Intersections of quadrics and Hamiltonian-minimal Lagrangian submanifolds

based on joint work with Andrey Mironov

Taras Panov

Lomonosov Moscow State University

International Conference "Analysis and Singularities"
dedicated to the 75th anniversary of V. I. Arnold
Steklov Institute, Moscow, 17-21 December 2012

Motivation

Intersection theory for Hamiltonian deformations of Lagrangian submanifds

Motivation

Intersection theory for Hamiltonian deformations of Lagrangian submanifds

Conjecture (Arnold–Givental)

Let $i_t(N)$ be a transversal Hamiltonian deformation of a Lagrangian submanifold $N \subset M$, $\dim N = n$. Then for any $t > 0$,

$$\#(N \cap i_t(N)) \geqslant \sum_{k=0}^{n} b_k(N; \mathbb{Z}_2).$$

Motivation

Intersection theory for Hamiltonian deformations of Lagrangian submanifds

Conjecture (Arnold–Givental)

Let $i_t(N)$ be a transversal Hamiltonian deformation of a Lagrangian submanifold $N \subset M$, $\dim N = n$. Then for any $t > 0$,

$$\#(N \cap i_t(N)) \geqslant \sum_{k=0}^{n} b_k(N;\mathbb{Z}_2).$$

This is a generalisation of Arnold conjecture on the minimum number of fixed points for a Hamiltonian symplectomorphism.

Motivation

Intersection theory for Hamiltonian deformations of Lagrangian submanifds

Conjecture (Arnold–Givental)

Let $i_t(N)$ be a transversal Hamiltonian deformation of a Lagrangian submanifold $N \subset M$, $\dim N = n$. Then for any $t > 0$,

$$\#(N \cap i_t(N)) \geqslant \sum_{k=0}^{n} b_k(N; \mathbb{Z}_2).$$

This is a generalisation of Arnold conjecture on the minimum number of fixed points for a Hamiltonian symplectomorphism.

Inspired by some progress related to these conjectures in the early 1990s, Y.-G. Oh initiated the study of the stability properties of Lagrangian submanifolds under Hamiltonian deformations in Kähler manifolds. This led to the notion of Hamiltonian minimality (H-minimality), the symplectic analogue of the minimality in Riemannian geometry.

Basics

Let M be a Kähler manifold with symplectic form ω, $\dim_{\mathbb{R}} M = 2n$.

Basics

Let M be a Kähler manifold with symplectic form ω, $\dim_{\mathbb{R}} M = 2n$.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

Basics

Let M be a Kähler manifold with symplectic form ω, $\dim_{\mathbb{R}} M = 2n$.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

A vector field ξ on M is Hamiltonian if the 1-form $\omega(\,\cdot\,,\xi)$ is exact.

Basics

Let M be a Kähler manifold with symplectic form ω, $\dim_{\mathbb{R}} M = 2n$.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

A vector field ξ on M is Hamiltonian if the 1-form $\omega(\,\cdot\,,\xi)$ is exact.

A Lagrangian immersion $i\colon N \looparrowright M$ is Hamiltonian minimal (H-minimal) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, i.e.

$$\frac{d}{dt}\,\mathrm{vol}(i_t(N))\big|_{t=0} = 0,$$

where $i_t(N)$ is a Hamiltonian deformation of $i(N) = i_0(N)$.

Overview

Explicit examples of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{C}P^m$ were constructed in the work of Yong-Geun Oh, Castro–Urbano, Hélein–Romon, Amarzaya–Ohnita, among others.

Overview

Explicit examples of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{C}P^m$ were constructed in the work of Yong-Geun Oh, Castro–Urbano, Hélein–Romon, Amarzaya–Ohnita, among others.

In 2003 Mironov suggested a universal construction providing an H-minimal Lagrangian immersion in $\mathbb{C}^m$ from an intersection of special real quadrics.

Overview

Explicit examples of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{C}P^m$ were constructed in the work of Yong-Geun Oh, Castro–Urbano, Hélein–Romon, Amarzaya–Ohnita, among others.

