\partial y_j}, \quad \dot{y}_j = \lambda_j x_j + \varepsilon \frac{\partial h_1}{\partial y_j}. \qquad (4)$$

KAM – Theorem B. For most $\lambda \in Q$, eq. (4) has an invar. n-torus $T^\varepsilon(I)$, which is ε^a-close to $T(I)$, and is filled in with time-QP solutions of (4)

Thm A = Thm B

§2. 1d ham. PDE

(Analogy of Thm B. Example: 1d NLS)

$$\dot{u}(t,x) = i(-u_{xx} + V(x)u), \quad x \in (0,1) \qquad (2.1)$$
$$u(0) = u(1) = 0.$$

This is a linear ham. PDE:

$\dot{u} = i\, \nabla h(u)$, $\quad h = \frac{1}{2}\int_0^1 (u_x^2 + V(x)u^2)\, dx.$

Consider $A = -\frac{\partial^2}{\partial x^2} + V(x)$; let $\varphi_1(x), \varphi_2(x), \ldots$ be its eigenfunctions with eigenvalues $\lambda_1, \lambda_2, \ldots$.

Solutions of (2,1) are

$u(t,x) = \sum_{j=1}^{\infty} u_j\, e^{i\lambda_j t}\, \varphi_j(x)$. They are almost-periodic in t.

Fix any $n < \infty$, and $I = (I_1, \ldots, I_n) \in \mathbb{R}^n_+$.

Set

$$T^n(I) = \left\{ \sum_{j=1}^{n} u_j \varphi_j(x) \;\middle|\; \tfrac{1}{2}|u_j|^2 = I_j,\ j = 1, 2, \ldots, n \right\}.$$

This is n-torus, invar. for (2.1), filled in with time-QP solutions.

Let $V = V(x; a)$, $a \in Q \in \mathbb{R}^n$.

Then $A = A_a$, $\varphi_j = \varphi_j(x,a)$, $\lambda_j(a)$, $T^n_a(I)$.

Assume that $V(x;a)$ as a function of a is in a general position

Consider small ham. perturbation:

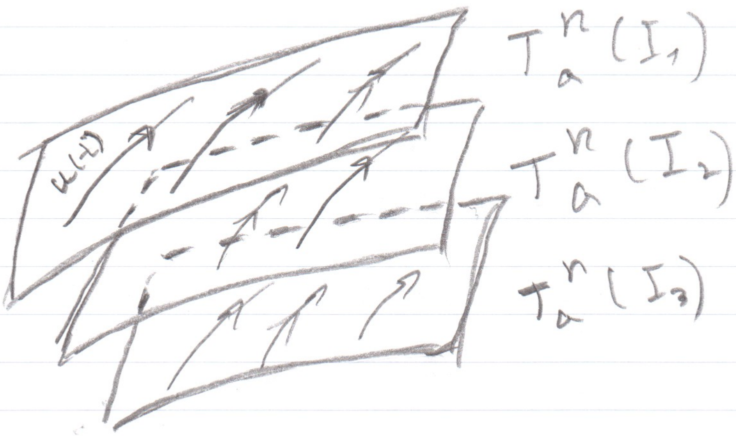
$$\dot u(t,x) = i\left(-u_{xx} + V(x;a)u + \varepsilon f(|u|^2, x)u\right), \qquad (2.2)$$

$$u(0) = u(1) = 0.$$

f – real function, analytic in $|u|^2$, smooth in x.

Theorem 2B. 1) For most $a \in U$, eq. (2.2) has an invar. torus $T_a^{n\,\varepsilon}(I)$ which is ε^b-close to $T_a^n(I)$. It is filled in with time-QP solutions.

2) All Lyapunov exponents of these solutions are zero.



„Proof"

Run the KAM-procedure. There will be small divisors. The worst of them are

$$\lambda_{j_1}(a) - \lambda_{j_2}(a) + \sum_{k=1}^{n} \lambda_k(a) \cdot s_k ,$$

$j_1 > j_2 > n$; s_k – integers.

But $\lambda_j \sim j^2$. So $\lambda_{j_1} - \lambda_{j_2} \gtrsim j_2$.