In 2003 Mironov suggested a universal construction providing an H-minimal Lagrangian immersion in $\mathbb{C}^m$ from an intersection of special real quadrics.

The same intersections of real quadrics are known to toric geometers and topologists as (real) moment-angle manifolds. They appear, for instance, as the level sets of the moment map in the construction of Hamiltonian toric manifolds via symplectic reduction.

Overview

Explicit examples of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{C}P^m$ were constructed in the work of Yong-Geun Oh, Castro–Urbano, Hélein–Romon, Amarzaya–Ohnita, among others.

In 2003 Mironov suggested a universal construction providing an H-minimal Lagrangian immersion in $\mathbb{C}^m$ from an intersection of special real quadrics.

The same intersections of real quadrics are known to toric geometers and topologists as (real) moment-angle manifolds. They appear, for instance, as the level sets of the moment map in the construction of Hamiltonian toric manifolds via symplectic reduction.

Here we combine Mironov's construction with the methods of toric topology to produce new examples of H-minimal Lagrangian **embeddings** with interesting and complicated topology.

A convex polytope in $\mathbb{R}^n$ is obtained by intersecting m halfspaces:

$$P = \bigl\{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\bigr\}.$$

Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is obtained by intersecting m halfspaces:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\}.$$

Suppose each $F_i = P \cap \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0\}$ is a facet (m facets in total).

Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is obtained by intersecting m halfspaces:

$$P = \bigl\{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \dots, m\bigr\}.$$

Suppose each $F_i = P \cap \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\}$ is a facet (m facets in total).

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \bigl(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \dots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\bigr).$$

Polytopes and moment-angle manifolds

A convex polytope in $\mathbb{R}^n$ is obtained by intersecting m halfspaces:

$$P = \bigl\{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\bigr\}.$$

Suppose each $F_i = P \cap \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\}$ is a facet (m facets in total).

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \bigl(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\bigr).$$

Then i_P is monomorphic, and $i_P(P) \subset \mathbb{R}^m$ is the intersection of an n-plane with $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \stackrel{i_Z}{\longrightarrow} & \mathbb{C}^m & \qquad & (z_1,\ldots,z_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \stackrel{i_P}{\longrightarrow} & \mathbb{R}^m_\geqslant & & (|z_1|^2,\ldots,|z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & & (z_1,\ldots,z_m) \\
\downarrow & & \downarrow \mu & & \downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2,\ldots,|z_m|^2)
\end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

A polytope P is simple if exactly $n = \dim P$ facets meet at each vertex.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \stackrel{i_Z}{\longrightarrow} & \mathbb{C}^m \\ \downarrow & & \downarrow \mu \\ P & \stackrel{i_P}{\longrightarrow} & \mathbb{R}^m_{\geqslant} \end{array} \qquad \begin{array}{c} (z_1, \ldots, z_m) \\ \downarrow \\ (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

A polytope P is simple if exactly $n = \dim P$ facets meet at each vertex.

Proposition

If P is simple, then $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1,\ldots,z_m) \\ \downarrow & & \downarrow \mu & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant & & (|z_1|^2,\ldots,|z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

A polytope P is simple if exactly $n = \dim P$ facets meet at each vertex.

Proposition

If P is simple, then $\mathcal{Z}_P$ is a smooth manifold of dimension $m+n$.

Proof.

Write $i_P(\mathbb{R}^n)$ by $m-n$ linear equations in $(y_1,\ldots,y_m) \in \mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by quadrics. □

$\mathcal{Z}_P$ is the moment-angle manifold corresponding to P.

$\mathcal{Z}_P$ is the moment-angle manifold corresponding to P.

Similarly, by considering

$$\begin{array}{ccccc} \mathcal{R}_P & \longrightarrow & \mathbb{R}^m & & (u_1,\ldots,u_m) \\ \downarrow & & \downarrow \mu & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant & & (u_1^2,\ldots,u_m^2) \end{array}$$

we obtain the real moment-angle manifold $\mathcal{R}_P$.

$\mathcal{Z}_P$ is the moment-angle manifold corresponding to P.

Similarly, by considering

$$\begin{array}{ccccc}
\mathcal{R}_P & \longrightarrow & \mathbb{R}^m & & (u_1,\ldots,u_m) \\
\downarrow & & \downarrow \mu & & \downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (u_1^2,\ldots,u_m^2)
\end{array}$$

we obtain the real moment-angle manifold $\mathcal{R}_P$.

Example

$P = \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -\gamma_1 x_1 - \gamma_2 x_2 + 1 \geqslant 0\}$, $\gamma_1, \gamma_2 > 0$ (a 2-simplex). Then
$\mathcal{Z}_P = \{(z_1,z_2,z_3) \in \mathbb{C}^3 \colon \gamma_1|z_1|^2 + \gamma_2|z_2|^2 + |z_3|^2 = 1\}$ (a 5-sphere),
$\mathcal{R}_P = \{(u_1,u_2,u_3) \in \mathbb{R}^3 \colon \gamma_1|u_1|^2 + \gamma_2|u_2|^2 + |u_3|^2 = 1\}$ (a 2-sphere).

Torus actions

Have intersections of quadrics

$$\mathcal{Z}_P = \{\boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \gamma_1 |z_1|^2 + \cdots + \gamma_m |z_m|^2 = c\},$$
$$\mathcal{R}_P = \{\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c\}$$

where $\gamma_1, \ldots, \gamma_m$ and c are vectors in $\mathbb{R}^{m-n}$.

Torus actions

Have intersections of quadrics

$$\mathcal{Z}_P = \{ \boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \gamma_1 |z_1|^2 + \cdots + \gamma_m |z_m|^2 = c \},$$
$$\mathcal{R}_P = \{ \boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c \}$$

where $\gamma_1, \ldots, \gamma_m$ and c are vectors in $\mathbb{R}^{m-n}$.

Assume that the polytope P is rational. Then have two lattices:
$\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle \subset \mathbb{R}^n$ and $L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}$.

Torus actions

Have intersections of quadrics

$$\mathcal{Z}_P = \{\boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \gamma_1|z_1|^2 + \cdots + \gamma_m|z_m|^2 = c\},$$
$$\mathcal{R}_P = \{\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c\}$$

where $\gamma_1, \ldots, \gamma_m$ and c are vectors in $\mathbb{R}^{m-n}$.

Assume that the polytope P is rational. Then have two lattices:
$\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle \subset \mathbb{R}^n$ and $L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}$.

Consider the $(m-n)$-torus $T_P = \bigl\{\bigl(e^{2\pi i\langle \gamma_1, \varphi\rangle}, \ldots, e^{2\pi i\langle \gamma_m, \varphi\rangle}\bigr) \in \mathbb{T}^m\bigr\}$,
i.e. $T_P = \mathbb{R}^{m-n}/L^*$, and set $D_P = \frac{1}{2}L^*/L^* \cong (\mathbb{Z}_2)^{m-n}$.

Torus actions

Have intersections of quadrics

$$\mathcal{Z}_P = \{\boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \gamma_1 |z_1|^2 + \cdots + \gamma_m |z_m|^2 = c\},$$
$$\mathcal{R}_P = \{\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c\}$$

where $\gamma_1, \ldots, \gamma_m$ and c are vectors in $\mathbb{R}^{m-n}$.

Assume that the polytope P is rational. Then have two lattices:
$\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle \subset \mathbb{R}^n$ and $L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}$.

Consider the $(m-n)$-torus $T_P = \big\{ \big(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle} \big) \in \mathbb{T}^m \big\}$,
i.e. $T_P = \mathbb{R}^{m-n}/L^*$, and set $D_P = \frac{1}{2} L^*/L^* \cong (\mathbb{Z}_2)^{m-n}$.