That is, only few of these divisors are small. Excluding "bad" parameters a, we achieve that these few also are $\gtrsim 1$.

So the KAM-machine runs!

References: SK 1987, 1988
C. E. Wayne 1990

This holds for MANY other 1d hamiltonian PDE. Incl. perturbations of KdV. (SK, 1988) (this corresponds to Thm A).

§ 3. nd hamiltonian PDE.

Analogy of Thm B; example

$$\dot{u}(t,x) = i(-\Delta u + V(x,a) * u) + \varepsilon i f(|u|^2, x) u, \quad x \in \mathbb{T}^d \qquad (3.1)$$

$a \in Q \Subset \mathbb{R}^n$ – parameter.

$\lambda_s(a)$, $s \in \mathbb{Z}^d$ – eigenvalues; eigenfunctions are $e^{is\cdot x}$.

Choose any n nodes $s_1, \dots, s_n \in \mathbb{Z}^d$.

Consider

$$T_a^n(I) = \Big\{ \sum_{k=1}^n u_k e^{is_k \cdot x} \;\Big|\; \tfrac{1}{2}|u_k|^2 = I_k \ \forall k = 1, \dots, n \Big\}.$$

This torus is invariant for linear eq. $(3.1)|_{\varepsilon=0}$, is filled in with time-QP solutions.

Thm 3B. 1) For most $a \in Q$, eq. (3.1) has an invariant n-torus, which is ε^b-close to $T_a^n(I)$, filled in with time-QP solutions.

2) Their Lyapunov exponents are 0.

REFERENCES. 1) – J. Bourgain, 98, 04

1)+2) – SK & H. Eliasson, 2010.

This is HARD, and is not general. Why?

In (HE-SK) proof, the worst small divisors are

$$\lambda_{m_1}(a) - \lambda_{m_2}(a) + \sum_{k=1}^{n} \lambda_{s_k}(a) \cdot \ell_k,$$

$m_1, m_2 \in \mathbb{Z}^d \setminus \{s_1, s_2, \dots, s_n\}$, ℓ_k - integers.

Now, $\lambda_m \sim |m|^2$. So, ∞-often $\lambda_{m_1} - \lambda_{m_2} \simeq 0$, and we have ∞-many VERY small divisors!

This is HARD!

Results of M. Berti.

Can we work with eq. with multiplicative potential? — Yes, a bit. M. Berti used ideas of Bourgain to prove KAM for

$$\dot{u}(t,x) = i(-\Delta u + V(x)u) +$$
$$+ \varepsilon i f(\omega t; |u|^2, x)\, u; \quad x \in \mathbb{T}^d.$$

$\omega \in \mathbb{R}^n$ — parameter of the problem

Analogy of Thm A.

Consider n/e beam equation:

$$\ddot{u} = -\Delta^2 u - au + \gamma u^3, \qquad (3.2)$$
$$x \in \mathbb{T}^d, \quad 0 < a \le 1$$

Eigenvalues for $(3.2)_{\gamma=0}$: $\lambda_s = \sqrt{|s|^4 + a}$, $s \in \mathbb{Z}^d$.

Choose any n nodes $s_1, \dots, s_n$, such that $|s_j| \ne |s_k|$ if $j \ne k$.
Consider correspond. invar. n-torus $T_a^n(I)$ for linear eq. $(3.2)_{\gamma=0}$.
Assume $|I| = \varepsilon \ll 1$

Thm 3A (B. Grebert, HE, SK, 2012 – ...)

1) for most of $a \in (0,1)$ eq. $(3.2)_{\gamma=1}$ has an invar. n-torus, ε^6-close to $T_a^n(I)$. It is filled in with time – QP solutions of (3.2).

2) If $d \le 2$, then all their Lyapunov exponents are 0.

3). If $d \ge 3$, then for some choise of nodes $s_1, \dots, s_n$ these solutions have positive Lyapunov exponents.

Problem: Do the same for n/e wave eq.

$$\ddot{u} = \Delta u - au + u^3, \quad x \in \mathbb{T}^d!$$