Proposition

The $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ almost freely.

Main construction

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u}, \varphi) \mapsto \boldsymbol{u} \cdot \varphi = \left(u_1 e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, u_m e^{2\pi i \langle \gamma_m, \varphi \rangle}\right).$$

Main construction

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u}, \varphi) \mapsto \boldsymbol{u} \cdot \varphi = \left(u_1 e^{2\pi i \langle \gamma_1, \varphi\rangle}, \ldots, u_m e^{2\pi i \langle \gamma_m, \varphi\rangle}\right).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.
Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Main construction

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u}, \varphi) \mapsto \boldsymbol{u} \cdot \varphi = (u_1 e^{2\pi i\langle\gamma_1,\varphi\rangle}, \ldots, u_m e^{2\pi i\langle\gamma_m,\varphi\rangle}).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.
Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Lemma

$f\colon \mathcal{R}_P \times T_P \to \mathbb{C}^m$ induces an immersion $j\colon N_P \looparrowright \mathbb{C}^m$.

Main construction

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u},\varphi) \mapsto \boldsymbol{u}\cdot\varphi = \big(u_1 e^{2\pi i\langle\gamma_1,\varphi\rangle},\ldots,u_m e^{2\pi i\langle\gamma_m,\varphi\rangle}\big).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.
Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Lemma

$f\colon \mathcal{R}_P \times T_P \to \mathbb{C}^m$ induces an immersion $j\colon N_P \looparrowright \mathbb{C}^m$.

Theorem (Mironov)

The immersion $j\colon N_P \looparrowright \mathbb{C}^m$ is H-minimal Lagrangian.

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ is an embedding?

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ *is an embedding?*

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \dots \cap F_{i_n}.$$

Theorem

The following conditions are equivalent:

1. *$j\colon N_P \to \mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold;*
2. *the $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ freely.*
3. *P is a Delzant polytope.*

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

Theorem

The following conditions are equivalent:

1. *$j\colon N_P \to \mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold;*
2. *the $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ freely.*
3. *P is a Delzant polytope.*

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

Theorem

The following conditions are equivalent:

1. *$j\colon N_P \to \mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold;*
2. *the $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ freely.*
3. *P is a Delzant polytope.*

Question

When $j\colon N_P \looparrowright \mathbb{C}^m$ is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

Theorem

The following conditions are equivalent:

1. *$j\colon N_P \to \mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold;*
2. *the $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ freely.*
3. *P is a Delzant polytope.*

Get an H-minimal Lagrangian submanifold N_P in $\mathbb{C}^m$ for any Delzant n-polytope P with m facets!

Get an H-minimal Lagrangian submanifold N_P in $\mathbb{C}^m$ for any Delzant n-polytope P with m facets!

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:

Get an H-minimal Lagrangian submanifold N_P in $\mathbb{C}^m$ for any Delzant n-polytope P with m facets!

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:

- simplices and cubes in all dimensions;

Get an H-minimal Lagrangian submanifold N_P in $\mathbb{C}^m$ for any Delzant n-polytope P with m facets!

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:

- simplices and cubes in all dimensions;
- products and face truncations;

Get an H-minimal Lagrangian submanifold N_P in $\mathbb{C}^m$ for any Delzant n-polytope P with m facets!

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:

- simplices and cubes in all dimensions;
- products and face truncations;
- associahedra (Stasheff polytopes), permutahedra, and generalisations.

Examples

Example (one quadric)

Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$.

Examples

Example (one quadric)

Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$.
$\mathcal{R}_{\Delta^{m-1}}$ is given by a single quadric

$$\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$$

with $\gamma_i > 0$, i.e. $\mathcal{R}_{\Delta^{m-1}} \cong S^{m-1}$.

Examples

Example (one quadric)

Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$.
$\mathcal{R}_{\Delta^{m-1}}$ is given by a single quadric

$$\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$$

with $\gamma_i > 0$, i.e. $\mathcal{R}_{\Delta^{m-1}} \cong S^{m-1}$.
Then

$$N \cong S^{m-1} \times_{\mathbb{Z}_2} S^1 \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orient. of } S^{m-1}, \\ \mathcal{K}^m & \text{if } \tau \text{ reverses the orient. of } S^{m-1}, \end{cases}$$

where τ is the involution and $\mathcal{K}^m$ is an m-dimensional Klein bottle.

Proposition (one quadric)

We obtain an H-minimal Lagrangian embedding of $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}_2} S^1$ *in* $\mathbb{C}^m$ *whenever* $\gamma_1 = \cdots = \gamma_m$ *in* $\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$.

Proposition (one quadric)

We obtain an H-minimal Lagrangian embedding of $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}_2} S^1$ *in* $\mathbb{C}^m$ *whenever* $\gamma_1 = \cdots = \gamma_m$ *in* $\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$.
The topology of $N_{\Delta^{m-1}} = N(m)$ *depends on the parity of* m*:*

$$N(m) \cong S^{m-1} \times S^1 \qquad \text{if } m \text{ is even,}$$
$$N(m) \cong \mathcal{K}^m \qquad \text{if } m \text{ is odd.}$$

Proposition (one quadric)

We obtain an H-minimal Lagrangian embedding of $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}_2} S^1$ in $\mathbb{C}^m$ whenever $\gamma_1 = \cdots = \gamma_m$ in $\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c$.
The topology of $N_{\Delta^{m-1}} = N(m)$ depends on the parity of m:

$$\begin{aligned} N(m) &\cong S^{m-1} \times S^1 && \text{if } m \text{ is even,} \\ N(m) &\cong \mathcal{K}^m && \text{if } m \text{ is odd.} \end{aligned}$$

The Klein bottle $\mathcal{K}^m$ with even m does *not* admit Lagrangian embeddings in $\mathbb{C}^m$ [Nemirovsky, Shevchishin].

Theorem (two quadrics)

Let $m - n = 2$, *i.e.* $P \simeq \Delta^{p-1} \times \Delta^{q-1}$.

Theorem (two quadrics)

Let $m - n = 2$, *i.e.* $P \simeq \Delta^{p-1} \times \Delta^{q-1}$.

- $\mathcal{R}_P$ *is diffeomorphic to* $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ *given by*

$$\begin{aligned} u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \cdots + u_p^2 \qquad\qquad &= 1, \\ u_1^2 + \ldots + u_k^2 \qquad\qquad + u_{p+1}^2 + \cdots + u_m^2 &= 2, \end{aligned}$$

where $p + q = m$, $0 < p < m$ *and* $0 \leqslant k \leqslant p$.

Theorem (two quadrics)

Let $m - n = 2$*, i.e.* $P \simeq \Delta^{p-1} \times \Delta^{q-1}$*.*

- $\mathcal{R}_P$ *is diffeomorphic to* $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ *given by*

$$\begin{aligned} u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \cdots + u_p^2 & & = 1, \\ u_1^2 + \ldots + u_k^2 & + u_{p+1}^2 + \cdots + u_m^2 & = 2, \end{aligned}$$

  where $p + q = m$*,* $0 < p < m$ *and* $0 \leqslant k \leqslant p$*.*

- *If* $N_P \to \mathbb{C}^m$ *is an embedding, then* N_P *is diffeomorphic to*

$$N_k(p,q) = \mathcal{R}(p,q) \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} (S^1 \times S^1),$$

  where the two involutions act on $\mathcal{R}(p,q)$ *by*

  $\psi_1 \colon (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, -u_{k+1}, \ldots, -u_p, u_{p+1}, \ldots, u_m)$*,*
  $\psi_2 \colon (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, u_{k+1}, \ldots, u_p, -u_{p+1}, \ldots, -u_m)$*.*

Theorem (two quadrics)

Let $m - n = 2$, *i.e.* $P \simeq \Delta^{p-1} \times \Delta^{q-1}$.

- $\mathcal{R}_P$ *is diffeomorphic to* $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ *given by*

$$\begin{aligned} u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \cdots + u_p^2 & &= 1,\\ u_1^2 + \ldots + u_k^2 & + u_{p+1}^2 + \cdots + u_m^2 &= 2, \end{aligned}$$

where $p + q = m$, $0 < p < m$ *and* $0 \leqslant k \leqslant p$.

- *If* $N_P \to \mathbb{C}^m$ *is an embedding, then* N_P *is diffeomorphic to*

$$N_k(p,q) = \mathcal{R}(p,q) \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} (S^1 \times S^1),$$

where the two involutions act on $\mathcal{R}(p,q)$ *by*

$\psi_1 \colon (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, -u_{k+1}, \ldots, -u_p, u_{p+1}, \ldots, u_m)$,

$\psi_2 \colon (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, u_{k+1}, \ldots, u_p, -u_{p+1}, \ldots, -u_m)$.

There is a fibration $N_k(p,q) \to S^{q-1} \times_{\mathbb{Z}_2} S^1 = N(q)$ *with fibre* $N(p)$.

Example (three quadrics)

In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

Example (three quadrics)

In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest case is $n = 2$ and $m = 5$: a Delzant pentagon.

Example (three quadrics)

In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest case is $n = 2$ and $m = 5$: a Delzant pentagon.

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3 \times S^4$.

Example (three quadrics)

In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest case is $n = 2$ and $m = 5$: a Delzant pentagon.

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3 \times S^4$.

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Proposition

Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.

Proposition

Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^m$ which is the total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters into the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

Proposition

Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^m$ which is the total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters into the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}_P$ and $\mathcal{Z}_P$ is even more complicated.

Generalisation to toric manifolds

Consider 2 sets of quadrics:

$$\mathcal{Z}_\Gamma = \{\boldsymbol{z} \in \mathbb{C}^m \colon \sum\nolimits_{k=1}^{m} \gamma_k |z_k|^2 = \boldsymbol{c}\}, \quad \gamma_k, \boldsymbol{c} \in \mathbb{R}^{m-n};$$
$$\mathcal{Z}_\Delta = \Big\{\boldsymbol{z} \in \mathbb{C}^m \colon \sum\nolimits_{k=1}^{m} \delta_k |z_k|^2 = \boldsymbol{d}\Big\}, \quad \delta_k, \boldsymbol{d} \in \mathbb{R}^{m-\ell};$$

s. t. the polytopes corresponding to $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ *and* $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ are Delzant.

Generalisation to toric manifolds

Consider 2 sets of quadrics:

$$\mathcal{Z}_\Gamma = \{\boldsymbol{z} \in \mathbb{C}^m \colon \textstyle\sum_{k=1}^m \gamma_k |z_k|^2 = \boldsymbol{c}\}, \quad \gamma_k, \boldsymbol{c} \in \mathbb{R}^{m-n};$$

$$\mathcal{Z}_\Delta = \Big\{\boldsymbol{z} \in \mathbb{C}^m \colon \textstyle\sum_{k=1}^m \delta_k |z_k|^2 = \boldsymbol{d}\Big\}, \quad \delta_k, \boldsymbol{d} \in \mathbb{R}^{m-\ell};$$

s. t. the polytopes corresponding to $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ *and* $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ are Delzant.

Define $\mathcal{R}_\Gamma$, $T_\Gamma \cong \mathbb{T}^{m-n}$, $D_\Gamma \cong \mathbb{Z}_2^{m-n}$, $\mathcal{R}_\Delta$, $T_\Delta \cong \mathbb{T}^{m-\ell}$, $D_\Delta \cong \mathbb{Z}_2^{m-\ell}$ as before.

Generalisation to toric manifolds

Consider 2 sets of quadrics:

$$\mathcal{Z}_\Gamma = \{\boldsymbol{z} \in \mathbb{C}^m \colon \sum_{k=1}^m \gamma_k |z_k|^2 = \boldsymbol{c}\}, \quad \gamma_k, \boldsymbol{c} \in \mathbb{R}^{m-n};$$
$$\mathcal{Z}_\Delta = \Big\{\boldsymbol{z} \in \mathbb{C}^m \colon \sum_{k=1}^m \delta_k |z_k|^2 = \boldsymbol{d}\Big\}, \quad \delta_k, \boldsymbol{d} \in \mathbb{R}^{m-\ell};$$

s. t. the polytopes corresponding to $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ *and* $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ are Delzant.

Define $\mathcal{R}_\Gamma$, $T_\Gamma \cong \mathbb{T}^{m-n}$, $D_\Gamma \cong \mathbb{Z}_2^{m-n}$, $\mathcal{R}_\Delta$, $T_\Delta \cong \mathbb{T}^{m-\ell}$, $D_\Delta \cong \mathbb{Z}_2^{m-\ell}$ as before.

The idea is to use the first set of quadrics to produce a toric manifold M via symplectic reduction, and then use the second set of quadrics to define an H-minimal Lagrangian submanifold in M.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M$, $\quad \dim N = n$.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M, \quad \dim N = n.$

Theorem

N is an H-minimal Lagrangian submanifold in M.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M, \quad \dim N = n$.

Theorem

N is an H-minimal Lagrangian submanifold in M.

Idea of proof.

□

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M, \quad \dim N = n$.

Theorem

N is an H-minimal Lagrangian submanifold in M.

Idea of proof.

Consider $\widetilde{M} := M /\!\!/ T_\Delta = (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta)$.

□

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M, \quad \dim N = n$.

Theorem

N is an H-minimal Lagrangian submanifold in M.

Idea of proof.

Consider $\widetilde{M} := M /\!\!/ T_\Delta = (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta)$. Then

$$\widetilde{N} := N/T_\Delta = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta) = \widetilde{M}$$

is a minimal (totally geodesic) submanifold.

□

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ a toric manifold, $\dim M = 2n$.

Real points $\mathcal{R}_\Gamma / D_\Gamma \subset \mathcal{Z}_\Gamma / T_\Gamma = M$.
$R := (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma$ subset of real points of M, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M, \quad \dim N = n$.

Theorem

N is an H-minimal Lagrangian submanifold in M.

Idea of proof.

Consider $\widetilde{M} := M /\!\!/ T_\Delta = (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta)$. Then

$$\widetilde{N} := N / T_\Delta = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta) = \widetilde{M}$$

is a minimal (totally geodesic) submanifold.
According to [Y. Dong], $N \subset M$ is H-minimal. □

Example

Example

1. If $m - \ell = 0$, i.e. $\mathcal{Z}_\Delta = \varnothing$, then $M = \mathbb{C}^m$ and we get the original construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$.

Example

1. If $m-\ell=0$, i.e. $\mathcal{Z}_\Delta=\varnothing$, then $M=\mathbb{C}^m$ and we get the original construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$.
2. If $m-n=0$, i.e. $\mathcal{Z}_\Gamma=\varnothing$, then N is set of real points of M. It is minimal (totally geodesic).

Example

1. If $m-\ell=0$, i.e. $\mathcal{Z}_\Delta=\varnothing$, then $M=\mathbb{C}^m$ and we get the original construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$.
2. If $m-n=0$, i.e. $\mathcal{Z}_\Gamma=\varnothing$, then N is set of real points of M. It is minimal (totally geodesic).
3. If $m-\ell=1$, i.e. $\mathcal{Z}_\Delta\cong S^{2m-1}$, then we get H-minimal Lagrangian submanifolds in $M=\mathbb{C}P^{m-1}$.

Reference

Andrey Mironov and Taras Panov. *Intersections of quadrics, moment-angle manifolds, and Hamiltonian-minimal Lagrangian embeddings*. Preprint (2011); arXiv:1103.4970.